ON IMAGINARY ROOTS OF EQUATIONS
(translated from the French by J. Rosenberg)

AUGUSTIN CAUCHY (PUBLISHED 1820)

That it's always possible to decompose a polynomial [with real coefficients] into real factors of the first or second degree, or, in other terms, that every equation involving a [non-constant] rational function of the variable x can be satisfied by a[t least one] real or imaginary[1] value of this variable: this is a proposition that has already been proved in various ways. Messrs. Lagrange, Laplace, and Gauss have used diverse methods for establishing it, and I myself have given a demonstration founded on considerations analogous to those used by Gauss. But, in each of the methods which I have cited, one pays special attention to the degree of the given equation, and sometimes in fact one has to go back to the case of an equation of **higher** degree. These considerations seem foreign to the question, and Mr. Lagrange already noted this (*Théorie des Nombres, Part I, §14*) in suggesting the idea of expanding things in [infinite] series. I have arrived, in following the same idea, at a demonstration which seems as direct and simple as one could possibly desire. I will explain it here in a few words.

Let $f(x)$ be any [non-constant] polynomial [with real coefficients] in x. If one substitutes for x an imaginary value $u + v\sqrt{-1}$, one will have

$$f(u + v\sqrt{-1}) = P + Q\sqrt{-1}, \tag{1}$$

P and Q being real functions of u and v. In addition, if one writes[2]

$$P + Q\sqrt{-1} = R(\cos T + \sqrt{-1}\sin T), \tag{2}$$

R will be the *modulus of the imaginary expression* $P + Q\sqrt{-1}$, and its value will be given by the equation

$$R^2 = P^2 + Q^2. \tag{3}$$

J. de l'École Polytechnique, XVIII^e Cahier, **11** (1820), 411; *Œuvres Complètes*, II^e Série, Tome I, 258–263

[1]By an **imaginary** number, Cauchy means a complex number $a + bi$ with any real a and b, not just one with $a = 0$. The notation i was not yet standard for one of the square roots of -1, and Cauchy writes simply $\sqrt{-1}$. Of course, $-\sqrt{-1}$ is also a square root of -1.

[2]using the polar form of a complex number

Having said this, the theorem to be demonstrated is that one can always find real values of u and v satisfying the two equations $P = 0$ and $Q = 0$, or equivalently, the single equation $R = 0$. It's important therefore to know what are the various values that the function R can take on, and how this function varies with u and with v. We will do this in what follows.

Suppose that the quantities u and v are increased by the amounts h and k, respectively, and let ΔP, ΔQ, ΔR be the corresponding changes in P, Q, R. The equations (3) and (1) become respectively

$$(R+\Delta R)^2 = (P+\Delta P)^2 + (Q+\Delta Q)^2, \tag{4}$$

$$\begin{aligned}(P+\Delta P) + (Q+\Delta Q)\sqrt{-1} &= f(u+v\sqrt{-1}+h+k\sqrt{-1})\\ &= f(u+v\sqrt{-1}) + (h+k\sqrt{-1})f_1(u+v\sqrt{-1})\\ &\quad + (h+k\sqrt{-1})^2 f_2(u+v\sqrt{-1}) + \dots,\end{aligned} \tag{5}$$

f_1, $f_2, \dots$ designating new functions.[3] To deduce from equation (5) the values of $P + \Delta P$ and of $Q + \Delta Q$, it suffices to rewrite the right-hand side in the form $p + q\sqrt{-1}$. We'll do this by substituting for $f(u + v\sqrt{-1})$ its value $R(\cos T + \sqrt{-1}\sin T)$, and by setting, in addition,

$$\begin{aligned} h+k\sqrt{-1} &= \rho(\cos\theta + \sqrt{-1}\sin\theta),\\ f_1(u+v\sqrt{-1}) &= R_1(\cos T_1 + \sqrt{-1}\sin T_1),\\ f_2(u+v\sqrt{-1}) &= R_2(\cos T_2 + \sqrt{-1}\sin T_2),\\ &\dots. \end{aligned}$$

After making these reductions, the equation (5) becomes

$$\begin{aligned}(P+\Delta P) + (Q+\Delta Q)\sqrt{-1} = R\cos T + R_1\rho\cos(T_1+\theta) + R_2\rho^2\cos(T_2+2\theta) + \dots\\ + \left[R\sin T + R_1\rho\sin(T_1+\theta) + R_2\rho^2\sin(T_2+2\theta) + \dots\right]\sqrt{-1},\end{aligned} \tag{6}$$

and one concludes that

$$\begin{aligned} P+\Delta P &= R\cos T + R_1\rho\cos(T_1+\theta) + R_2\rho^2\cos(T_2+2\theta) + \dots,\\ Q+\Delta Q &= R\sin T + R_1\rho\sin(T_1+\theta) + R_2\rho^2\sin(T_2+2\theta) + \dots;\end{aligned} \tag{7}$$

$$\begin{aligned}(R+\Delta R)^2 = \left[R\cos T + R_1\rho\cos(T_1+\theta) + R_2\rho^2\cos(T_2+2\theta) + \dots\right]^2\\ + \left[R\sin T + R_1\rho\sin(T_1+\theta) + R_2\rho^2\sin(T_2+2\theta) + \dots\right]^2.\end{aligned} \tag{8}$$

[3] Since everything here is a polynomial, there are only finitely many terms before the series terminates, and f_1, $f_2, \dots$ are also polynomials.

Suppose now that, for certain values of the variables u and v, the equation $R = 0$ is not satisfied. If, under this hypothesis, R_1 is not zero, the right-hand side of equation (8), ordered according to increasing powers of ρ, becomes

$$R^2 + 2RR_1\rho\cos(T_1 - T + \theta) + \dots;$$

and consequently, the quantity

$$(R + \Delta R)^2 - R^2,$$

or in other words the change in R^2 ordered according to increasing powers of ρ will have leading term

$$2RR_1\rho\cos(T_1 - T + \theta).$$

If, under the same hypothesis, R_1 is zero but R_2 is non-zero, the change in R^2 will have leading term

$$2RR_2\rho^2\cos(T_2 - T + 2\theta),$$

etc., etc. In general the leading term will have the form

$$2RR_n\rho^n\cos(T_n - T + n\theta),$$

if, for the given values of u and v, all the quantities $R_1, R_2, \dots$ vanish through R_{n-1}. Thus if one gives ρ very small positive values and θ arbitrary values, or, what amounts to the same thing, if one gives h and k values which are numerically very small,[4] then the change in R^2, in other words,

$$(R + \Delta R)^2 - R^2,$$

will be of the same sign as its leading term

$$2RR_2\rho^2\cos(T_2 - T + 2\theta),$$

and since one can choose the value of θ to make the sign of the last factor $\cos(T_2 - T + 2\theta)$, and thus of the whole expression, either positive or negative as one wishes, it follows that, in the case where the particular values chosen for u and v do not satisfy the equation $R = 0$, the corresponding value of R^2 can be neither a maximum nor a minimum. Hence, if we can assure ourselves, *a priori*, that R^2 must have a minimum value, then we will have to conclude that this minimum value is zero and thus that the equation $R = 0$ has a solution.

Now, R^2 will evidently have a minimum for some finite values of u and v if, for very large numerical values[5] of these variables , R^2 will eventually be greater than any given [positive] quantity. So if we let

$$u + v\sqrt{-1} = r(\cos z + \sqrt{-1}\sin z),$$

[4]By this Cauchy means values very small in absolute value, but possibly negative.
[5]This means values large in absolute value.

choosing very large numerical values of u and v will correspond precisely to taking r very large. Therefore, in order to show that the equation $R = 0$ has a solution for some finite values of u and v, it's necessary and sufficient to show that the quantity R^2 determined by the equations

$$R^2 = P^2 + Q^2,$$
$$P + Q\sqrt{-1} = f\left[r(\cos z + \sqrt{-1}\sin z)\right] \tag{10}$$

eventually becomes, for very large values of r, greater than any given number.

The above conclusion would persist more generally, whether or not the function $f(x)$ is defined everywhere. It requires only that P and Q be continuous[6] functions of the variables u and v and that the quantities R_1, $R_2, \ldots$ should never become infinite for finite values of the variables.

Suppose that the function $f(x)$ is given as a polynomial[7]

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

The equations (10) give

$$\begin{aligned} P + Q\sqrt{-1} &= f\left(r\cos z + r\sin z\sqrt{-1}\right) \\ &= a_0r^n\cos nz + a_1r^{n-1}\cos(n-1)z + \cdots + a_{n-1}r\cos z + a_n \\ &\quad + \left[a_0r^n\sin nz + a_1r^{n-1}\sin(n-1)z + \cdots + a_{n-1}r\sin z + a_n\right]\sqrt{-1}, \\ P &= a_0r^n\left[\cos nz + \frac{a_1}{a_0}\frac{\cos(n-1)z}{r} + \cdots + \frac{a_{n-1}}{a_0}\frac{\cos z}{r^{n-1}} + \frac{a_n}{a_0}\frac{1}{r^n}\right], \\ Q &= a_0r^n\left[\sin nz + \frac{a_1}{a_0}\frac{\sin(n-1)z}{r} + \cdots + \frac{a_{n-1}}{a_0}\frac{\sin z}{r^{n-1}}\right], \\ R^2 = P^2 + Q^2 &= a_0^2r^{2n}\left[1 + \text{term in } \frac{1}{r} + \cdots + \left(\frac{a_n}{a_0}\right)^2\frac{1}{r^{2n}}\right]. \end{aligned}$$

Now it's clear that for large values of r, the above value of R^2 will be larger than any given quantity. Therefore, in view of what was said above, one can find real values of u and v solving the equation $R = 0$, or equivalently, the two equations

$$P = 0, \qquad Q = 0.$$

By the way, the above method doesn't just apply to polynomial functions, and can be used also to determine for more general functions whether it's possible to solve the equation $f(x) = 0$ for some real or imaginary value of x.[8]

[6] Cauchy seems to be assuming the [false] assertion that any continuous function can be expanded in a convergent power series. This is of course not a problem for polynomials.

[7] presumably non-constant, so that one should assume $a_0 \neq 0$. In addition, all the coefficients are to be real.

[8] Cauchy seems to have in mind the case of a rational function, i.e., a ratio of two polynomials, but of course to solve $\frac{p(x)}{q(x)} = 0$, it's enough to solve $p(x) = 0$. If f is just an entire function, i.e., a function that can be expanded in a power series that converges everywhere, it's not necessarily true that $f(x) = 0$ has to have a solution. For instance $e^x = 0$ has no solutions even with x complex.