Mathematics 748H: Introduction to Homotopy Theory Exercise Set #4: Weak Equivalence, etc.

Jonathan Rosenberg

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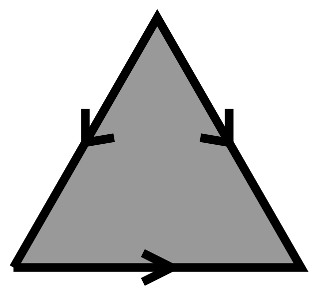
1. The "dunce hat" X is the space obtained by making identifications on the sides of a solid triangle as follows:



(The meaning of the figure is that all three sides of the figure are identified together, as indicated by the arrows.) Show that X can be regarded as a 2-dimensional CW-complex with one 0-cell, one 1-cell, and one 2-cell. Show that X is contractible. (Hint: It is easiest **not** to write down the contraction directly but instead to show there is a weak equivalence between X and the 2-disk.)

2. (This problem is taken from Spanier's book.) A space X is said to be *dominated* by a space Y if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \simeq 1_X$. (Note that there is no condition on $f \circ g$; the symbol $\simeq$ means "is homotopic to.") Suppose X is dominated by a CW-complex. Show that X is homotopy-equivalent to a (possibly different) CW-complex. (Hint: Use a CW-approximation to X and one of the forms of Whitehead's Theorem. By the way, it is **not** true that if a space X is dominated by a *finite* CW-complex, then it is homotopy-equivalent to a finite CW-complex. However, this is true if X is simply connected. Construction of counterexamples requires somewhat sophisticated algebra.)

3. (Graham Segal, *Publ. Math. IHES*, 1968) Let $\mathcal{C}$ be a small category. The *nerve* of the category is the simplicial set $\mathcal{NC}_\bullet$ defined as follows. $\mathcal{NC}_0$, the set

of "0-simplices," is just the set of objects of $\mathcal{C}$; $\mathcal{NC}_1$, the set of "1-simplices," is just the set of morphisms of $\mathcal{C}$; $\mathcal{NC}_2$, the set of "2-simplices," is just the set of commutative triangular diagrams of $\mathcal{C}$; etc. Let $B\mathcal{C} = |\mathcal{NC}_\bullet|$ denote the geometric realization of $\mathcal{NC}_\bullet$ in the sense of May, Ch. 16. Prove the following:

1. If J is the ordered set $\{0, 1\}$, regarded as a category, then $BJ = I$, the unit interval.

2. If $\mathcal{C}, \mathcal{C}'$ are small categories and $F\colon \mathcal{C} \to \mathcal{C}'$ is a functor, there is an induced map $BF\colon B\mathcal{C} \to B\mathcal{C}'$.

3. If $\mathcal{C}$, $\mathcal{C}'$ are small categories and $F_0, F_1\colon \mathcal{C} \to \mathcal{C}'$ are functors, and if $\varphi\colon F_0 \to F_1$ is a natural transformation, then the induced maps
$$BF_0,\, BF_1\colon B\mathcal{C} \to B\mathcal{C}'$$
are homotopic. (Regard φ as a functor $J \times \mathcal{C} \to \mathcal{C}'$, and use (1) and (2).)

4. Deduce that if $\mathcal{C}$, $\mathcal{C}'$ are equivalent categories, then $B\mathcal{C}$ and $B\mathcal{C}'$ are homotopy-equivalent.

5. If $\mathcal{C}$ has exactly one morphism (an isomorphism) from any object to any other object, deduce that $B\mathcal{C}$ is contractible. (Hint: Show that $\mathcal{C}$ is equivalent to the trivial category with just one object and one morphism. Then use (4).)