MATH 748R, Spring 2012
Homotopy Theory
Homework Assignment #3:
Whitehead's Theorem and Weak Equivalence

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due Friday, February 24, 2012

1. Show from Whitehead's Theorem that if X and Y are CW complexes and $f\colon X \to Y$ is a weak homotopy equivalence, then f induces isomorphisms on all homology groups. (Actually this is true for arbitrary spaces also.)

2. Show that $\mathbb{RP}^2 \times S^3$ and $S^2 \times \mathbb{RP}^3$ have isomorphic homotopy groups in all degrees. However, show that their homology groups differ. (Hint: Both are compact manifolds, but one is orientable and the other is not.) Then use (1) to deduce that these spaces are *not* weakly homotopy-equivalent.

3. (Hatcher, section 4.1, problem 17) Show that if X and Y are CW complexes (each with a distinguished 0-cell as basepoint) with X m-connected and Y n-connected, then $(X \times Y, X \vee Y)$ is $(m + n + 1)$-connected, as is the smash product $X \wedge Y$.

4. Recall that we proved that $\pi_2(\mathbb{CP}^k) \cong \mathbb{Z}$ for all k. Consider the CW complex $X = S^2 \cup_n e^3$ with one 0-cell, one 2-cell, and one 3-cell, where the 3-cell is attached by a map $h\colon S^2 \to S^2$ representing n in $\pi_2(S^2) \cong \mathbb{Z}$. Compute the set $[X, \mathbb{CP}^k]$ of (based) homotopy classes of (based) maps $f\colon X \to \mathbb{CP}^k$, using the following scheme.[1] (This answer will turn out to be independent of k once $k \geq 2$.) First consider the possibilities for the homotopy class of $g = f|_{S^2}$ in $\pi_2(\mathbb{CP}^k)$. Then given g, note that it extends to a map of X into $\mathbb{CP}^k$ exactly when $g \circ h$ is null-homotopic. When this is the case, you need to see how many extensions there are up to homotopy. You'll need to treat the case $n = 0$ separately.

[1] Since X and $\mathbb{CP}^k$ are both simply connected, it actually doesn't matter whether one keeps track of the basepoint or not.