MATH 748R, Spring 2012
Homotopy Theory
Homework Assignment #2: Fibrations

Jonathan Rosenberg

due Friday, Feburary 17, 2012

1. (An easy slice theorem) Let X be a smooth n-manifold and let G be a compact Lie group acting smoothly and freely on X. Prove that X/G is a smooth manifold of dimension $n - \dim G$ and that the quotient map $p\colon X \to X/G$ is locally the projection of a product, hence a fiber bundle and a Serre fibration with fiber G. (Hint: use the implicit function theorem.)

2. Apply the result of (1) (with $G = SO(n-1)$) to conclude that the map $SO(n) \twoheadrightarrow S^{n-1}$, defined by $g \mapsto g \cdot v_0$, v_0 the north pole of the unit sphere in $\mathbb{R}^n$, is a fibration with fiber $SO(n-1)$. Deduce from the long exact homotopy sequence that the inclusion map $SO(n-1) \hookrightarrow SO(n)$ induces an isomorphism on π_j for $j \leq n-3$. Show from the example of $n = 3$ that it is *not* necessarily an isomorphism on π_{n-2}. Deduce that there is a *stable range* for the homotopy groups of $SO(n)$; $\pi_j(SO(n))$ is independent of n once $n > j+1$.

3. Apply the result of (1) (with $G = U(n-1)$) to conclude that the map $U(n) \twoheadrightarrow S^{2n-1}$, defined by $g \mapsto g \cdot v_0$, v_0 the north pole of the unit sphere in $\mathbb{C}^n$, is a fibration with fiber $U(n-1)$. Deduce from the long exact homotopy sequence that the inclusion map $U(n-1) \hookrightarrow U(n)$ induces an isomorphism on π_j for $j \leq 2n-3$. Show from the example of $n = 2$ that it is *not* necessarily an isomorphism on π_{2n-1}. (Hint: $SU(2)$ can be identified with S^3, and the universal cover of $U(2)$ is $SU(2) \times \mathbb{R}$.) Deduce that there is a *stable range* for the homotopy groups of $U(n)$; $\pi_j(U(n))$ is independent of n once $n > j/2$.