1. Use the theory of Postnikov systems to classify (up to homotopy equivalence) all CW complexes \( X \) with \( \pi_2(X) \cong \pi_3(X) \cong \mathbb{Z} \) and all other homotopy groups 0. (It should turn out there is a one-parameter family of such \( X \)'s; how do you distinguish them?)

*Solution.* These complexes come with a fibration

\[
\begin{array}{ccc}
K(\mathbb{Z}, 3) & \longrightarrow & X \\
p \downarrow & & \downarrow \\
K(\mathbb{Z}, 2) & & 
\end{array}
\]

and so are classified by the \( k \)-invariant

\[
k \in [K(\mathbb{Z}, 2), BK(\mathbb{Z}, 3)] = H^4(K(\mathbb{Z}, 2), \mathbb{Z}) = H^4(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}.
\]

So for each integer \( k \), we have a corresponding space \( X_k \) with this \( k \)-invariant. How does one distinguish them? Well, \( X_k \) comes with a Serre spectral sequence \( H^p(K(\mathbb{Z}, 2), H^q(K(\mathbb{Z}, 3), \mathbb{Z})) \Rightarrow H^{p+q}(X_k, \mathbb{Z}) \). By the Hurewicz Theorem, the Hurewicz map \( \pi_j(K(\mathbb{Z}, 3)) \rightarrow H_j(K(\mathbb{Z}, 3), \mathbb{Z}) \) is an isomorphism for \( j = 3 \) and is surjective for \( j = 4 \). Thus \( H^q(K(\mathbb{Z}, 3), \mathbb{Z}) = 0 \) for \( q = 1, 2, 4 \) and \( \cong \mathbb{Z} \) for \( q = 0, 3 \), so the bottom rows of the spectral sequence look like

\[
\begin{array}{cccccccc}
& & & & & & & \\
& Z & 0 & 0 & 0 & 0 & \cdots \\
& 0 & 0 & 0 & 0 & \cdots \\
& 0 & 0 & 0 & \cdots \\
& Z & 0 & \cdots \\
\end{array}
\]

\[\xRightarrow{p}\]

\[
\begin{array}{cccccccc}
& & & & & & & \\
& Z & 0 & 0 & 0 & 0 & \cdots \\
& 0 & 0 & 0 & 0 & \cdots \\
& Z & 0 & \cdots \\
\end{array}
\]
So we’ll always have $H^j(X_k, \mathbb{Z}) = 0$ for $q = 1$ and $\cong \mathbb{Z}$ for $q = 2$ (this is also obvious from the Hurewicz Theorem) and the first place anything interesting can happen is in total degree 3 or 4. If $k = 0$, $X_k \cong K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)$ and $d_4 = 0$, so $H^3(X_k, \mathbb{Z}) \cong \mathbb{Z}$. But the $k$-invariant in $H^4(K(\mathbb{Z}, 3), \mathbb{Z})$ turns out to be precisely the image of the standard generator of $H^3(K(\mathbb{Z}, 3), \mathbb{Z})$ in $H^4(K(\mathbb{Z}, 3), \mathbb{Z})$ under $d_4$. So if the $k$-invariant is non-zero, $H^3(X_k, \mathbb{Z}) = 0$ and $H^4(X_k, \mathbb{Z}) \cong \mathbb{Z}/k$. So we can read $k$ (up to sign) off from the cohomology of $X_k$. □

2. Classify up to homotopy all maps $K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$, and show that if you think of $K(\mathbb{Z}, 2)$ as $BS^1$, that they all arise as $B\phi$ for some homomorphism of topological groups $\phi$: $S^1 \to S^1$.

**Solution.** We have $[K(\mathbb{Z}, 2), K(\mathbb{Z}, 2)] \cong H^2(K(\mathbb{Z}, 2), \mathbb{Z}) \cong \mathbb{Z}$. So a map $K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$ is determined by the integer $k$ by which it multiplies on $H_2$ or $H^2$. Now for each $k$, we have the continuous homomorphism $\phi_k: z \mapsto z^k$ from $S^1$ to $S^1$, which induces multiplication by $k$ on $\pi_1(S^1)$. Since the function $B$ shifts homotopy groups up in dimension by 1, $B\phi_k$ induces multiplication by $k$ on $\pi_2$ or $H_2$. Thus all possible maps $BS^1 \to B^1$ arise as $B\phi$ for some continuous group homomorphism $\phi$. □

3. Classify up to homotopy all maps $K(\mathbb{Z}/2, 1) \to K(\mathbb{Z}, 2)$, and show that if you think of $K(\mathbb{Z}/2, 1)$ as $B(\mathbb{Z}/2)$, that they all arise as $B\phi$ for some group homomorphism $\phi$: $(\mathbb{Z}/2) \to S^1$.

**Solution.** Just as in the last problem, we have $[K(\mathbb{Z}/2, 1), K(\mathbb{Z}, 2)] \cong H^2(K(\mathbb{Z}/2, 1), \mathbb{Z}) \cong H^2(\mathbb{R}P^\infty, \mathbb{Z}) \cong \mathbb{Z}/2$. So there are exactly two homotopy classes of maps $K(\mathbb{Z}/2, 1) \to K(\mathbb{Z}, 2)$. (The based and unbased classifications are the same since both spaces are simple.) But there are two homomorphisms $\phi$: $(\mathbb{Z}/2) \to S^1$. The trivial homomorphism clearly induces the null-homotopic map. The injective homomorphism induces the map $E\mathbb{Z}/2 = \mathbb{S}^0 * \mathbb{S}^0 * \cdots \to S^1 * S^1 * \cdots = ES^1$ corresponding to the inclusion $S^0 \hookrightarrow S^1$ and this map is not equivariantly homotopically trivial. So both homotopy classes of maps $K(\mathbb{Z}/2, 1) \to K(\mathbb{Z}, 2)$ arise as $B\phi$ for some homomorphism $\phi$: $(\mathbb{Z}/2) \to S^1$. □

4. Let $X$ be a connected CW complex with a distinguished 0-cell as basepoint. Show that if $G$ is a discrete group, any homomorphism $\pi_1(X) \to G$ can be realized by a unique homotopy class of based maps $X \to K(G, 1)$. (This is a slight variant of a theorem proved in class.)

**Solution.** Without loss of generality we can assume $X$ has a single 0-cell. Then the 1-skeleton $X^1$ of $X$ can be assumed to be a wedge of $S^1$’s indexed by a set of generators $x_i$ for $\pi_1(X)$, and the 2-skeleton has additional 2-cells corresponding to the relations $r_j(x_1, x_2, \cdots)$ in a presentation for $\pi_1(X)$. Clearly there is a unique map $X^1 \to K(G, 1)$ sending each $x_i$ to $\phi(x_i) \in G = \pi_1(K(G, 1))$. This map can be extended to a map $X^2 \to K(G, 1)$ since each $[r_j]$ goes to 0 in $G$. Then the obstructions to extending to higher skeleta all vanish, since $K(G, 1)$ has all its higher homotopy groups $= 0$. Similarly, suppose $f$ and $f'$ are two maps $X \to K(G, 1)$ inducing the same map on $\pi_1$. Then $f|_{X^1}$ and $f'|_{X^1}$ are homotopic, since each $x_i$ goes to the same element of $G$ under the two maps. Again, the obstructions to a homotopy
\[ f|_X \simeq f'|_X \] live in \( H^n(X, \pi_1(K(G,1))) = 0 \) for \( n = 2 \), so we can extend by induction to a homotopy \( f \simeq f' \). □