ELECTRO-MAGNETIC DUALITY

and geometric Langlands for $U(1)$

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Structure of the Talk

1. Introduce the electro-magnetic duality
2. Introduce the abelian geometric Langlands
3. Find a dictionary between these two
4. Generalize

Let $M = \mathbb{R}^{3,1}$, and let A be a connection on a $U(1)$-principal bundle E of M, let $F = dA$. The equation of motion for Maxwell theory reads

$$d \star F = 0$$

Moreover the theory is invariant under the transformation $F \mapsto \star F$ with $\star^2 = -1$ for Lorentzian manifold. Thus if we set x^0 be the time-like coordinate and $x^i, i = 1, 2, 3$ be the space-like coordinate,

$$E_i = F_{0i}, \quad B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$$

Then this $\star$ duality induces the famous electric-magnetic duality

$$E_i \mapsto B_i, \quad B_i \mapsto -E_i$$

Action of Maxwell's theory

Let M be a Lorentzian 4-manifold, the Lagrangian for the previous equation of motion can be seen as

$$S(A) = \frac{1}{2e^2} \int_X F \wedge \star F$$

We can add a term that depends only on the topology of M and E, namely $\int_X F \wedge F$ as it's essentially calculating c_1^2 of the line bundle associated to the $U(1)$ bundle. So

$$S(A) = \frac{1}{2e^2} \int_X F \wedge \star F + \frac{\theta}{8\pi^2} \int_X F \wedge F$$

has the same equation of motion. But it will affect the quantum theory since we need

$$Z = \sum_E \int \mathcal{D}A e^{iS(A)}$$

and the second term gives weight to topologically different E.

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One key fact on some symmetry

Since now if we just do $F \mapsto \star F$ the symmetry is not preserved in $S(A)$, so the naive electro-magnetic duality failed on quantum level.

But note that

$$\frac{1}{8\pi^2}\int_X F \wedge F \in \mathbb{Z},$$

and $e^{2\pi i \mathbb{Z}} = 1$, so if we changes $\theta \to \theta + 2\pi$ it won't change the quantum theory.

We may consider the 4-manifold $M = \mathbb{R}^4$ and the following action since we change to Euclidean metric:

$$S_E(A) = \frac{1}{2e^2}\int_X F \wedge \star F - \frac{i\theta}{8\pi^2}\int_X F \wedge F$$

Deduction of symmetry

Then for the quantum theory, we have the following equations

$$
\begin{aligned}
Z &= \sum_E \int \mathcal{D}A e^{-S_E(A)} \\
&= \sum_E \int \mathcal{D}F e^{-S_E(A)} \delta(dF) && \text{(closed F)} \\
&= \sum_E \int \mathcal{D}F \mathcal{D}B e^{-S_E(A)+i\int_X BdF} && \text{(delta function)} \\
&= \sum_E \int \mathcal{D}B e^{-\frac{1}{2\hat{e}^2}\int G\wedge\star G + \frac{i\hat{\theta}}{8\pi^2}\int_X G\wedge G} && \text{(Gaussian)}
\end{aligned}
$$

Here $G = dB$ and $\frac{\hat{\theta}}{2\pi} + \frac{2\pi i}{\hat{e}} = -\left(\frac{\theta}{2\pi} + \frac{2\pi i}{e}\right)^{-1}$. So if we set $\tau = \frac{\theta}{2\pi} + \frac{2\pi i}{e}$, $\tau \mapsto -1/\tau$ and $F \mapsto G$ will be a symmetry in quantum level for $M = \mathbb{R}^4$.

Note: Previous $\theta \to \theta + 2\pi$ will become $\tau \to \tau + 1$.

These two will form a $PSL(2, \mathbb{Z})$ group, and the electro-magnetic duality only study the duality of $\tau \to -1/\tau$.

For general dimension four manifold M the second step fails as not every closed 2-form is exact. One has to insert extra delta functions in the summation, but luckily, Witten found that the effect of these delta-functions can be reproduced by letting B to be a connection 1-form on an arbitrary principal $U(1)$ bundle and sum over all possible bundles. The proof can be found in his paper "On S-duality in Abelian Gauge theory".

Montonen-Olive Duality conjecture

Let E be a G-principal bundle on a four-manifold X and let the Yang-Mills action to be

$$S(A) = \frac{1}{2e^2}\int_X \mathrm{Tr}(F \wedge \star F) + \frac{\theta}{8\pi^2}\int_X \mathrm{Tr}(F \wedge F)$$

where $F = dA + A \wedge A \in \Omega^2(ad(E))$ is the curvature for connection A. Then set $n_{\mathfrak{g}}$ be maximal multiplicity of edge Dynkin diagram.

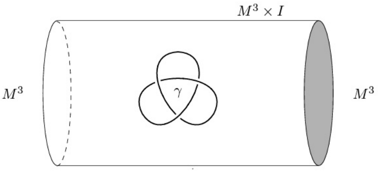
$$Z(X, G, \tau) = Z(X, {}^L G, -\frac{1}{n_{\mathfrak{g}}\tau}), \quad {}^L G \text{ is Langlands dual}$$

and moreover, for any observables $O_1, \ldots, O_n$ for the G-Yang-Mills, there exists $\tilde{O}_1, \ldots \tilde{O}_n$ and we have equation correlation function

$$\langle O_1, \ldots, O_n \rangle_{X,G,\tau} = \langle \tilde{O}_1, \ldots \tilde{O}_n \rangle_{X,{}^L G,-\frac{1}{n_{\mathfrak{g}}\tau}}$$

Observables

We have the Wilson operators and t 'Hooft operator. Wilson operators consider a line in the space-time $M^3 \times I$:



For a connection A, the operator is

$$W_R(\gamma) = \mathrm{Tr}_R(\mathrm{Hol}_\gamma(A))$$

where R is a representation of G. In physics it represents inserting electric charges.

We have the t 'Hooft operator $H_R(\gamma)$, which are "magnetic monopoles" or fields allowing singularity alongside γ.

More mathematical interpretation

Consider the slice of Hilbert space $\mathcal{H} = L^2(\mathcal{C}(M^3))$ for each time, where $\mathcal{C}(M^3)$ denotes the collection of line bundles on M^3 with connections modulo gauge transformation. Then we just need to look at how operators acts on this Hilbert space.

$$\mathcal{C}(M^3) \cong \Lambda \times T \times V$$

1. $\Lambda = H^2(M, \mathbb{Z})$ lattice of possible line bundle
2. $T =$ flat connections $= H^1(M, U(1))$,
3. V an infinite dimensional vector space (don't care about)

t 'Hooft operator

We're considering fields $\mathcal{C}(M \times I \backslash \gamma)$, i.e. the connections are possibly singular along γ. So we have excised the knot, and this has introduced a new boundary component of our 4-manifold: the link of the knot (boundary of tubular neighborhood of the knot) which looks like $S^2 \times S^1$. Then we can look at connections which have a specific integral over this. In particular, we can ask for

$$\frac{1}{2\pi i} \int_{S^2} F = 1$$

We call this a magnetic monopole with charge 1, and since $\mathcal{H}$ evolves over time with this monopole inserted, the resulting transformation is called the t 'Hooft operator. Then one can find that t 'Hooft operator simply shifts the Λ by $[\gamma] \in H^2(M, \mathbb{Z})$.

Wilson operator

The Wilson operators are are eigenfunctions for the action of the space of flat connections:

$$T_b = H^1(M, U(1)) \text{ with discrete topology on } U(1)$$

that acts on M by tensoring. The eigenfunctions are given by multiplying by the monodromy along γ: by moving a flat connection around γ we get a value in $U(1)$ by taking monodromy. So Wilson operators are charaters for the torus T_b.

We know from the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$ there is a morphism $H^1(M, U(1)) \to H^2(M, \mathbb{Z})$, there will be decomposition of torus action in terms of this grading on the Hilbert space

$$\mathcal{H} = \bigoplus_{e \in H^2(M,\mathbb{Z})} \mathcal{H}_e$$

and e are called electric charges.

S-duality for Electrodynamics

The S-duality can be thought of as performing Fourier transform with respect to our torus, specifically

$$L^2(\mathcal{C}_{U(1)}(M)) \cong L^2(\mathcal{C}_{U(1)^\vee}(M)).$$

Specifically, we have on the F and $\star F$ sides we have associated grading

$$\Lambda_B \times T_E \times V \text{ and } \Lambda_E \times T_B \times V$$

on the left the Wilson operators are diagonalized, and on the right the t'Hooft operators are diagonalized.

In general non-abelian group G these are too hard to talk about, so we need find an "easier" method of discovering the operators.

On the other hand, the geometric Langlands correspondence come from a very old story: the Abel Jacobi map.

For a smooth Riemann surface C, we can consider the Abel-Jacobi map, by mapping a point p on C to its Jacobian $J(C) = (H^{1,0})^*/(H_1(C,\mathbb{Z})$ by sending

$$i = \int_{p_0} : C \to J(C); \quad p \mapsto \int_{p_0}^{p}$$

But this depends on a choice of basepoint. Later on we usualy use the Picard group $\mathrm{Pic}(C) =$ isomorphism classes of line bundles on curves by

$$i : C \to \mathrm{Pic}^1(C), \quad p \mapsto \mathcal{O}_C(p)$$

where Pic^i means the degree i line bundles.

Hecke Correspondence

In a broader sense, we have

$$h : C \times \mathrm{Pic}^d \to \mathrm{Pic}^{d+1} \quad p \times L \mapsto L(p)$$

and this is generalized to the Hecke correspondence in the Langlands program.

We may consider the category of G-local system on X, namely the morphism $\rho : \pi_1(X) \to G$. Then we have the following:

Baby Geometric Langlands correspondence

For an abelian group G, the induced map $i^* : Loc(J(C), G) \to Loc(C, G)$ is an isomorphism of category.

Proof: We know that $\pi_1(J) = H_1(C, \mathbb{Z}) = \pi_1(C)/[\pi_1(C), \pi_1(C)]$, so any morphism of $\pi_1(C)$ to G will factors through the abelianization $\pi_1(J)$.

Hecke Correpondence, II

Reformulation of the baby case: for any abelian local system l on C, there exists an abelian local system $c(l)$ on $\mathrm{Pic}^0(C)$ such that

$$h^*(c(l)) = pr_1^* l \otimes pr_2^*(c(l))$$

So that l is like an "eigenvalue", $c(l)$ is like "eigenfunction", and in general it's called Hecke eigensheaf.

The Hecke correpondence will in general be more complicated than this.

$$\begin{tikzcd} & Hecke \arrow[dl] \arrow[dr, "h"] & \\ C \times Bun & & Bun \end{tikzcd}$$

The geometric Langlands roughly states that $h^* c(l) \cong l \boxtimes c(l)$ in some sense.

Relation with electro-magnetic duality

The idea of Kapustin-Witten is to use the topological field theory we talked about last time on Maxwell theory, i.e. we take the classical Yang-Mills action, add some fields, and find the SUSY algebra. Then we have some Q with $Q^2 = 0$ and we look at the observables that are in the Q-cohomology of the theory.

Before, our space of fields consisted of a line bundle and a choice of connection $\nabla = d + A$, now we add

1. 1-form ϕ on the manifold (Higgs field)
2. a complex scalar u, and
3. four fermions (odd fields) (we will not pay much attention to these).

Now, in the A-twist, the 3-manifold M^3 gets attached to the cohomology of this space: $H^\bullet(C(M))$. We have a much bigger space of fields now, but it doesn't actually make a difference at the level of the cohomology, since introducing these new fields didn't change the topology of the space. But as it turns out, we don't want ordinary cohomology, we want cohomology which is equivariant with respect to the automorphisms of the connections, i.e. we want:

$$\mathcal{H} = H^\bullet(\text{fields/gauge equivalence}) = H^\bullet_{U(1)}(\mathcal{C}(M)).$$

So $\mathcal{H} = H^\bullet(\Lambda \times T \times V \times BU(1)) = H^\bullet(\mathrm{Pic}(M))$.

B-twist

As for the B-twist, we think of

$$\nabla + i\phi = d + (A + i\phi)$$

as a connection on a $\mathbb{C}^*$-bundle rather than $U(1)$-bundle. This looks like a flat $\mathbb{C}^*$-local system on M if we think about the Q-vanishing criterion. The local system $\pi_1(M) \to \mathbb{C}^*$ factors through its abelianization $\Lambda = H_1(M)$, so

$$Loc_{\mathbb{C}^*}(M) \cong \Lambda \times |Loc_{\mathbb{C}^*}(M)|$$

and this turns out to just be the degree 0 part of the duality between these A and B-twists,

Summary

The S-duality for Maxwell theory sums up to be the following

A-twist	B-twist
topology of Pic	topology of $Loc_{\mathbb{C}^*}(M)$
$\Lambda = H^2(M, \mathbb{Z})$	$T_C^\vee = \lvert Loc_{\mathbb{C}^*}(M) \rvert$
t'Hooft operator	Wilson operators
Create magnetic monopole	Create electric particle

If we set $M = C \times \mathbb{R}$, then we may recover our previous baby Langlands correspondence.

More general G-Yang-Mills

One should study not just 2 twists, but a $\mathbb{CP}^1$-family of twists, given by $Q = uQ_l + vQ_r$. Since the things annihilated by Q are also invariant under λQ, so we have this $\mathbb{CP}^1$ family. We use $t = u/v$ as the affine coordinate. It's a miracle that the action of the N=4 super-Yang-Mills, after the reduction, can be written as

$$I = \{Q, V\} + \frac{i\Psi}{4\pi} \int F \wedge F$$

The first part is Q-exact and the second part is topological, where

$$\Psi = \frac{\theta}{2\pi} + \frac{t^2 - 1}{t^2 + 1} \frac{4\pi i}{e^2}$$

More general G-Yang-Mills

We are also adding the Higgs field ϕ and a scalar field σ. The Q-closed condition will imply immediately the Kapustin-Witten equations

$$\begin{cases} (F - \phi \wedge \phi + t\phi)^+ & = 0 \\ (F - \phi \wedge \phi - t^{-1}\phi)^- & = 0 \\ D \star \phi & = 0 \end{cases}$$

Here $()^+, ()^-$ means the self-dual/anti-self-dual part of the 2-form.

The interesting case for geometric Langlands is $t = i$, where we can set $\mathcal{A} = A + i\phi$ with curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$ and the equation is just

$$\mathcal{F} = 0, \quad D \star \phi = 0.$$

The first of these equations is invariant under the complexified gauge transformations, while the second one is not. The moduli space is unchanged if one drops the second equation and considers the space of solutions of the equation $\mathcal{F} = 0$ modulo $G_{\mathbb{C}}$ gauge.

S-duality for this case

The S-duality map tells us the $t = i, \theta = 0$ will be mapped to $t = 1, \theta = 0$. In that part the equation will be

$$F - \phi \wedge \phi + \star D\phi = 0, \quad D \star \phi = 0$$

They resemble both the Hitchin equations in 2d and the Bogomolny equations in 3d.

Reduction to $2D$

Now we consider $X = \Sigma \times C$, where C and Σ are Riemann surfaces. We will assume that C has no boundary and has genus $g > 1$, while Σ may have a boundary.

As Vafa and collaboratos have figured out in 1994, the 2D field thoery here is a σ-model whose target space is the Hitchin moduli space $M_{\mathrm{Hit}}(G, C)$ of stable Higgs bundles, defined by

$$F - \phi \wedge \phi = 0$$
$$D \star \phi = 0$$

More precisely, this is true for t=1. It turns out that this 2D TFT is an A-model.

On the other hand, if instead we choose the dual gauge group ${}^L G$ and $t = i$, then the gauge theory on Σ turns out to be a σ-model with target $M_{\mathrm{Hit}}({}^L G, C)$, which happens to be a B-model 2D TFT.