Fermat's Challenge Problem on $109x^2 + 1 = y^2$

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numlib::contfrac(sqrt(109))
```

$$10+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{4+\ldots}}}}}}}}}}$$

```
a0 := numlib::sqrt2cfrac(109)[1];
a1 := numlib::sqrt2cfrac(109)[2]
```

[10]

[2, 3, 1, 2, 4, 1, 6, 6, 1, 4, 2, 1, 3, 2, 20]

From this we can get the series of "convergents", from which we can look for a solution to $109x^2 + 1 = y^2$. The recursive relations are $h_n = a_n h_{n-1} + h_{n-2}$, $k_n = a_n k_{n-1} + k_{n-2}$, starting with $a_0 = 10$, $h_{-2} = 0$, $h_{-1} = 1$, $k_{-2} = 1$, $k_{-1} = 0$.

```
hprev := 1: hcurr := 10: kprev := 0: kcurr := 1:
for i from 1 to 30 do
   j := i mod 15:
   if j = 0 then j := 15 end_if:
   h1 := hcurr: k1 := kcurr:
   hcurr := a1[j]*h1+hprev:
   kcurr := a1[j]*k1+kprev:
   print([hcurr, kcurr, hcurr^2 - 109*kcurr^2]);
   hprev := h1: kprev := k1:
end_for
```

[21, 2, 5]

[73, 7, −12]

[94, 9, 7]

[261, 25, −4]

[1138, 109, 15]

[1399, 134, −3]

[9532, 913, 3]

[58591, 5612, −15]

[68123, 6525, 4]

[331083, 31712, −7]

[730289, 69949, 12]

[1061372, 101661, −5]

[3914405, 374932, 9]

[8890182, 851525, −1]

[181718045, 17405432, 9]

[372326272, 35662389, −5]

[1298696861, 124392599, 12]

[1671023133, 160054988, −7]
[4640743127, 444502575, 4]
[20233995641, 1938065288, −15]
[24874738768, 2382567863, 3]
[169482428249, 16233472466, −3]
[1041769308262, 99783402659, 15]
[1211251736511, 116016875125, −4]
[5886776254306, 563850903159, 7]
[12984804245123, 1243718681443, −12]
[18871580499429, 1807569584602, 5]
[69599545743410, 6666427435249, −9]
[158070671986249, 15140424455100, 1]
[3231012985468390, 309474916537249, −9]
15140424455100

So we found the smallest solution to Fermat's challenge:

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158070671986249^2 - 109*15140424455100^2
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Not something you'd get by trial and error!