Here Begins Chapter Twelve.

We therefore divide chapter twelve on problems of abaci into nine parts.

Of which the first is on summing series of number, and certain other similar problems.

The second is on proportions of numbers by the rule of four proportions.

The third is on problems of trees, and other similar problems which have solutions.

The fourth is on the finding of purses.

The fifth is on the buying of horses among company members according to given proportions.

The sixth is on travellers, and the problems that have resemblance to the problems of travellers.

The seventh is on false position and rules of variation.

The eighth is on certain problems of divination.

The ninth is on the doubling of squares and certain other problems.

Here Ends the Table of Contents for the XIIth Chapter.
Here Begins the First Part on Summing Series of Numbers.

When you wish to sum a given series of numbers which increases by some given number, as increasing by ones, or twos, or threes, or any other numbers, then you multiply half the number of numbers in the series times the sum of the first and last numbers in the series, or you multiply half the sum of the first and last numbers in the series by the number of numbers in the series, and you will have the proposition. For example, I wish to sum 7 numbers that increase by threes from seven up to 31, namely 7, 10, 13, and so forth up to 31. The number of

				19				
			16		22			
		13				25		
	10						28	
7								31

the aforesaid numbers is indeed 9, that is there are nine numbers in the aforesaid series, of which the first is the seven. The remaining number of numbers is however eight, which is had for a third of 24 which remains of 31 when 7 is subtracted. Therefore the sum of the extremes, namely the 7 and the 31 is 38; therefore if you multiply half the [p167] 9 by the 38, or half the 38 by the 9, then the result is 171 for the sum of the posed series of nine numbers; indeed by this rule can be found the sums of the series written below that we shall demonstrate in yet another way.

On the Same in Another Way.

If you wish to sum a series of numbers which ascends in order by ones beginning with one, or increases by twos beginning with 2, or increases by any other number beginning with that number, then you divide the last number by the first number, and you add one to the quotient, and you will keep the result; you multiply it by half the last number, or you multiply the last number by half the kept number. For example, I wish to sum all of the numbers which run from 1 to 60; I therefore shall divide the 60 by the 1, and to the quotient I add 1; there will be 61 that I shall multiply by half the 60, or I shall multiply the 60 by half of the 61; there results 1830 for the sum of the said series. Similarly if you wish to sum the series that runs from two to 60 by twos, that is the even numbers, then you divide the 60 by the 2, and you add 1 to the quotient; there will be 31 that you multiply by half of the 60. Similarly if you wish to sum the series from 3 to 60, increasing by threes, namely 3, 6, 9, and so forth, then you multiply one plus one third of the 60, namely 21, by half of the 60; there will be 630, and you understand how to proceed in any remaining similar problems.

And if you wish to sum only some of the numbers running from 1 up to any number, then you can proceed by the prior rule. Or by what is the same thing, you multiply half the sum of the extremes by the number of numbers, and you will have the proposition. For example, if you wish to sum the odd numbers that run from 1 up to 19, then you multiply half the sum of the extremes, namely will have the proposition. For example, if you wish to sum the odd numbers that run from 1 up to 19, then you multiply half the sum of the extremes, namely 10, by the number of odd numbers in the series. There are ten odd numbers which run from 1 up to 19; the product will be 100 for the said sum.

[*On the Sum of Squares.*]

However if you wish to have the sum of the squares of all numbers in order from the square of the unit, namely from one up to the square of any number, we say up to the square of ten, of which the square is 100, then you put the 10

aside, and before it you put the succeeding number, namely 11, and the sum of them both, namely 21, you put below them, and you multiply the 10 by the 11 and by the 21, and you divide the triple product by 6, and by the 1 which is the difference between the 10 and the 11, and you will have 285 for the said sum; and it will always be possible to cancel the 6 by which the product is divided. And if you wish to have the sum of squares which are made from the odd numbers up to the square of nine, then you put before the 9 the successor of the 9, that is 11, and the sum of them, namely 20, you put below them, and you multiply the three numbers together, and you divide the triple product by 12, that is by 6, and by the 2 that is the difference between the 9 and the 11, and you cancel, namely a third of the 9 you multiply by a fourth of the 20; there will be 15 that you multiply by the 11; there will be 165, and this is the sum. And if you wish to have the sum of the squares which are made from the even numbers in order from the square of the two, which is 4, up to the square of the ten, which is 100, then you put the 10 and the succeeding even number, namely the 12, and the sum of them, namely 22, aside. And from the abovesaid rule you take a twelfth of the triple product of the numbers which will be the even numbers in order from the square of the two, which is 4, up to the square of the ten, which is 100, then you put the 10 and the succeeding even number, namely the 12, and the sum of them, namely 22, aside. And from the abovesaid rule you take a twelfth of the triple product of the numbers which will be the sought sum, but you will cancel the $\frac{1}{12}$, and you will have 220. Similarly you can have the sum of all the squares which are made from numbers increasing by threes, or fours, or any other number. And if you wish to have the sum of the squares [p168] which are made from numbers increasing by fours beginning with the square of four, which is 16, up to the square of any number, and we say up to the square of 20, that is 400, then you put first the 20, and you write the succeeding number in the series, namely the 24; below them indeed you put 44, namely the sum of them, and you will multiply the 20 and the 24 and the 44, and you divide the triple product by 6, and by the increase number, that is 4; you will multiply the 20 by a fourth of fourth of a sixth of 24, namely by 1, and by the 44; the quotient will be 880 for the sum, and thus one goes on. I proved indeed geometrically that this is the said sum of squares in the book I composed upon squares.

On Two Travellers, One of Whom Goes after the Other with an Increasing Pace.

The rules for the summing of series were indeed shown; now truly applications of them are shown, namely as was said. There are two men who propose to go on a long journey, and one will go 20 miles daily. The other truly goes 1 mile the first day, 2 the second, 3 the third, and so on always one more mile daily to the end when they meet; it is sought for how many days the first is followed, which is found thus: namely, when the 20 is doubled there results 40 from which you subtract 1; there remains 39, and this amount of days he is followed; he who goes daily 20 miles goes in these 39 days 20 times 39 miles, which make 780 miles. The other man truly in the same 39 days goes as many miles as are in the sum of the numbers which run from one up to 39, which sum is found similarly from the multiplication of the 20 by the 39.

More on Two Travellers, One of Whom Follows the Other with Increasing Numbers.

days 21

Also if it is proposed that one man goes daily 21 miles, and the other truly goes with increasing odd numbers of miles beginning with one, and with continuing successive odd numbers, then it will be clear that he follows for 21 days. If we take 21 odd numbers in order, then there will be the sum of them from one up to 41; whence the sum of the odd numbers which increase from one up to 41 is the product of the 21 by itself.

On Two Travellers, One of Whom Goes after the Other by Even Numbers.

days 29

Truly if it is proposed that one goes daily 30 miles, and the other truly goes after by increasing even numbers, then it is done thus. You subtract 1 from the 30; there remains 29, and a total of 29 days he follows. Because there are 29 even numbers increasing from two up to 58, and because the sum of the even numbers up to 58 results from the multiplication of the 29 times the 30, it will not be doubted that he follows for 870 days.

When One Man Goes after Another by Increasing Threes or Some Other Number.

Truly if it is proposed that one goes daily some number of miles, that can be integrally divided by the increase number of the series by which the other follows, which increases by threes, or fours, or fives, or any other number, then it is done thus: the number of miles that the first man goes daily you divide by the increase number of the other, and the quotient is doubled, and from the doubled sum is subtracted 1; the residue will be the amount of days for which he follows. For example, it is put that one goes daily [p169] 60 miles, and the other truly goes with an increase of threes, that is in the first day 3 miles, in the second 6, in the third 9, and so forth; you divide the 60 by the 3; there will be 20 that you double; there will be 40, from which you subtract one; there remains 39, and for this amount of days he will follow; 39 is the number of numbers which increase by threes up to the triple of 39, that is 117. The sum of the numbers which increase by threes, from 3 up to 117, indeed results from the multiplication of the 39 times the 60, as is found by the first rule. And he who daily goes 60 miles, goes similarly for 39 times 60 miles in the 39 days.

days 39