MATH 748R, Spring 2012
Homotopy Theory
Homework Assignment #6:
The Serre Spectral Sequence and Applications

Jonathan Rosenberg

Solutions

1. Show that an even-dimensional sphere cannot be the total space of a spherical fibration over a sphere (regardless of the dimensions of the fiber and the base). Also show that if an odd-dimensional sphere of dimension $2n+1$ is the total space of a spherical fibration over a sphere, then the base must have dimension $n+1$ and the fiber must have dimension n. (This case of course is possible; think of S^3 as an S^1-bundle over S^2, or of S^7 as an S^3-bundle over S^4.)

   Solution. First suppose that S^{2n} is a fibration over S^k, with fiber S^m. We can't have $k = 1$, since any map $S^{2n} \to S^1$ is null-homotopic (since $S^1 = K(\mathbb{Z}, 1)$ and S^{2n} is simply connected). So the base of the fibration is simply connected and we have a Serre spectral sequence $H_p(S^k, H_q(S^m, \mathbb{Z})) \Rightarrow H_{p+q}(S^{2n}, \mathbb{Z})$. We need something non-zero in E^2 in total degree $p + q = 2n$, and everything else in other positive total degrees must die in E^∞. Since we start with $\mathbb{Z}$'s in total degrees k, m, and $k+m$, and differentials have bidegree $(-r, r-1)$, the only possibility is to have $m = k - 1$ and $k + m = 2n$, which is impossible, since this implies $2m = 2n-1$. The case where we start with an odd sphere S^{2n+1} and $n > 0$ is similar, but this time it is possible to have $m = k - 1$ and $k + m = 2n + 1$, so $2m = 2n$, $m = n$, and $k = m + 1 = n + 1$, as long as $d_{n+1}\colon E^{n+1}_{n+1,0} \to E^{n+1}_{0,n}$ is an isomorphism. Finally, there is one and only one case where the base is not simply connected: S^1 is a bundle over S^1 with fiber $S^0 = \mathbb{Z}/2$, via the covering map $S^1 \to S^1$ of degree 2. $\square$

2. Recall that we used the Serre spectral sequence in homology to show in class that $H_j(\Omega S^n, \mathbb{Z}) = 0$ for j not a multiple of $n - 1$ and that $H_j(\Omega S^n, \mathbb{Z}) \cong \mathbb{Z}$ for j a multiple of $n - 1$. Use the Serre spectral sequence in *cohomology* to show that for n odd, $H^*(\Omega S^n, \mathbb{Q})$ is a polynomial ring on a single generator in degree $n - 1$. What goes wrong with the argument if you use integral instead of rational coefficients?

   Solution. Let $n > 1$ and let $A = \bigoplus_j H^j(\Omega S^n, \mathbb{Q})$ be the rational cohomology ring of ΩS^n. The Serre spectral sequence for the path fibration of S^n has $E_2^{p,q} = \bigwedge_{\mathbb{Q}}(x) \otimes A$ and must have $E_\infty^{p,q} = 0$ except with $p = q = 0$. Here $\bigwedge_{\mathbb{Q}}(x) = \mathbb{Q}[x]/(x^2)$ is the cohomology ring of S^n, with x in degree n, and the usual rules of graded commutativity apply, so x commutes with all of A if n is even, commutes with A^{even} if n is odd, but anticommutes with A^{odd} if n is odd. Since $E_2^{p,q} = 0$ unless $p = 0$ or $p = n$, the only possible differential is d_n, which sends $E_2^{0,q} = A^q$ to $E_2^{n,q-n+1} = \mathbb{Q}x \otimes A^{q-n+1}$. And since we know by Hurewicz that $A^q = 0$ for $0 < q < n-1$ and that $\dim A^{n-1} = 1$, there must be a class $y \in A^{n-1}$ such that $d_n(y) = x$. Furthermore, by the argument given in class (induction on q and the fact that d_n must be an isomorphism), $\dim A^q = 0$ unless q is a multiple of $n-1$, and $\dim_{\mathbb{Q}} A^{k(n-1)} = 1$ for all k. If n is odd, $n-1$ is even so the cup product powers of y are potentially non-zero. If y does not generate a polynomial ring, then there is some smallest k for which $y^k \neq 0$ and $y^{k+1} = 0$. (No other dependence relations are possible since the powers of y live in different degrees.) By the derivation property, $d_n(y^{k+1}) = (k+1)y^k x \neq 0$, which contradicts the assumption that $y^{k+1} = 0$.

Working over $\mathbb{Z}$, the argument for a polynomial ring breaks down, since y^k can be non-zero and still not generate $H^{k(n-1)}(\Omega S^n) \cong \mathbb{Z}$ integrally. (It may only be a multiple of a generator.) □

3. Use the Serre spectral sequence in cohomology to give another proof (not dependent on Poincaré duality) that the cohomology ring of $\mathbb{CP}^n$ is $\mathbb{Z}[u]/(u^{n+1})$, where u has degree 2. Show similarly that the cohomology ring of $\mathbb{HP}^n$ is $\mathbb{Z}[u]/(v^{n+1})$, where v has degree 4. (Use the Hopf fibrations $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ and $S^3 \to S^{4n+3} \to \mathbb{HP}^n$.)

   Solution. First consider the fibration $S^1 \to S^{2n+1} \to \mathbb{CP}^n$. Since he cohomology of S^1 is torsion free, E_2 of the Serre spectral sequence is $H^p(\mathbb{CP}^n, \mathbb{Z}) \otimes H^q(S^1, \mathbb{Z})$, which is non-zero only for $q = 0$ or 1. Thus there is only one possible differential, d_2. Let y be the usual generator of $H^1(S^1, \mathbb{Z})$, viewed as living in $E_2^{0,1}$. Let $d(y) = u \in H^2(\mathbb{CP}^n, \mathbb{Z})$. Then $d(uy) = d(u)y + ud(y) = 0 + u \cdot u = u^2$. Similarly, $d(u^2 y) = u^3$, etc. Since the spectral sequence must converge to the cohomology of S^{2n+1}, $E_\infty^{p,q} = E_3^{p,q} = 0$ unless $p = q = 0$ or $p = 2n$, $q = 1$. We claim this implies that u^k must generate $H^{2k}(\mathbb{CP}^n, \mathbb{Z})$ for $k \leq n$, and $H^{2k+1}(\mathbb{CP}^n, \mathbb{Z})$ must vanish for all k. Indeed, $H^1(\mathbb{CP}^n, \mathbb{Z}) = 0$ either by simple connectivity or because there is nothing in bidegree $(-1, 1)$ to kill it. And d_2 must be an isomorphism from $\mathbb{Z}y$ to $E_2^{2,1}$. So $u \neq 0$ and generates $H^2(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}$. Next, $H^3(\mathbb{CP}^n, \mathbb{Z}) = 0$ since there is nothing in bidegree $(1, 1)$ to cancel it, and d_2 must be an isomorphism from $E_2^{2,1} = \mathbb{Z}uy$ to $E_2^{4,0} = H^4(\mathbb{CP}^n, \mathbb{Z})$. So H^4 is infinite cyclic and generated by u^2. We continue this way to show that the powers of u generate all the cohomology ring. Of course $u^{n+1} = 0$ since $\mathbb{CP}^n$ is $2n$-dimensional and u^{n+1} lives in degree $2n+2$. So $H^*(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}[u]/(u^{n+1})$.

   The case of $\mathbb{HP}^n$ is exactly analogous, except that since the fiber of the Hopf fibration $S^{24+3} \to \mathbb{HP}^n$ is S^3, y is replaced by z, the usual generator of $H^3(S^3)$, and the only differential is d_4, of bidegree $(4, -3)$. The class u is replaced by $v = d_4(z) \in H^4(\mathbb{HP}^n)$. To get the induction started, $E_2^{p,0} = H^p(\mathbb{HP}^n, \mathbb{Z}) = 0$ for $j = 1, 2, 3$, either because the usual cell decomposition of $\mathbb{HP}^n$ has no cells in these dimensions or because there is nothing to kill them in the $q = 3$ row. $\square$