MATH 748R, Spring 2012
Homotopy Theory
Homework Assignment #4: Applications of the Homotopy Excision and Hurewicz Theorems

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1. **The Relative Hurewicz Theorem**. Starting from the absolute version of the Hurewicz Theorem proved in class, prove the relative version: that if $n \geq 2$ and A is a simply connected subcomplex of a connected CW complex X and (X, A) is $(n-1)$-connected, then $H_i(X, A) = 0$ for $i \leq n-1$ and the Hurewicz map $\pi_n(X, A) \to H_n(X, A)$ is an isomorphism.

2. **A construction of Whitehead and Pontryagin**. Recall that we proved from the exact sequence of the fibration $S^1 \to S^3 \to S^2$ that $\pi_2(S^2) \cong \mathbb{Z}$, and proved from the Homotopy Excision Theorem that the suspension map $\pi_3(S^2) \to \pi_4(S^3)$ is surjective. Homotopy Excision also proves that the further suspension homomorphisms

$$\pi_4(S^3) \to \pi_5(S^4) \to \pi_6(S^5) \to \cdots$$

are all isomorphisms. However, this left open the question of what $\pi_4(S^3)$ actually is. Whitehead constructed a homomorphism now called the *J-homomorphism* $\pi_j(SO(k)) \to \pi_{j+k}(S^k)$ which can be used to construct an element of order 2 in $\pi_4(S^3)$, since $SO(3) \cong \mathbb{RP}^3$ has fundamental group $\mathbb{Z}/2$. In fact the J-homomorphism is an isomorphism from $\pi_1(SO(3))$ to $\pi_4(S^3)$; the inverse map $\pi_4(S^3) \to \pi_1(SO(3))$ is given by a construction of Pontryagin. Fill in as much as you can of the following sketch:

   (a) We start with Pontryagin's construction. Let $f\colon S^4 \to S^3$. Without loss of generality we may assume f is smooth. By Sard's Theorem, we can pick a regular value $z \in S^3$, and $f^{-1}(z)$ is (by the Implicit Function Theorem) a compact submanifold of S^3 of dimension $4 - 3 = 1$. In other words, it is a finite union of circles. These circles acquire orientations from the usual orientations of S^4 and S^3. Furthermore, if we fix a frame (oriented basis for the tangent space) at z, pulling this back gives a *framing* of $f^{-1}(z)$, i.e., a smoothly varying family of frames, for the 3-dimensional *normal bundle* of $f^{-1}(z)$. Since the normal bundle is necessarily trivial (why?), if $f^{-1}(z)$ is connected, this gives us a map $S^1 \to GL^+(3,\mathbb{R}) \simeq SO(3)$. (Here $GL^+(3,\mathbb{R})$ is the group of 3×3 matrices with positive determinant; it may be identified with the set of oriented frames of $\mathbb{R}^3$, and has a deformation retraction down to $SO(3)$ by *polar decompostion* from linear algebra.) Show that in this way one gets a well-defined map $\pi_4(S^3) \to \pi_1(SO(3)) \cong \mathbb{Z}/2$. (There are lots of things to check. See the last chapter of Milnor, *Topology from the Differentiable Viewpoint*, if you get stuck. If $f^{-1}(z)$ has multiple components, just add the corresponding elements of $\pi_1(SO(3))$.)

(b) Now consider Whitehead's construction. A class in $\pi_1(SO(3))$ corresponds to a homotopy class of maps $S^1 \to SO(3) \subset \mathrm{Diff}^+(S^2)$ (via the transitive action of $SO(3)$ on S^2 by rotations), and thus to a map $h\colon S^1 \times S^2 \to S^2$. Note that $S^1 \times S^2 = (S^1 \vee S^2) \cup_f e^3$, where the attaching map $[f] \in \pi_2(S^1 \vee S^2) \cong \bigoplus_n \mathbb{Z}e_n$ can be identified with $e_1 - e_0$ (why?). When we suspend, Σf becomes homotopically trivial and so $\Sigma(S^1 \times S^2) \simeq S^2 \vee S^3 \vee S^4$. (Prove this by using the fact that $(S^2 \times S^3, S^2 \vee S^3)$ is 4-connected to compute $\pi_3(S^2 \vee S^3)$, and then see where the suspension map takes $[f] \in \pi_2(S^1 \vee S^2)$.) Take the composite $S^4 \hookrightarrow \Sigma(S^1 \vee S^2) \xrightarrow{\Sigma h} S^3$, and show that in this way we get a homomorphism $\pi_1(SO(3)) \to \pi_4(S^3)$.

(c) (extra credit) See if you can show the maps of Pontryagin in (a) and of Whitehead in (b) are inverse to one another.