MATH 748R, Spring 2012
Homotopy Theory
Homework Assignment #7: Serre Classes and Applications

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Solutions

1. Suppose X is a connected CW complex with $\pi_{2k}(X) \cong \mathbb{Z}$ for $2k \geq 2$ even and with $\pi_{2k+1}(X) = 0$ for $k \geq 0$. Thus the Postnikov system of X is built out of $K(\mathbb{Z}, 2k)$'s. Show that the rational cohomology ring of X is a polynomial ring over $\mathbb{Q}$ on generators in degrees $2, 4, 6, \cdots$. (The theorem is not vacuous; it turns out, as we will see later, that BU satisfies the hypothesis.) Hint: use what we proved about rational cohomology of $K(\mathbb{Z}, n)$.

   Solution. We claim by induction on n that the rational cohomology ring of X_{2n}, the Postnikov approximation to X based on the homotopy groups through degree $2n$, is a polynomial ring $\mathbb{Q}[x_s, \cdots, x_{2n}]$. The result then follows since $H^*(X, \mathbb{Q}) = \varinjlim H^*(X_{2n}, \mathbb{Q}) = \mathbb{Q}[x_s, \cdots, x_{2n}, \cdots]$.

   To begin the induction, note that $X_2 = K(\mathbb{Z}, 2) \simeq \mathbb{CP}^\infty$, whose cohomology ring is a polynomial ring on one generator x_2 in degree 2. For the inductive step, assume the result is true for X_{2n-2}, and consider X_{2n}. Since $\pi_{2n-1}(X) = 0$, we have a principal fibration

$$\begin{array}{ccc} K(\mathbb{Z}, 2n) & \longrightarrow & X_{2n} \\ & & \downarrow \\ & & X_{2n-2}. \end{array}$$

   Apply the Serre spectral sequence in rational cohomology. Since all the rational cohomology of both fiber and base is concentrated in even degrees, there cannot be any nonzero differentials, so

$$H^*(X_{2n}, \mathbb{Q}) \cong H^*(X_{2n-2}, \mathbb{Q}) \otimes H^*(K(\mathbb{Z}, 2n), \mathbb{Q}),$$

   even as $\mathbb{Q}$-algebras. Since $H^*(K(\mathbb{Z}, 2n), \mathbb{Q}) = \mathbb{Q}[x_{2n}]$, the inductive step follows. □

2. The second stable homotopy group of spheres is $\pi_2^s = \varinjlim \pi_{n+2}(S^n)$, which by the Freudenthal Suspension Theorem can be computed as $\pi_6(S^4)$. Investigate this group as follows.

   (a) Observe from the fibration $S^3 \to S^7 \to S^4$ that $\pi_6(S^4) \cong \pi_5(S^3)$, so that the stable range is already achieved with $\pi_5(S^3)$.

   Solution. The long exact sequence of the given fibration gives $0 = \pi_6(S^7) \to \pi_6(S^4) \to \pi_5(S^3) \to \pi_6(S^7) = 0$, so $\pi_6(S^4) \cong \pi_5(S^3)$. □

   (b) Recall that there is a homotopy fibration

$$K(\mathbb{Z}, 2) \to F \xrightarrow{p} S^3, \tag{1}$$

   where $p\colon F \to S^3$ is the homotopy fiber of the map $S^3 \to K(\mathbb{Z}, 3)$ inducing an isomorphism on π_3. Thus F is 3-connected. Show from the spectral sequence of (1) (using the derivation property for the differentials and the fact that the cohomology ring of $\mathbb{CP}^\infty$ is a polynomial ring on one generator in degree 2) that $H^{2k}(F, \mathbb{Z}) = 0$ for $2k$ even and that $H^{2k+1}(F, \mathbb{Z}) \cong \mathbb{Z}/k$ for $2k + 1 \geq 5$ odd. In particular, $H^5(F, \mathbb{Z}) \cong \mathbb{Z}/2$, which is how we showed that $\pi_4(S^3) \cong \mathbb{Z}/2$.

   Solution. The spectral sequence for (1) in integral cohomology has $E_2^{p,q} = H^p(S^3, \mathbb{Z}) \otimes H^q(\mathbb{CP}^\infty, \mathbb{Z}) = \mathbb{Z}[x, y]/(y^2)$, where x is a polynomial generator in bidegree $(0, 2)$ and y is a generator for $H^3(S^3)$ in bidegree $(3, 0)$. Furthermore, we know the spectral sequence converges to the cohomology of the 4-connected space F. Thus x and y have to cancel out. Since the spectral sequence has only two columns, with $p = 0$ and $p = 3$, the only nonzero differential is d_3, and we must have $d_3 x = y$ (once sign conventions are properly chosen). Then by the derivation property, $d_3(x^k) = k\, x^{k-1} y$, and d_3 is injective on the $p = 0$ column (except when $q = 0$) and we obtain $E_\infty^{0,q} = 0$, $q > 0$, and $E_\infty^{3,2(k-1)} \cong \mathbb{Z}/k$. Thus $\widetilde{H}^{\text{even}}(F, \mathbb{Z}) = 0$ and $H^{2k+1}(F, \mathbb{Z}) \cong \mathbb{Z}/k$. Since all the reduced cohomology is torsion, the homology groups are the same, but shifted down by one in degree. □

   (c) To compute $\pi_5(S^3)$, carry this process one step further; let $p'\colon F' \to F$ be the homotopy fiber of the map $F \to K(\mathbb{Z}/2, 4)$ inducing an isomorphism on π_4, and show that you get a homotopy fibration

$$K(\mathbb{Z}/2, 3) \to F' \xrightarrow{p'} F. \tag{2}$$

   Solution. By (b), $H_4(F, \mathbb{Z}) \cong \mathbb{Z}/2$, so also $\pi_4(F) \cong \mathbb{Z}/2$. Thus there is a map $F \to K(\mathbb{Z}/2, 4)$ inducing an isomorphism on π_4; we let $p'\colon F' \to F$ be its homotopy fiber. The exact sequence of $F' \xrightarrow{p'} F \to K(\mathbb{Z}/2, 4)$ shows that we can also view p' as a homotopy fibration with homotopy fiber $\Omega K(\mathbb{Z}/2, 4) = K(\mathbb{Z}/2, 3)$. □

   (d) To finish the calculation, use the spectral sequence of the fibration (2) and the fact that F' is 4-connected. Modulo the Serre class of 2-primary finite groups, show p' is a homotopy equivalence, and thus that F' has no cohomology other than 2-primary torsion below degree 7. Conclude that $\pi_5(S^3)$ is a 2-primary finite group.

Solution. Modulo 2-primary torsion groups, $K(\mathbb{Z}/2,3)$ is acyclic and so p' is a homotopy equivalence. Now $\widetilde{H}^*(F,\mathbb{Z})$ is concentrated in odd degrees and begins with $\mathbb{Z}/2$ in degree 5 and $\mathbb{Z}/3$ in degree 7. So modulo 2-primary torsion, $\widetilde{H}^*(F,\mathbb{Z})$ begins with a $\mathbb{Z}/3$ in degree 7. In particular, $\pi_5(S^3) = \pi_5(F)$ coincides modulo 2-primary torsion with $H_5(F')$, and so $\pi_5(S^3)$ is a 2-primary torsion group. □

(e) Finally, compute the 2-primary torsion in $\pi_5(S^3)$ by using the spectral sequence of (2) with $\mathbb{F}_2$ coefficients. You will need to observe that $H^i(F,\mathbb{F}_2) \cong 0$ for $i = 6$ and that $H^i(K(\mathbb{Z}/2,3),\mathbb{F}_2) \cong \mathbb{F}_2$ for $i = 3,4,5$. This last fact can be deduced from path fibrations and the facts that $\Omega K(\mathbb{Z}/2,3) \simeq K(\mathbb{Z}/2,2)$, $\Omega K(\mathbb{Z}/2,2) \simeq \mathbb{RP}^\infty$, $H^*(\mathbb{RP}^\infty,\mathbb{F}_2) \cong \mathbb{F}_2[a]$, where a is the canonical element in degree 1.

Solution. We know that $\pi_5(S^3) \cong \pi_5(F) \cong \pi_5(F') \cong H_5(F',\mathbb{Z}) \cong H^6(F',\mathbb{Z})$ (the last step by the UCT, since the homology is torsion). On the other hand, we know $H^6(F,\mathbb{Z})$ by (b), so we can compute $H^6(F',\mathbb{Z})$ using the spectral sequence for (2) once we compute the low-dimensional cohomology of $K(\mathbb{Z}/2,3)$.

The cohomology of $K(\mathbb{Z}/2,1) = \mathbb{RP}^\infty$ (with $\mathbb{F}_2$ coefficients) is $\mathbb{F}_2[a]$. Then from the path fibration of $K(\mathbb{Z}/2,2)$, we get a spectral sequence with $E_2^{p,q} = H^p(K(\mathbb{Z}/2,2),\mathbb{F}_2)\otimes_{\mathbb{F}_2}\mathbb{F}_2[a]$ and with $E_\infty^{p,q} = 0$ unless $p = q = 0$. Furthermore, by Hurewicz and the UCT, $H^1(K(\mathbb{Z}/2,2),\mathbb{F}_2) = 0$ and $H^2(K(\mathbb{Z}/2,2),\mathbb{F}_2) \cong \mathbb{F}_2$. In fact, the integral (reduced) cohomology must begin with $H^3(K(\mathbb{Z}/2,2),\mathbb{Z}) \cong \mathbb{Z}/2$. So if x generates $H^2(K(\mathbb{Z}/2,2),\mathbb{F}_2)$, $d_2(a) = x$ and $d_2(a^k) = ka^{k-1}x$, which is zero if k is even and generates $E_2^{2,k-1}$ if k is odd. Similarly $d_2(ax) = x^2$, must be nonzero in order to kill off ax, etc., so all powers of x are nonzero. Since a^2 cannot survive to E_∞, $d_3(a^2) = y$ for some y generating $H^3(K(\mathbb{Z}/2,2),\mathbb{F}_2)$. Since a^2 can't survive to E_∞, $d_3(a^2) = y$ must be nonzero and must generate $H^3(K(\mathbb{Z}/2,2),\mathbb{F}_2)$. Since a^2y must also die eventually, $d_3(a^2y) = y^2$, and d_3 is the only differential that can be nonzero on a^2y, we similarly have $y^2 \neq 0$, etc. Finally, $d_3(a^4) = 0$ so a^4 must map nontrivially under d_5, which forces there to be another generator z of $H^*(K(\mathbb{Z}/2,2),\mathbb{F}_2)$ in degree 5 (aside from $xy = d_2(ay)$). So in low degrees at least, $H^*(K(\mathbb{Z}/2,2),\mathbb{F}_2)$ looks like a polynomial ring on x in degree 2, y in degree 3, and z in degree 5.

Now we can compute the low-dimensional cohomology of $H^*(K(\mathbb{Z}/2,3),\mathbb{F}_2)$ with the same technique. The path fibration of $K(\mathbb{Z}/2,3)$ gives a spectral sequence with $E_2^{p,q} = H^p(K(\mathbb{Z}/2,3),\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^q(K(\mathbb{Z}/2,2),\mathbb{F}_2)$ and with $E_\infty^{p,q} = 0$ unless $p = q = 0$. We know $H_q(K(\mathbb{Z}/2,3),\mathbb{Z}) = 0$ for $0 < q < 3$ and $H_3(K(\mathbb{Z}/2,3),\mathbb{Z}) \cong \mathbb{Z}/2$ by Hurewicz, so $E_2^{p,0} \cong \mathbb{F}_2$ for $p = 3$ and $E_2^{p,0} = 0$ for $p = 1,2$. We also know that $H^4(K(\mathbb{Z}/2,3),\mathbb{Z}) \cong \mathbb{Z}/2$, and reducing mod 2 shows that $\dim E_\infty^{4,0} \geq 1$. Proceeding as before shows that $H^*(K(\mathbb{Z}/2,3),\mathbb{F}_2)$ is, at least in low degrees, a polynomial algebra on generators b in degree 3, c in degree 4 and e in degree 5, with $d_3(x) = b$, $d_4(y) = c$, and $d_5(x^2) = e$.

Finally, we can return to the spectral sequence of the fibration (2). Note that since F' is 4-connected, terms of total degree 3 or 4 in E_2 cannot survive to E_∞. In integral cohomology, the bottom row is $E_2^{p,0} = H^p(F, \mathbb{Z})$, which has a $\mathbb{Z}$ when $p = 0$, then a $\mathbb{Z}/2$ when $p = 5$ and a $\mathbb{Z}/3$ when $p = 7$. The next row that is not identically zero is $E_2^{p,4} = H^p(F, \mathbb{F}_2)$, which is $\mathbb{F}_2$ for $p = 0$, 4, 5 and 0 for $p = 1, 2, 3$, and 6. The only way to get the necessary cancellation is to have d_5 map $H^4(K(\mathbb{Z}/2, 3), \mathbb{Z})$ isomorphically onto $H^5(F, \mathbb{Z}) \cong \mathbb{Z}/2$. However there is no way to cancel $H^6(K(\mathbb{Z}/2, 3), \mathbb{Z})$, which is a 2-primary torsion group, since $H^7(F, \mathbb{Z}) \cong \mathbb{Z}/3$, so we get $\pi_5(S^3) \cong H_5(F', \mathbb{Z}) \cong H^6(F', \mathbb{Z}) \cong H^6(K(\mathbb{Z}/2, 3), \mathbb{Z})$. Since $H^6(K(\mathbb{Z}/2, 3), \mathbb{F}_2) \cong \mathbb{F}_2 b^2$ and $\beta b = c$, where β is the Bockstein (which is a derivation), $\beta(b^2) = 2b(\beta b) = 0$ and b^2 is the reduction of an integral class. So $H^6(K(\mathbb{Z}/2, 3), \mathbb{Z}) \cong \mathbb{Z}/2^r$ for some $r \geq 1$. To finish the calculation, we need to check that the order of $H^6(K(\mathbb{Z}/2, 3), \mathbb{Z})$ is not bigger than 2.

To do this, one can go back over the calculation of the cohomology of $K(\mathbb{Z}/2, 2)$ and $K(\mathbb{Z}/2, 3)$ in low dimensions, but this time with integral cohomology instead of $\mathbb{F}_2$ cohomology. Except for the $\mathbb{Z}$ in degree 0, the integral cohomology of $K(\mathbb{Z}/2, 1)$ is a polynomial ring $\mathbb{F}_2[a^2]$. (The generator in degree 2 can be identified with the cup-square of $a \in H^1(\mathbb{RP}^\infty, \mathbb{F}_2)$, since this is what it reduces to mod 2.) Similarly, when we look at the integral cohomology of $K(\mathbb{Z}/2, 2)$, x in degree 2 is not present but we do have y of order 2 in H^3. (This reduces mod 2 to the y we had before in $\mathbb{F}_2$ cohomology.) In the spectral sequence in integral cohomology for the path fibration of $K(\mathbb{Z}/2, 3)$, as before, $d_3(a^2) = y$. But now the $q > 0$ rows look different from the $q = 0$ row, since the former look like $\mathbb{F}_2[x, y, z] \otimes_{\mathbb{F}_2} \mathbb{F}_2 a^2$ and the latter starts with y in degree 3. Note that $H^4(K(\mathbb{Z}/2, 2), \mathbb{Z})$ must vanish since there is nothing in $E_2^{p,3-p}$ that could kill it. The one question mark is $H^5(K(\mathbb{Z}/2, 2), \mathbb{Z})$. d_3 is 0 on a^4 in $E_3^{0,4}$ but has to kill the $\mathbb{Z}/2$ in $E_3^{2,2}$. So the $\mathbb{Z}/2$ in position $(0, 4)$ survives to E_4 and must map nontrivially under d_5 (there is nowhere else for it to go). That means that $E_5^{4,0} \cong \mathbb{Z}/2$, but since we already cancelled a $\mathbb{Z}/2$ from $E_3^{2,2}$, that means by process of elimination that $H^5(K(\mathbb{Z}/2, 2), \mathbb{Z}) \cong \mathbb{Z}/4$ (surprise!).

Finally, we're down to computing $H^6(K(\mathbb{Z}/2, 3), \mathbb{Z})$ from the spectral sequence of the path fibration of $K(\mathbb{Z}/2, 3)$. In total degree 3 in E_2, we have just a $\mathbb{Z}/2$ in position $(0, 3)$. In total degree 4, we have just a $\mathbb{Z}/2$ in position $(4, 0)$, which cancels the $\mathbb{Z}/2$ in position $(0, 3)$ via d_4. In total degree 5, $E_2^{5,0} = H^5(K(\mathbb{Z}/2, 3), \mathbb{Z}) = 0$ since there is nothing to cancel it, so the only term of total degree 5 is the $\mathbb{Z}/4$ in the $(0, 4)$ position. In total degree 6, we have $E_2^{6,0}$, which we're trying to compute, and $E_2^{3,3} = H^3(K(\mathbb{Z}/2, 3), \mathbb{Z}/2) \cong \mathbb{Z}/2$. Now d_3 must send the generator of $E_3^{5,0} \cong \mathbb{Z}/4$ to the generator of $E_3^{3,3}$, leaving behind a $\mathbb{Z}/2$ in $E_4^{4,0}$. The only place this can go in a later stage is to cancel $H^6(K(\mathbb{Z}/2, 3), \mathbb{Z})$ under d_6, so $H^6(K(\mathbb{Z}/2, 3), \mathbb{Z}) \cong \mathbb{Z}/2$ and $\pi_5(S^3) \cong \mathbb{Z}/2$. $\square$