MATH 748R, Spring 2012
Homotopy Theory
Homework Assignment #7:
Serre Classes and Applications

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due Friday, April 27, 2012

1. Suppose X is a connected CW complex with $\pi_{2k}(X) \cong \mathbb{Z}$ for $2k \geq 2$ even and with $\pi_{2k+1}(X) = 0$ for $k \geq 0$. Thus the Postnikov system of X is built out of $K(\mathbb{Z}, 2k)$'s. Show that the rational cohomology ring of X is a polynomial ring over $\mathbb{Q}$ on generators in degrees $2, 4, 6, \cdots$. (The theorem is not vacuous; it turns out, as we will see later, that BU satisfies the hypothesis.) Hint: use what we proved about rational cohomology of $K(\mathbb{Z}, n)$.

2. The second stable homotopy group of spheres is $\pi_2^s = \varinjlim \pi_{n+2}(S^n)$, which by the Freudenthal Suspension Theorem can be computed as $\pi_6(S^4)$. Investigate this group as follows.

   (a) Observe from the fibration $S^3 \to S^7 \to S^4$ that $\pi_6(S^4) \cong \pi_5(S^3)$, so that the stable range is already achieved with $\pi_5(S^3)$.

   (b) Recall that there is a homotopy fibration
   $$K(\mathbb{Z}, 2) \to F \xrightarrow{p} S^3, \qquad (1)$$
   where $p\colon F \to S^3$ is the homotopy fiber of the map $S^3 \to K(\mathbb{Z}, 3)$ inducing an isomorphism on π_3. Thus F is 3-connected. Show from the spectral sequence of (1) (using the derivation property for the differentials and the fact that the cohomology ring of $\mathbb{CP}^\infty$ is a polynomial ring on one generator in degree 2) that $H^{2k}(F, \mathbb{Z}) = 0$ for $2k$ even and that $H^{2k+1}(F, \mathbb{Z}) \cong \mathbb{Z}/k$ for $2k + 1 \geq 5$ odd. In particular, $H^5(F, \mathbb{Z}) \cong \mathbb{Z}/2$, which is how we showed that $\pi_4(S^3) \cong \mathbb{Z}/2$.

   (c) To compute $\pi_5(S^3)$, carry this process one step further; let $p'\colon F' \to F$ be the homotopy fiber of the map $F \to K(\mathbb{Z}/2, 4)$ inducing an isomorphism on π_4, and show that you get a homotopy fibration
   $$K(\mathbb{Z}/2, 3) \to F' \xrightarrow{p'} F. \qquad (2)$$

(d) To finish the calculation, use the spectral sequence of the fibration (2) and the fact that F' is 4-connected. Modulo the Serre class of 2-primary finite groups, show p' is a homotopy equivalence, and thus that F' has no cohomology other than 2-primary torsion below degree 7. Conclude that $\pi_5(S^3)$ is a 2-primary finite group.

(e) Finally, compute the 2-primary torsion in $\pi_5(S^3)$ by using the spectral sequence of (2) with $\mathbb{F}_2$ coefficients. You will need to observe that $H^i(F, \mathbb{F}_2) \cong 0$ for $i = 6$ and that $H^i(K(\mathbb{Z}/2, 3), \mathbb{F}_2) \cong \mathbb{F}_2$ for $i = 3, 4, 5$. This last fact can be deduced from path fibrations and the facts that $\Omega K(\mathbb{Z}/2, 3) \simeq K(\mathbb{Z}/2, 2)$, $\Omega K(\mathbb{Z}/2, 2) \simeq \mathbb{RP}^\infty$, $H^*(\mathbb{RP}^\infty, \mathbb{F}_2) \cong \mathbb{F}_2[a]$, where a is the canonical element in degree 1.