MATH 748R, Spring 2012
Homotopy Theory
Homework Assignment #5: Eilenberg-MacLane Spaces and Obstruction Theory

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Solutions

1. Use the theory of Postnikov systems to classify (up to homotopy equivalence) all CW complexes X with $\pi_2(X) \cong \pi_3(X) \cong \mathbb{Z}$ and all other homotopy groups 0. (It should turn out there is a one-parameter family of such X's; how do you distinguish them?)

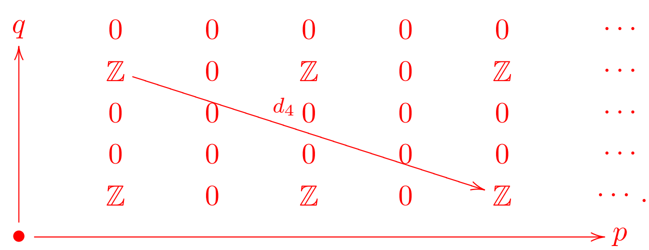
   Solution. These complexes come with a fibration

$$\begin{array}{ccc} K(\mathbb{Z},3) & \longrightarrow & X \\ & & \downarrow p \\ & & K(\mathbb{Z},2) \end{array}$$

   and so are classified by the k-invariant

$$k \in [K(\mathbb{Z},2), BK(\mathbb{Z},3)] = H^4(K(\mathbb{Z},2),\mathbb{Z}) = H^4(\mathbb{CP}^\infty,\mathbb{Z}) \cong \mathbb{Z}.$$

   So for each integer k, we have a corresponding space X_k with this k-invariant. How does one distinguish them? Well, X_k comes with a Serre spectral sequence $H^p(K(\mathbb{Z},2), H^q(K(\mathbb{Z},3),\mathbb{Z}) \Rightarrow H^{p+q}(X_k,\mathbb{Z})$. By the Hurewicz Theorem, the Hurewicz map $\pi_j(K(\mathbb{Z},3)) \to H_j(K(\mathbb{Z},3),\mathbb{Z})$ is an isomorphism for $j = 3$ and is surjective for $j = 4$. Thus $H^q(K(\mathbb{Z},3),\mathbb{Z}) = 0$ for $q = 1, 2, 4$ and $\cong \mathbb{Z}$ for $q = 0, 3$, so the bottom rows of the spectral sequence look like



So we'll always have $H^j(X_k, \mathbb{Z}) = 0$ for $q = 1$ and $\cong \mathbb{Z}$ for $q = 2$ (this is also obvious from the Hurewicz Theorem) and the first place anything interesting can happen is in total degree 3 or 4. If $k = 0$, $X_k \simeq K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)$ and $d_4 = 0$, so $H^3(X_k, \mathbb{Z}) \cong \mathbb{Z}$. But the k-invariant in $H^4(K(\mathbb{Z}, 3), \mathbb{Z})$ turns out to be precisely the image of the standard generator of $H^3(K(\mathbb{Z}, 3), \mathbb{Z})$ in $H^4(K(\mathbb{Z}, 3), \mathbb{Z})$ under d_4. So if the k-invariant is non-zero, $H^3(X_k, \mathbb{Z}) = 0$ and $H^4(X_k, \mathbb{Z}) \cong \mathbb{Z}/k$. So we can read k (up to sign) off from the cohomology of X_k. $\square$

2. Classify up to homotopy all maps $K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$, and show that if you think of $K(\mathbb{Z}, 2)$ as BS^1, that they all arise as $B\phi$ for some homomorphism of topological groups $\phi\colon S^1 \to S^1$.

   Solution. We have $[K(\mathbb{Z}, 2), K(\mathbb{Z}, 2)] \cong H^2(K(\mathbb{Z}, 2), \mathbb{Z}) \cong \mathbb{Z}$. So a map $K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$ is determined by the integer k by which it multiplies on H_2 or H^2. Now for each k, we have the continuous homomorphism $\phi_k\colon z \mapsto z^k$ from S^1 to S^1, which induces multiplication by k on $\pi_1(S^1)$. Since the functor B shifts homotopy groups up in dimension by 1, $B\phi_k$ induces multiplication by k on π_2 or H_2. Thus all possible maps $BS^1 \to B^1$ arise as $B\phi$ for some continuous group homomorphism ϕ. $\square$

3. Classify up to homotopy all maps $K(\mathbb{Z}/2, 1) \to K(\mathbb{Z}, 2)$, and show that if you think of $K(\mathbb{Z}/2, 1)$ as $B(\mathbb{Z}/2)$, that they all arise as $B\phi$ for some group homomorphism $\phi\colon (\mathbb{Z}/2) \to S^1$.

   Solution. Just as in the last problem, we have $[K(\mathbb{Z}/2, 1), K(\mathbb{Z}, 2)] \cong H^2(K(\mathbb{Z}/2, 1), \mathbb{Z}) \cong H^2(\mathbb{RP}^\infty, \mathbb{Z}) \cong \mathbb{Z}/2$. So there are exactly two homotopy classes of maps $K(\mathbb{Z}/2, 1) \to K(\mathbb{Z}, 2)$. (The based and unbased classifications are the same since both spaces are simple.) But there are two homomorphisms $\phi\colon (\mathbb{Z}/2) \to S^1$. The trivial homomorphism clearly induces the null-homotopic map. The injective homomorphism induces the map $E\mathbb{Z}/2 = S^0 * S^0 * \cdots \to S^1 * S^1 * \cdots = ES^1$ corresponding to the inclusion $S^0 \hookrightarrow S^1$ and this map is *not* equivariantly homotopically trivial. So both homotopy classes of maps $K(\mathbb{Z}/2, 1) \to K(\mathbb{Z}, 2)$ arise as $B\phi$ for some homomorphism $\phi\colon (\mathbb{Z}/2) \to S^1$. $\square$

4. Let X be a connected CW complex with a distinguished 0-cell as basepoint. Show that if G is a discrete group, any homomorphism $\pi_1(X) \to G$ can be realized by a unique homotopy class of based maps $X \to K(G, 1)$. (This is a slight variant of a theorem proved in class.)

   Solution. Without loss of generality we can assume X has a single 0-cell. Then the 1-skeleton X^1 of X can be assumed to be a wedge of S^1's indexed by a set of generators x_i for $\pi_1(X)$, and the 2-skeleton has additional 2-cells corresponding to the relations $r_j(x_1, x_2, \cdots)$ in a presentation for $\pi_1(X)$. Clearly there is a unique map $X^1 \to K(G, 1)$ sending each x_i to $\phi(x_i) \in G = \pi_1(K(G, 1))$. This map can be extended to a map $X^2 \to K(G, 1)$ since each $[r_j]$ goes to 0 in G. Then the obstructions to extending to higher skeleta all vanish, since $K(G, 1)$ has all its higher homotopy groups $= 0$. Similarly, suppose f and f' are two maps $X \to K(G, 1)$ inducing the same map on π_1. Then $f|_{X^1}$ and $f'|_{X^1}$ are homotopic, since each x_i goes to the same element of G under the two maps. Again, the obstructions to a homotopy

$f|_{X^n} \simeq f'|_{X^n}$ live in $H^n(X, \pi_1(K(G,1))) = 0$ for $n = 2$, so we can extend by induction to a homotopy $f \simeq f'$. $\square$