

MATHEMATICS 748R: HOMOTOPY THEORY
EXERCISE SET #1: FUNDAMENTALS
OF COFIBRATIONS AND FIBRATIONS
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1. Let X be a (Hausdorff) topological space, A a subspace.
 - (i) Show that if there is a retraction r from X onto A , then A is closed in X .
 - (ii) (May, problem 1, p. 46) Deduce that if the inclusion of A into X is a cofibration, then A is closed in X . Hint: Use the characterization of cofibrations, together with (i), applied not to (X, A) but to some other pair.
 - (iii) Show that if A is contractible and the inclusion $A \hookrightarrow X$ is a cofibration, then there is a retraction from X onto A .
 - (iv) Show that not every inclusion of compact Hausdorff spaces is a cofibration. In particular, let $X = [0, 1] \times [0, 1]$ and let A be the union of the line segments from the origin to the points $(0, 1)$ and $(1/n, 1)$, n a positive integer. (This example is in Whitehead's book.) Show that A is contractible, but that there is no retraction from X onto A . Deduce from (iii) that the inclusion of A into X is not a cofibration.

2. (Whitehead Ch. I, Exercise 8) Suppose $A \hookrightarrow X$ is a cofibration and Y is contractible.
 - (i) Show every map $f : A \rightarrow Y$ has an extension $g : X \rightarrow Y$.
 - (ii) Show that any two maps $f_0, f_1 : X \rightarrow Y$ which agree when restricted to A are homotopic (rel. A). Hint: First show

$$(X \times \{0, 1\}) \cup (A \times [0, 1]) \hookrightarrow X \times [0, 1]$$

is an cofibration; then use (i).

- (iii) (an application to analysis, not in Whitehead) Suppose V is a normed vector space over \mathbb{R} . Deduce that every uniformly bounded map $A \rightarrow V$ has an extension to X satisfying the same uniform bound.

3. Suppose $p : X \rightarrow B$ is a fibration and Y is contractible. Show that every map $f : Y \rightarrow B$ has a lifting $g : Y \rightarrow X$.