GENERAL RESEARCHES ON THE MORTALITY AND THE MULTIPLICATION OF THE HUMAN SPECIES
(translated from the French by J. Rosenberg)

Leonhard Euler (published 1760)

1. The registers of deaths and births at each age, which are published in various places every year, pose so many different questions about the mortality and the multiplication of the human species, that one couldn't list them all. However, these questions are mostly interdependent, so that once one has developed the theory of one or two of them, all the others can be similarly answered. As the solutions must be extracted from the registers mentioned above, one should note that these registers differ quite a bit according to the diversity of the towns, villages, and provinces from which they are taken, and hence the solutions of all of these questions must be different depending on the data on which they are based. That's why I propose here to give a general theory of these questions, without limiting myself to results based on the data from a certain locality; consequently, it will be easy to apply the theory to any desired set of data.

2. Now, I observe first that all these questions depend in general on two hypotheses; once these are well fixed, it will be easy to solve them all. I will call the first *the hypothesis of mortality*, from which one can determine how many, out of a certain number of people all born at the same time, will still be alive after any number of years has elapsed. Here, the multiplication of the species hasn't been taken into account, and therefore we need the second hypothesis, which I will call *the hypothesis of multiplication*, by which one can determine how the number of people alive increases or decreases in the course of a year. This second hypothesis depends on the number of marriages and on the fertility rate, whereas the first depends on the vitality of people at various ages.

I. Hypothesis of Mortality

3. For the first hypothesis, let's consider any number N of infants born at the same time; I will denote the number of those still living after 1 year by $(1)N$, those still living after 2 years by $(2)N$, after 3 years by $(3)N$, after 4 years by $(4)N$, and so on.[1] This is the general notation that I use for indicating how the number of people born at one time decreases successively; these will have values depending on the climate and way of life[2] of

Mémoires de l'Académie des Sciences de Berlin **16** (1760), 144–164; *Opera Omnia*, Series 1, vol. 7, 79–100.

[1] This is Euler's original notation. More typical modern notation would be to call the **fraction** still living after n years, say, $p(n)$; the **number** still living after n years would be $p(n)N$.

[2] We would say lifestyle and habits.

the people involved. Nevertheless, one can note that the numbers indicated by

$$(1), (2), (3), (4), (5), \text{etc.}$$

constitute a decreasing sequence of fractions, of which the largest, (1), is less than one; and when one continues past the 100th term of this sequence, they decrease so rapidly that they almost vanish entirely. For, if out of 100 million people, none attain the age of 125 years, then the term (125) is less than $\frac{1}{100{,}000{,}000}$.

4. Once one has established for a certain locale, by a sufficiently large number of observations, the values of the fractions (1), (2), (3), (4), etc., one can solve quite a number of questions that one ordinarily poses about the probability of human life. First, it is evident, if the number of infants born at a certain time is N, that, according to probability, there will die in each year as many as indicated in the following table:

between 0 years and 1 year, $N - (1)N$ will die,
between 1 year and 2 years, $(1)N - (2)N$ will die,
between 2 years and 3 years, $(2)N - (3)N$ will die,
between 3 years and 4 years, $(3)N - (4)N$ will die,
between 4 years and 5 years, $(4)N - (5)N$ will die,
etc.

And since of the number N there will probably still be $(n)N$ living after n years, it must be that the number who will die before the n-th year is $= N - (n)N$. After this remark, I will now give the solutions to the following questions.

1st Question. *5. Given a certain number of persons, all of the same age, find how many will probably still be living after a certain number of years.*

Suppose there are M people with the same age of m years and one asks how many will probably still be living after n years. Let one set

$$M = (m)N,$$

so as to have $N = \frac{M}{(m)}$, where N is the number of infants born at the same time, of which there remain alive M after m years. Then, of this same number there will probably be $(m+n)N$ still living $m+n$ years after their birth and n years after the given time. Thus the number asked for in the question is

$$= \frac{(m+n)}{(m)} M;$$

or after n years, there will probably still be living this many out of M persons all presently of age m.

Thus, it is probable that out of the number M of m-year olds, the number

$$\left(1 - \frac{(m+n)}{(m)}\right) M$$

will die before n years have elapsed.

2nd Question. 6. *Find the probability that a person of a certain age will still be alive after a certain number of years.*

Say the person in question is m years old, and one wants the probability that this person will still be alive after n years. If we imagine M people of this age, since there will probably still be $\frac{(m+n)}{(m)}M$ living after n years, the probability that the given person will be among this number is

$$= \frac{(m+n)}{(m)}.$$

Thus, the probability that this person will die before the end of these n years is

$$1 - \frac{(m+n)}{(m)}.$$

On the other hand, the hope that this person has of not dying in the next $m+n$ years is to the fear of dying in the same interval as

$$\frac{(m+n)}{(m)-(m+n)}.$$

Thus, the hope will surpass the fear if $(m+n) > \frac{1}{2}(m)$, and the fear will be better founded if $(m+n) < \frac{1}{2}(m)$. The hope and fear will be equal if $(m+n) = \frac{1}{2}(m)$.

3rd Question. 7. *One asks for the probability that a man of a certain age will die in the course of a given year.*

In other words, one supposes that the person is m years old, and one asks for the probability that he will live to the age of n, but that he will die before reaching the age of $n+1$. To find this probability, consider a large number M of persons of the same age, and setting $M = (m)N$ and $N = \frac{M}{(m)}$, there will be $\frac{(n)}{(m)}M$ people living to the age of n and $\frac{(n+1)}{(m)}M$ living to the age of $n+1$. Thus there will probably die in the course of this year

$$\frac{(n)-(n+1)}{(m)}M;$$

and hence, the probability that the given person will be among this number is

$$= \frac{(n)-(n+1)}{(m)}.$$

From this it is evident, that for the same person to die between the ages of n and $n+\nu$, the probability will be

$$= \frac{(n)-(n+\nu)}{(m)}.$$

Also, for the given person to die on a given day of the proposed year, the probability will be

$$= \frac{(n)-(n+1)}{365(m)}.$$

If the question is asked of a newborn infant, one has only to replace the fraction (m) by 1.

4th Question. *8. To find the age to which a person of given age can hope to live, so that it is equally probable that he will die before and after this age.*

Suppose the person in question is m years old and that he can hope to live to the age of z, which we have to find. Now, since the probability that he will survive to this age is $= \frac{(z)}{(m)}$, the probability that he will die before this age is $= 1 - \frac{(z)}{(m)}$. Therefore, since the two probabilities are to be equal, we have the equation

$$\frac{(z)}{(m)} = 1 - \frac{(z)}{(m)}.$$

Thus $(z) = \frac{1}{2}(m)$, from which it is easy to find the number (z) from a table of values of the fractions

$$(1), (2), (3), (4), (5), (6), \text{etc.};$$

for one can see when (z) is half of the given (m).

Having found this z, one calls the time $z - m$ the *force of life* of a person of age m.

5th Question. *9. To determine the fair amount of an old-age pension to be paid to people at a fixed rate, starting at a certain retirement age and continuing until their death, in exchange for a sum which they will have invested in advance.*

Imagine M people all of age m, and that each one pays an amount a;[3] this makes a fund of amount $= Ma$. Let x be the amount which each person is paid each year as long as he is still alive; then after one year the fund has to pay out

$$\frac{(m+1)}{(m)} Mx,$$

after two years[4]

$$\frac{(m+2)}{(m)} Mx,$$

after three

$$\frac{(m+3)}{(m)} Mx,$$

and so on.

Now, supposing that the fund is invested at 5% interest, a sum S payable after n years is only worth at present $\left(\frac{20}{21}\right)^n S$; but to make our determination more general, let us assume that a sum S increases by interest to the amount of λS in one year; then $\frac{1}{\lambda}$ will take the place of $\frac{20}{21}$, and a sum S payable at the end of n years is only worth at present $\frac{S}{\lambda^n}$. From this, one obtains the following calculations:

	one has to pay	worth at present
after 1 year	$\frac{(m+1)}{(m)} Mx,$	$\frac{(m+1)}{(m)} M \frac{x}{\lambda},$
after 2 years	$\frac{(m+2)}{(m)} Mx,$	$\frac{(m+2)}{(m)} M \frac{x}{\lambda^2},$

[3] Evidently Euler assumes each person deposits a lump sum at the time of retirement; otherwise one could make allowances for growth of the fund from the time of investment to the time of retirement.

[4] Euler should have said "at the end of the second year," but I'm translating literally.

after 3 years $\frac{(m+3)}{(m)}Mx$, $\frac{(m+3)}{(m)}M\frac{x}{\lambda^3}$,
etc.; etc.

Thus, fairness demands that all these sums reduced to the present time should equal the amount of the entire fund Ma, from which one obtains the equation

$$a = \frac{x}{(m)}\left(\frac{(m+1)}{\lambda} + \frac{(m+2)}{\lambda^2} + \frac{(m+3)}{\lambda^3} + \frac{(m+4)}{\lambda^4} + \text{etc.}\right),$$

and hence, the amount that the fund has to pay yearly to each of the investors is

$$x = \frac{(m)a}{\frac{(m+1)}{\lambda} + \frac{(m+2)}{\lambda^2} + \frac{(m+3)}{\lambda^3} + \frac{(m+4)}{\lambda^4} + \text{etc.}}.$$

Knowing the values of all the fractions (1), (2), (3), etc., it is easy to compute the sum x corresponding to any given age m, as a function of the given interest rate.

6th Question. *10. When those involved are newborn infants and their old-age pensions are only be paid after they reach a certain retirement age, determine the amount of these pensions.*

Suppose that one pays a sum a for each newborn infant and that they are only to receive their pensions when they reach the age on n years; after this time, they receive an annual sum x which one has to determine. Taking into account the interest as before, one arrives at the equation

$$a = x\left(\frac{(n)}{\lambda^n} + \frac{(n+1)}{\lambda^{n+1}} + \frac{(n+2)}{\lambda^{n+2}} + \frac{(n+3)}{\lambda^{n+3}} + \text{etc.}\right),$$

which gives

$$x = \frac{a}{\frac{(n)}{\lambda^n} + \frac{(n+1)}{\lambda^{n+1}} + \frac{(n+2)}{\lambda^{n+2}} + \frac{(n+3)}{\lambda^{n+3}} + \text{etc.}}.$$

From this is it evident that such a pension scheme can be very advantageous and that a person, when he reaches a certain age, can benefit from a considerable income, for very little initial expense, throughout the rest of his life.

11. All of these questions can be answered easily, as soon as one knows the values of the fractions (1), (2), (3), (4), etc., which depend so much on climate and also on way of life; also one should remark that these values are different for the two sexes, so that one can't make precise determinations in general. Now, to conclude these observations, one understands easily that it is necessary to use a large sample, covering all sorts of persons; and in this regard, it is important not just to use registers of vital statistics that begin with children over the age of 1.[5] For first of all, one can't consider these infants as newborns, as most of them had certainly already escaped the dangers of infancy; and furthermore, the statistics wouldn't take into account children of a weak complexion, which would skew the pension calculations. Therefore, the values of our fractions (1), (2), (3), etc. which

[5] As can be seen from the data quoted below by Euler, infant mortality in his time was much larger than 20%, and thus in most localities it probably seemed a waste of time to keep statistics on newborn or sickly infants.

one would obtain from most statistics are undoubtedly too large, especially as regards the first few years. Nevertheless, since one has to do pension calculations on the basis of these statistics and not just on the true mortality rate, I will append the values of our fractions as can be obtained from the observations of Mr. Kersseboom. Since this table is based on selected infants who had already survived the first few months after their birth, if one wants to apply the theory to all the newborn infants in a certain city or province, these numbers should be reduced to take into account the high mortality of infants soon after birth. But we'll deal with this correction when we consider multiplication.