ON CONTINUOUS MAPPING OF A LINE ONTO A PLANAR SURFACE (translated from the German by J. Rosenberg)

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Peano [Math. Annalen **36** (1890), 157] has recently shown through arithmetical considerations how the points of a line can be continuously mapped onto the points of a piece of a planar surface. Such a mapping can be described through very explicit functions, as we will see with the following geometrical argument. The line segment to be mapped, say an interval of length 1, we first divide into 4 equal pieces 1, 2, 3, 4; and the piece of surface, which we take to be a solid square of sides of length 1, we divide with straight cuts into 4 smaller squares of equal area (Figure 1). Secondly, we divide further each of the four line segments into 4 equal pieces, so that we have 16 segments now numbered 1, 2, 3, ... , 16; at the same time we divide each of the 4 squares 1, 2, 3, 4 into 4 smaller squares, and thus end up with 16 equal squares, which we number with the integers 1, 2, 3, ... , 16 in such a way that any two consecutive squares have a side in common (Figure 2). Imagine this construction repeated—Figure 3 shows the next step—thus it's easy to see for each point of the line segment there is determined a unique point of the square. One has only to take note of which subsegments of the line the given point lies in. The correspondingly numbered squares lie one inside the other, and enclose in the limit a unique point of the planar surface. This is the point that corresponds to the given point. *The mapping determined in this way is well-defined and continuous; conversely, to a given point of the square, there correspond one, two, or four points of the line.* It is worth noting, that with suitable modification of the line segments in the square in the above construction, one can easily *construct a well-defined and continuous mapping, such that to any point in the image there correspond at most* **three** *points.*

The functions defined above are at the same time simple examples of functions which are everywhere continuous and nowhere differentiable.

The mechanical meaning of the mapping we have discussed is the following: *a point can continuously vary, so that in finite time it passes through all the points of a planar surface.* Also one can—with suitable modification of the line segments in the square—at the same time arrange *that at an infinite dense set of points in the square, there is a well-determined direction of motion (both forwards and backwards).*

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As seen from the analytical representation of the mapping functions, it follows from the continuity, by a generalization of a theorem of K. Weierstraß, that these functions can be represented by infinite series of entire rational [i.e., polynomial] functions, which on the entire interval converge absolutely and uniformly.