MATH 744, FALL 2010
HOMEWORK ASSIGNMENT #7 SOLUTIONS

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(a) Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Verify that $\mathfrak{g}$ is simple of dimension $n^2 - 1$, with the subalgebra $\mathfrak{h}$ of diagonal matrices of trace 0 as a Cartan subalgebra. Find all the roots α (there are $n^2 - n$ of them) and for each one compute the corresponding element H_α. In what way does this example generalize what we did for SU(3)? **Solution.** Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and let $\mathfrak{h}$ be the diagonal matrices in $\mathfrak{g}$. As usual, let e_{ij} be the matrix with a 1 in the (i, j)-entry and 0's everywhere else, so that

$$\begin{cases} [e_{ij}, e_{kl}] = e_{ij}e_{kl} - e_{kl}e_{ij} = \delta_{jk}e_{il} - \delta_{il}e_{kj}. \\ [e_{ii} - e_{jj}, e_{kl}] = (\delta_{ik} - \delta_{il} - \delta_{jk} + \delta_{jl})e_{kl}. \end{cases} \tag{1}$$

Thus $\mathfrak{h}$ is maximal abelian, and of course it consists of semisimple elements (since all elements of $\mathfrak{h}$ are diagonal), so it's a Cartan subalgebra. Note that $\dim \mathfrak{h} = n - 1$ and $\dim \mathfrak{g} = n^2 - 1$.

To show $\mathfrak{g}$ is simple, first let $\mathfrak{g}_{ij} = \mathbb{C}e_{ij}$ for $i \neq j$. There are $n^2 - n$ of these one-dimensional subspaces, and they are eigenspaces for the adjoint action of $\mathfrak{h}$ by (1). So these are the root spaces, and the roots can be identified with pairs (i, j) with $1 \leq i, j \leq n$, $i \neq j$. These act as linear functionals on $\mathfrak{h}$ by $\alpha_{(i,j)}(\operatorname{diag}(x_1, \cdots, x_n)) = x_i - x_j$. Given any ideal $\mathfrak{s}$ in $\mathfrak{g}$, it must be normalized by $\mathfrak{h}$, so $\mathfrak{s}$ is the direct sum of $\mathfrak{s} \cap \mathfrak{h}$ with a sum of root spaces. We already know $\mathfrak{sl}(2, \mathbb{C})$ is simple, so assume $n > 2$. If $\mathfrak{s}$ contains the root space $\mathfrak{g}_{ij}$, $i \neq j$, then it contains $[\mathfrak{g}_{ij}, \mathfrak{g}_{jk}] = \mathfrak{g}_{ik}$ for all $k \neq i, j$, and similarly it contains $[\mathfrak{g}_{ij}, \mathfrak{g}_{ki}] = \mathfrak{g}_{kj}$ for all $k \neq i, j$. Then $\mathfrak{s}$ in fact contains *every* root space, since we can get to every root space by a chain of such brackets. Then it contains all of $\mathfrak{h}$, since $[\mathfrak{g}_{ij}, \mathfrak{g}_{ji}]$ contains $e_{ii} - e_{jj}$, and such elements span $\mathfrak{h}$. On the other hand, if $\mathfrak{s}$ contains a non-zero element H of $\mathfrak{h}$, then $\alpha(H) \neq 0$ for some root, and then $\mathfrak{s}$ contains the root space $\mathfrak{g}_\alpha$, since $[H, \mathfrak{g}_\alpha] = \alpha(H)\mathfrak{g}_\alpha = \mathfrak{g}_\alpha$. So $\mathfrak{g}$ is simple.

Next, pick $\alpha = \alpha_{(i,j)}$ with $i \neq j$. The element H_α must be a multiple of $[e_{ij}, e_{ji}] = e_{ii} - e_{jj}$, and since $[e_{ii} - e_{jj}, e_{ij}] = 2e_{ij}$, $H_\alpha = e_{ii} - e_{jj}$. So the elements H_α generalize the elements $e_{11} - e_{22}$ and $e_{22} - e_{33}$ that showed up in the theory of $SU(3)$.

(b) Continuing with Exercise (a) above, show that one gets a system of $n - 1$ simple roots i associated to the pairs $(i, i + 1)$, $i < n$. Compute the Killing form B restricted to $\mathfrak{h}$, by computing $B(H_{\alpha_i}, H_{\alpha_j})$ for all the simple roots with $1 \leq i, j \leq n - 1$. You should find that the matrix is tridiagonal. **Solution.** Now let's show that the pairs $(i, i + 1)$ with $1 \leq i \leq n - 1$ index a set of simple roots, corresponding to the set of positive roots $\{\alpha_{(i,j)} \mid 1 \leq i < j \leq n\}$. Indeed, if $i \neq j$, either $i < j$ (i.e., the root $\alpha_{(i,j)}$ is positive) or else $i > j$ and $\alpha_{(j,i)} = -\alpha_{(i,j)}$ is positive. Furthermore, if $i < j$, then $\alpha_{(i,j)} = \alpha_{(i,i+1)} + \alpha_{(i+1,i+2)} + \cdots + \alpha_{(j-1,j)}$ can be written uniquely as a linear combination of roots of this form, with nonnegative integer coefficients. So the $\alpha_{(i,i+1)}$ are a set of simple roots.

Finally, let's compute the Killing form $B(H_i, H_j)$, where H_i is shorthand for $H_{\alpha_{(i,i+1)}}$. Note first that H_i acts trivially on the root space $\mathfrak{g}_{kl}$ *unless* k or l coincides with either i or $i + 1$. Thus if $i - j$ is not 0, 1, or -1, $B(H_i, H_j) = 0$, since there are no root spaces for which the action of $\operatorname{ad}(H_i)$ and of $\operatorname{ad}(H_j)$ are simultaneously non-zero. It's also obvious that $B(H_i, H_i)$ is independent of i, since the Weyl group S_n contains a reflection $(1, i)$ conjugating H_i to H_1. So

$$\begin{aligned}
B(H_i, H_i) &= B(H_1, H_1) = \mathrm{Tr}(\mathrm{ad}(H_1)^2) \\
&= \sum_{k<l} \mathrm{Tr}_{\mathfrak{g}_{kl}}(\mathrm{ad}(H_1)^2) + \mathrm{Tr}_{\mathfrak{g}_{lk}}(\mathrm{ad}(H_1)^2) \\
&= 2\sum_{k<l} (\alpha_{(k,l)}(H_1))^2 \\
&= 2(\alpha_{(1,2)}(H_1))^2 + 2\sum_{k=3}^{n} (\alpha_{(1,k)}(H_1))^2 + 2\sum_{k=3}^{n} (\alpha_{(2,k)}(H_1))^2 \\
&= 2\Big(2^2 + (n-2)\cdot 1^2 + (n-2)\cdot 1^2\Big) \\
&= 2(2n) = 4n.
\end{aligned}$$

Similarly, if $|i - j| = 1$, then $B(H_i, H_j) = B(H_1, H_2)$ since there is an element of W conjugating the pair (i, j) to $(1, 2)$. Also note that for $k < l$, $\alpha_{(k,l)}(H_1)$ is only nonzero if $k = 1$ or 2, and $\alpha_{(k,l)}(H_2)$ is only nonzero if $(k, l) = (1, 2)$ or $(1, 3)$ or if $k = 2$ or 3. So in this case,

$$\begin{aligned}
B(H_i, H_j) &= \mathrm{Tr}(\mathrm{ad}(H_1)\,\mathrm{ad}(H_2)) \\
&= \sum_{k<l} \mathrm{Tr}_{\mathfrak{g}_{kl}}(\mathrm{ad}(H_1)\,\mathrm{ad}(H_2)) + \mathrm{Tr}_{\mathfrak{g}_{lk}}(\mathrm{ad}(H_1)\,\mathrm{ad}(H_2)) \\
&= 2\sum_{k<l} \alpha_{(k,l)}(H_1)\alpha_{(k,l)}(H_2) \\
&= 2\alpha_{(1,2)}(H_1)\alpha_{(1,2)}(H_2) + 2\alpha_{(1,3)}(H_1)\alpha_{(1,3)}(H_2) + 2\sum_{k=3}^{n} \alpha_{(2,k)}(H_1)\alpha_{(2,k)}(H_2) \\
&= 2\Big(2\cdot(-1) + 1\cdot 1 + (-1)\cdot 2 + \sum_{k=4}^{n} (-1)\cdot(1)\Big) \\
&= -2\Big(3 + (n-3)) = -2n.
\end{aligned}$$

Thus the matrix $B(H_i, H_j)$ is

$$(2n)\begin{pmatrix} 2 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 \end{pmatrix}.$$