MATH 748R, Spring 2012
Homotopy Theory
Homework Assignment #8:
Vector Bundles and Characteristic Classes

Jonathan Rosenberg

due Monday, May 14, 2012

1. A vector bundle $p\colon E \to X$ is called *stably trivial* if there is a trivial bundle E' over X such that $E \oplus E'$ is also trivial. While this concept makes sense for real bundles also, the rest of this problem will deal only with complex vector bundles for simplicity. If X is compact Hausdorff and $p\colon E \to X$ corresponds to the homotopy class of $f\colon X \to BU(n)$, where n is the rank of E, show that stable triviality is equivalent to $\varphi \circ f$ being null-homotopic for some N, where $\varphi\colon BU(n) \to BU(n+N)$ is induced by the inclusion $U(n) \hookrightarrow U(n+N)$ (via the block direct sum with the $N \times N$ identity matrix). Prove that if E is a *line bundle*, i.e., $n = 1$, then E is stably trivial if and only if it is trivial. (Hint: use the cohomology of $BU(n+N)$ and what you know about the homotopy type of $BU(1)$.)

2. Now show that there is a stably trivial complex bundle of rank 2 over S^5 that is not trivial. Here is an outline:

   (a) First show that the homotopy groups of $U(2)$ are, except for π_1, the same as for S^3. Thus $\pi_4(U(2)) \cong \pi_5(BU(2)) \cong \mathbb{Z}/2$. The bundle you want corresponds to the generator of this group.

   (b) From the long exact sequence of the fibration $U(2) \to U(3) \to S^5$, show that $\pi_4(U(3))$ is either 0 or $\mathbb{Z}/2$.

   (c) Now you want to show that $\pi_4(U(3)) = 0$. You can do this as follows. First show that the universal cover of $U(3)$ is $SU(3) \times \mathbb{R}$, so it's enough to show that $\pi_4(SU(3)) = 0$. Also observe from the fibration $SU(2) \to SU(3) \to S^5$ that the integral cohomology ring of $SU(3)$ is an exterior algebra over $\mathbb{Z}$ on generators x in degree 3 and y in degree 5.

   (d) Suppose you can show that the fibration $SU(2) \to SU(3) \to S^5$ *does not split*, i.e., that there is no map $S^5 \to SU(3)$ such that the composite $S^5 \to SU(3) \to S^5$ is the identity.

Deduce that that $\pi_4(SU(3)) = 0$. Hint: a splitting would be an element of $\pi_5(SU(3))$ mapping to the generator of $\pi_5(S^5)$. Then use the long exact sequence.

(e) To finish the argument and deduce that $\pi_4(SU(3)) = 0$ and thus that the bundle you constructed in (a) is nontrivial but stably trivial, you need to show that $SU(3)$ is *not* homotopy equivalent to the product $S^3 \times S^5$. For this purpose consider the fibration $SO(3) \to SU(3) \to M^5$, where $SO(3)$ includes in $SU(3)$ as the real unitary matrices, and M is the five-dimensional homogeneous space $SU(3)/SO(3)$. Note that M is simply connected with $\pi_2(M) \cong \pi_1(SO(3)) \cong \mathbb{Z}/2$, so by Poincaré duality, it has the same homology groups as S^5 except for a $\mathbb{Z}/2$ in degree 2. In particular, it's rationally homotopy equivalent to S^5. So the map $\mathbb{RP}^3 \cong SO(3) \hookrightarrow SU(3)$ must be injective on $\pi_3(SO(3)) = \mathbb{Z}$. Now if $SU(3)$ were homotopy equivalent to $S^3 \times S^5$, then the map $\mathbb{RP}^3 \cong SO(3) \hookrightarrow SU(3)$ would have to factor through the S^3 factor (since any map $\mathbb{RP}^3 \to S^5$ is null homotopic), and since M is the homotopy cofiber of this map, M would have to split (up to homotopy) as a product of S^5 and the cofiber X of a map $\mathbb{RP}^3 \to S^3$. This is impossible since it would force M to have homology in dimension 7, which is bigger than its dimension. (Apply the Künneth Theorem using the facts that $H_5(S^5) \cong \mathbb{Z}$ and $H_2(X) \cong \pi_2(X) \cong \mathbb{Z}/2$.)