MATH 748R, Spring 2012
Homotopy Theory
Homework Assignment #8:
Vector Bundles and Characteristic Classes

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Solutions

1. A vector bundle $p\colon E \to X$ is called *stably trivial* if there is a trivial bundle E' over X such that $E \oplus E'$ is also trivial. While this concept makes sense for real bundles also, the rest of this problem will deal only with complex vector bundles for simplicity. If X is compact Hausdorff and $p\colon E \to X$ corresponds to the homotopy class of $f\colon X \to BU(n)$, where n is the rank of E, show that stable triviality is equivalent to $\varphi \circ f$ being null homotopic for some N, where $\varphi\colon BU(n) \to BU(n+N)$ is induced by the inclusion $U(n) \hookrightarrow U(n+N)$ (via the block direct sum with the $N \times N$ identity matrix). Prove that if E is a *line bundle*, i.e., $n = 1$, then E is stably trivial if and only if it is trivial. (Hint: use the cohomology of $BU(n+N)$ and what you know about the homotopy type of $BU(1)$.)

   Solution. If ε denotes the trivial line bundle $X \times \mathbb{C} \xrightarrow{\text{pr}_1} X$, then $\varepsilon^N = \overbrace{\varepsilon \oplus \cdots \oplus \varepsilon}^{N}$ is the trivial bundle of rank N, and stable triviality of E means $E \oplus \varepsilon^N$ is trivial for some N. Now if $f \to \text{Gr}(n, M)$ classifies E, then $E \oplus \varepsilon^N$ is the bundle whose fiber over $x \in X$ is $f(x) \oplus \mathbb{C}^N$, which is the subspace of $\mathbb{C}^{M+N}$ given by $\phi_{M,N} \circ f(x)$, where $\phi_{M,N}\colon \text{Gr}(n, M) \to \text{Gr}(n+N, M+N)$ sends $V \subseteq \mathbb{C}^M$ to $V \oplus \mathbb{C}^N \subseteq \mathbb{C}^M \oplus \mathbb{C}^N = \mathbb{C}^{M+N}$. If we identify $\text{Gr}(n, M)$ with $U(M)/\big(U(n) \times U(M-n)\big)$ and $\text{Gr}(n+N, M+N)$ with $U(M+N)/\big(U(n+N) \times U(M+N-n)\big)$, then $\phi_{M,N}$ is not precisely the map induced by the inclusion $U(n) \hookrightarrow U(n+N)$, but is homotopic to it, since the two maps just differ by conjugation by a matrix in $U(M+N)$ sending $\mathbb{C}^n \oplus \mathbb{C}^N \oplus 0^{M-n}$ to $\mathbb{C}^n \oplus 0^{M-n} \oplus \mathbb{C}^N$, and this matrix can be homotoped to the identity in $U(M+N)$, so the result follows.

   Now suppose E is a line bundle. We have $BU(1) \simeq \mathbb{CP}^\infty = K(\mathbb{Z}, 2)$, so E is classified by an element in $H^2(X, \mathbb{Z})$, by the universal property of Eilenberg-Mac Lane spaces, which is just $c_1(E)$. For any N, $c(E \oplus \varepsilon^N) = c(E)c(\varepsilon^N) = c(E)$, so $c_1(E \oplus \varepsilon^N) = c_1(E)$. So if $E \oplus \varepsilon^N$ is trivial, its c_1 is trivial, and thus $c_1(E) = 0$, so E is trivial. □

2. Now show that there is a stably trivial complex bundle of rank 2 over S^5 that is not trivial. Here is an outline:

   (a) First show that the homotopy groups of $U(2)$ are, except for π_1, the same as for S^3. Thus $\pi_4(U(2)) \cong \pi_5(BU(2)) \cong \mathbb{Z}/2$. The bundle you want corresponds to the generator of this group.

   Solution. Any unitary matrix $A \in U(N)$ can be written as $\begin{pmatrix} \det A & 0 \\ 0 & 1_{N-1} \end{pmatrix} A'$, for a unique $A' \in SU(N)$. This shows that $U(N)$ is homeomorphic to $U(1) \times SU(N)$ and thus its universal cover is homeomorphic to $\mathbb{R} \times SU(N)$. In particular, $\pi_j(U(N)) \cong \pi_j(SU(N))$ for $j \geq 2$. But

   $$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha,\, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta^2| = 1 \right\},$$

   which can be identified with the unit sphere in $\mathbb{C}^2$, or in other words S^3. So $\pi_5(BU(2)) \cong \pi_4(U(2)) \cong \pi_4(SU(2)) = \pi_4(S^3) \cong \mathbb{Z}/2$. This means there is a unique nontrivial rank 2 complex vector bundle over S^5. $\square$

   (b) From the long exact sequence of the fibration $U(2) \to U(3) \to S^5$, show that $\pi_4(U(3))$ is either 0 or $\mathbb{Z}/2$.

   Solution. The exact sequence of the fibration gives $\pi_5(S^5) \to \pi_4(U(2)) \to \pi_4(U(3)) \to \pi_4(S^5) = 0$. Since $\pi_4(U(2)) \cong \mathbb{Z}/2$ by (a), that means $\pi_4(U(3)) \cong \mathbb{Z}/2$ or 0. $\square$

   (c) Now you want to show that $\pi_4(U(3)) = 0$. You can do this as follows. First show that the universal cover of $U(3)$ is $SU(3) \times \mathbb{R}$, so it's enough to show that $\pi_4(SU(3)) = 0$. Also observe from the fibration $SU(2) \to SU(3) \to S^5$ that the integral cohomology ring of $SU(3)$ is an exterior algebra over $\mathbb{Z}$ on generators x in degree 3 and y in degree 5.

   Solution. That the universal cover of $U(3)$ is $SU(3) \times \mathbb{R}$ was proved above, so $\pi_4(U(3)) = \pi_4(SU(3))$. Also, $SU(3)$ acts transitively on the unit sphere in $\mathbb{C}^3$, and the stabilizer of the north pole can be identified with $SU(2)$, so we get a fibration $SU(2) \to SU(3) \to S^5$. The spectral sequence of this fibration has $E_2^{p,q} \neq 0$ only when $p = 0$ or 5 and when $q = 0$ or 3, so all differentials must vanish. Hence $E_2 = E_\infty$. Since E_2 is torsion free, there are no extension problems to worry about. So $H^*(SU(3), \mathbb{Z}) \cong \bigwedge_{\mathbb{Z}}(x) \otimes \bigwedge_{\mathbb{Z}}(y)$, with x of degree 3 and y of degree 5. $\square$

   (d) Suppose you can show that the fibration $SU(2) \to SU(3) \to S^5$ *does not split*, i.e., that there is no map $S^5 \to SU(3)$ such that the composite $S^5 \to SU(3) \to S^5$ is the identity. Deduce that that $\pi_4(SU(3)) = 0$. Hint: a splitting would be an element of $\pi_5(SU(3))$ mapping to the generator of $\pi_5(S^5)$. Then use the long exact sequence.

   Solution. The fibration not splitting means there is no map $S^5 \to SU(3)$ such that the composite $S^5 \to SU(3) \to S^5$ is the identity. Since all spaces in sight are simply connected, there is no difference between based and unbased homotopy, and thus this means there is no element of $\pi_5(SU(3))$ which maps to the class of the identity in $\pi_5(S^5)$. In other words, the map $\pi_5(SU(3)) \to \pi_5(S^5)$ is not surjective, and thus the boundary map $\pi_5(S^5) \to \pi_4(SU(2)) \cong \mathbb{Z}/2$ is nonzero. Since $\pi_4(SU(2))$ has only one nonzero element, that means the boundary map is surjective and thus, by the exact sequence above, $\pi_4(SU(3)) = 0$. On the other hand, if the fibration were to split, then the map $\pi_5(SU(3)) \to \pi_5(S^5)$ *would be* surjective, and thus the boundary map $\pi_5(S^5) \to \pi_4(SU(2)) \cong \mathbb{Z}/2$ would be *zero*. So in that case we'd have $\pi_4(SU(3)) \cong \mathbb{Z}/2$. □

(e) To finish the argument and deduce that $\pi_4(SU(3)) = 0$ and thus that the bundle you constructed in (a) is nontrivial but stably trivial, you need to show that $SU(3)$ is *not* homotopy equivalent to the product $S^3 \times S^5$. For this purpose consider the fibration $SO(3) \to SU(3) \to M^5$, where $SO(3)$ includes in $SU(3)$ as the real unitary matrices, and M is the five-dimensional homogeneous space $SU(3)/SO(3)$. Note that M is simply connected with $\pi_2(M) \cong \pi_1(SO(3)) \cong \mathbb{Z}/2$, so by Poincaré duality, it has the same homology groups as S^5 except for a $\mathbb{Z}/2$ in degree 2. In particular, it's rationally homotopy equivalent to S^5. So the map $\mathbb{RP}^3 \cong SO(3) \hookrightarrow SU(3)$ must be injective on $\pi_3(SO(3)) = \mathbb{Z}$. Now if $SU(3)$ were homotopy equivalent to $S^3 \times S^5$, then the map $\mathbb{RP}^3 \cong SO(3) \hookrightarrow SU(3)$ would have to factor through the S^3 factor (since any map $\mathbb{RP}^3 \to S^5$ is null homotopic), and since M is the homotopy cofiber of this map, M would have to split (up to homotopy) as a product of S^5 and the cofiber X of a map $\mathbb{RP}^3 \to S^3$. This is impossible since it would force M to have homology in dimension 7, which is bigger than its dimension. (Apply the Künneth Theorem using the facts that $H_5(S^5) \cong \mathbb{Z}$ and $H_2(X) \cong \pi_2(X) \cong \mathbb{Z}/2$.)

Solution. If the fibration $SU(2) \to SU(3) \to S^5$ were to split, the splitting would give an equivalence $SU(3) \simeq S^3 \times S^5$. Now let $M = SU(3)/SO(3)$. From the long exact sequence of the fibration $SO(3) \to SU(3) \to M^5$, $0 = \pi_1(SU(3)) \to \pi_1(M) \to \pi_0(SO(3)) = 0$ and $0 = \pi_2(SU(3)) \to \pi_2(M) \to \pi_1(SO(3)) \to \pi_1(SU(3) = 0$. So M is simply connected and $\pi_2(M) \cong \pi_1(SO(3)) \cong \mathbb{Z}/2$. By Hurewicz, $H_1(M, \mathbb{Z}) = 0$ and $H_2(M, \mathbb{Z}) \cong \mathbb{Z}/2$. Now M is an oriented manifold of dimension $\dim SU(3) - \dim SO(3) = 8 - 3 = 5$, so by Poincaré duality, $H_3(M, \mathbb{Z}) \cong H^2(M, \mathbb{Z}) = 0$ (by UCT, using what we know about H_1 and H_2). Similarly $H_4(M, \mathbb{Z}) \cong H^1(M, \mathbb{Z}) = 0$. So M has the same homology groups as S^5, except for a $\mathbb{Z}/2$ in dimension 2. Furthermore, if $\mathcal{C}$ denotes the Serre class of finite abelian groups, then a degree 1 map $M \to S^5$ is a mod $\mathcal{C}$ homology equivalence of simply connected CW complexes, and is thus a mod $\mathcal{C}$ homotopy equivalence (by the mod $\mathcal{C}$ Hurewicz and Whitehead Theorems).

Now suppose $SU(3) \simeq S^3 \times S^5$ and consider the fibration $SO(3) \cong \mathbb{RP}^3 \to SU(3) \to M^5$. By assumption we can replace $SU(3)$ by $S^3 \times S^5$. Every map $\mathbb{RP}^3 \to S^5$ is null homotopic (since we can approximate any map by a smooth map, which can't be surjective by Sard's

Theorem, and thus lands in $S^5 \setminus \{\text{pt}\} = \mathbb{R}^5$, which is contractible). So the fibration looks like

$$\mathbb{RP}^3 \times \{\text{pt}\} \to S^3 \times S^5 \to M^5,$$

and M^5 must have the homotopy type of $S^5 \times X$, where we have a homotopy fibration

$$\mathbb{RP}^3 \to S^3 \to X.$$

The homotopy exact sequence reads in part $\pi_2(S^3) \to \pi_2(X) \to \pi_1(\mathbb{RP}^3) \to \pi_1(S^3)$, or $0 \to \pi_2(X) \to \mathbb{Z}/2 \to 0$, so $\pi_2(X) \cong \mathbb{Z}/2$. Similarly from the part of the exact sequence that reads $\pi_1(S^3) \to \pi_1(X) \to \pi_0(\mathbb{RP}^3)$, we see X is simply connected, so by Hurewicz, $H_2(X, \mathbb{Z}) \cong \mathbb{Z}/2$. Now $H_7(M^5, \mathbb{Z}) \cong H_7(S^5 \times X, \mathbb{Z}) \supseteq H_5(S^5, \mathbb{Z}) \otimes H_2(X, \mathbb{Z}) \cong \mathbb{Z}/2$, which contradicts the fact that M is a manifold of dimension 5. Thus the assumption that $SU(3) \simeq S^3 \times S^5$ must be false and so from the reasoning above, $\pi_4(SU(3)) = 0$. $\square$