MATH 748R, Spring 2012
Homotopy Theory
Homework Assignment #6:
The Serre Spectral Sequence and Applications

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due Monday, April 16, 2012

1. Show that an even-dimensional sphere cannot be the total space of a spherical fibration over a sphere (regardless of the dimensions of the fiber and the base). Also show that if an odd-dimensional sphere of dimension $2n+1$ is the total space of a spherical fibration over a sphere, then the base must have dimension $n+1$ and the fiber must have dimension n. (This case of course is possible; think of S^3 as an S^1-bundle over S^2, or of S^7 as an S^3-bundle over S^4.)

2. Recall that we used the Serre spectral sequence in homology to show in class that $H_j(\Omega S^n, \mathbb{Z}) = 0$ for j not a multiple of $n-1$ and that $H_j(\Omega S^n, \mathbb{Z}) \cong \mathbb{Z}$ for j a multiple of $n-1$. Use the Serre spectral sequence in *cohomology* to show that for $n \geq 3$ odd, $H^*(\Omega S^n, \mathbb{Q})$ is a polynomial ring on a single generator in degree $n-1$. What goes wrong with the argument if you use integral instead of rational coefficients?

3. Use the Serre spectral sequence in cohomology to give another proof (not dependant on Poincaré duality) that the cohomology ring of $\mathbb{CP}^n$ is $\mathbb{Z}[u]/(u^{n+1})$, where u has degree 2. Show similarly that the cohomology ring of $\mathbb{HP}^n$ is $\mathbb{Z}[u]/(v^{n+1})$, where v has degree 4. (Use the Hopf fibrations $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ and $S^3 \to S^{4n+3} \to \mathbb{HP}^n$.)