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APPLICATIONS OF ANALYSIS ON LIPSCHITZ MANIFOLDS

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I shall try in this paper to give a brief survey of a few recent and very exciting developments in the application of analysis on Lipschitz manifolds to geometric topology. As will eventually become apparent, this work involves both operator algebras (especially the connection between C^*-algebras and K-theory) and harmonic analysis (in the literal sense of analysis of harmonics, i.e., of the spectrum of the Laplacian) in the proofs, though not in the statements of most of the theorems. Some of these results could only be obtained with great difficulty (if at all) by more traditional topological methods. I will give references to the literature but no proofs. The parts of this work that are my own are joint work with Shmuel Weinberger [10].

1. BASIC PROPERTIES OF LIPSCHITZ MANIFOLDS

A Lipschitz manifold is defined to be a topological manifold with certain extra structure. The key features of this structure are that on the one hand it seems to be only slightly weaker than a smooth structure, so that one can still do analysis with it, and yet existence and even essential uniqueness of this extra structure is almost automatic in many situations that are very far from being smooth. I'll try to make these notions precise in the rest of this paper.

Recall that if (X_1,d_1) and (X_2,d_2) are metric spaces, a function $f:X_1 \to X_2$ is said to be *Lipschitz* if there exists a constant $C > 0$ such that $d_2(f(x),f(y)) \le Cd_1(x,y)$ for all x and y in X_1, or *bi-Lipschitz* if f is a homeomorphism and both f and f^{-1} are Lipschitz. From the point of view of real analysis, the condition of being Lipschitz should be viewed as a weakened version of differentiability. In fact, we shall rely constantly on the

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following classical theorem of Rademacher.

THEOREM. Let U be an open set in $\mathbf{R}^n$, $f: U \to \mathbf{R}^m$ a continuous function. Then f is Lipschitz if and only if the distributional partial derivatives $\partial f_j / \partial x_k$ $(1 \leq j \leq m, 1 \leq k \leq n)$ are all given by functions in $L^\infty(U)$ (with respect to Lebesgue measure).

This has several important consequences, the most notable (for our purposes) being the following.

COROLLARY. Let U,V be open sets in $\mathbf{R}^n$, and let $f: U \to V$ be a locally bi-Lipschitz homeomorphism. Then f preserves the class of Lebesgue measure.

Now we are ready to introduce Lipschitz manifolds.

Definition. A *Lipschitz manifold* M^n of dimension n is a second-countable locally compact Hausdorff space M equipped with a family of so-called Lipschitz coordinate charts $\phi_\alpha: U_\alpha \to \mathbf{R}^n$, satisfying the following conditions:

(a) the U_α's are open sets in M which cover M;

(b) each ϕ_α is a homeomorphism onto its image (an open set in $\mathbf{R}^n$); and

(c) the transition functions

$$\phi_\beta \circ \phi_\alpha^{-1}|_{\phi_\alpha(U_\alpha \cap U_\beta)}: \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are locally bi-Lipschitz (with respect to the usual metric on $\mathbf{R}^n$).

Of course, conditions (a) and (b) just state that M is a topological n-manifold. However, condition (c) together with the corollary above implies:

PROPOSITION. Any Lipschitz manifold has a canonical measure class of full support (namely, the class of Lebesgue measure in any coordinate chart).

It is this proposition which makes it possible to do analysis on Lipschitz manifolds, somewhat in the way one can do calculus on smooth manifolds. In particular, there are certain distinguished function spaces on a Lipschitz manifold, most importantly

Lip_{loc} (locally Lipschitz functions) and L^p_{loc}, $1 \leq p \leq \infty$.

When the manifold is compact, the subscript "loc" can be deleted, and the transition functions in a Lipschitz atlas can be taken to be bi-Lipschitz (not just locally).

Examples.

(1) Any smooth (in fact, C^1) manifold has a canonical Lipschitz structure, since differentiable functions are Lipschitz.

(2) Any PL (piecewise-linear) manifold has a canonical Lipschitz structure, since any PL function is Lipschitz.

However, the real usefulness of Lipschitz manifolds stems from the following deep and rather surprising theorem of Sullivan. There is also a version for manifolds with boundary, which we won't need and therefore won't bother to state.

THEOREM (Sullivan [13] - see also [17] for an exposition of the proof). Any topological manifold M^n with $n \neq 4$ has a Lipschitz structure, and any two such structures are related by a Lipschitzeomorphism (i.e., locally bi-Lipschitz homeomorphism) isotopic to the identity.

Remark. Recent developments in 4-manifold theory have shown that the restriction to the case $n \neq 4$ is necessary. In fact, work of Freedman, Donaldson, and others (as far as I know, still unpublished) shows there are topological 4-manifolds with no Lipschitz structure. It is even possible that in dimension 4, a Lipschitz structure is always equivalent to a smooth structure.

The proof of Sullivan's theorem is not very constructive, and shows that Lipschitz structures behave quite differently from PL structures. It is a feasible but non-trivial exercise to start with two homeomorphic PL-manifolds which are not PL-isomorphic (e.g., fake tori of dimension ≥ 5) and to write down an explicit Lipschitzeomorphism between them. This was done by Siebenmann in [20] - see also [19].

2. THE TELEMAN SIGNATURE OPERATOR

Throughout this section, M^n will denote a fixed compact connected Lipschitz manifold M of dimension n. Eventually we will also take M to be oriented and n to be even, though we don't need to assume this for the moment.

The key to doing analysis on M is the observation (due to Sullivan and first thoroughly exploited by Teleman) that although M may not have a tangent or cotangent bundle in the usual sense, it makes sense to talk of measurable "sections" of the cotangent bundle, or [illegible] of L^p differential forms. In a coordinate chart having $U \subseteq \mathbb{R}^n$, such a form is an expression

$$\sum_{1 \le i_1 < \dots < i_k \le n} f_{i_1 \dots i_k} \, dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

where $f_{i_1 \dots i_k} \in L^p(U)$. This notion is well-defined on M since Lipschitz changes of coordinates only involve multiplication by determinants of matrices of distributional partial derivatives of the transition functions, which all lie in L^∞. This same observation is also [illegible] in L^∞ [illegible] L^p to L^p.

Since we will want to do L^2 analysis on differential forms, we need a way of fixing an inner product on each fiber. Just as in the smooth case, this [illegible] a Riemannian metric. However, one quickly discovers that if one starts with a smooth Riemannian metric on an open set in $\mathbb{R}^n$ and makes a Lipschitz change of coordinates, the metric will become a metric that only varies measurably from point to point, but at least is bounded above and below by multiples of the standard metric. A Lipschitz Riemannian metric is something which has this form in any coordinate chart. Existence can be proved [illegible] by patching with a partition of unity, just as in the smooth case; such a metric gives rise to a [illegible] operator from L^2 forms to L^2 forms satisfying $** = (-1)^{k(n-k)}$, as well as to a Riemannian volume density in the orientable case. These make it possible to define a specific inner product on the L^2 differential forms. For simplicity we take M to be oriented, so that one can define the inner product by the familiar formula

$$\langle\alpha,\beta\rangle = \int_M \alpha \wedge * \bar{\beta}\,,$$

for α and β j-forms. We use the notation $L^2(M,\Lambda^j)$ for the Hilbert space of L^2-j-forms. Teleman [15] pointed out that one can now construct a closed, densely defined (unbounded) operator

$$d:L^2(M,\Lambda^j) \to L^2(M,\Lambda^{j+1})$$

satisfying $d^2 = 0$ as in the smooth case. The domain of d consists of those L^2-forms for which the distributionally defined exterior derivative (in any Lipschitz coordinate chart) also lies in L^2. The following theorem of Teleman and Hilsum asserts that the exterior derivative as so defined has all the usual properties.

THEOREM (Teleman [15, §§1-4], Hilsum [5]). The Hilbert space adjoint of d is given by the usual formula $d^* = \pm *d*$ (where the sign is - if n is even, $(-1)^{j-1}$ if n is odd). The operator $D = d+d^*$ is self-adjoint, and $(1+D^2)^{-1}$ is compact (even in the same Schatten class as in the smooth case). Finally, the de Rham and Hodge theorems hold: the operators d and D have closed range, the de Rham cohomology ker d/im d is naturally isomorphic to the singular cohomology $H^*(M,\mathbb{R})$, and every de Rham cohomology class has a unique harmonic representative (i.e., a unique representative in the kernel of $\Delta = D^2$).

A substantial amount of analysis goes into the proof, but the essence of the argument is to see what happens to the spectrum of the Laplacian on a smooth manifold if one uses a (non-smooth) Lipschitz Riemannian metric.

In any event, the theorem shows that we have a well-behaved first-order "elliptic" differential operator D on M. This operator has the further good property that $\mathrm{Lip}(M) \subseteq \mathrm{dom}(D)$; in fact, for any $\alpha \in \mathrm{dom}(D)$ and $f \in \mathrm{Lip}(M)$,

$$f\alpha \in \mathrm{dom}(D) \text{ and } D(f\alpha) = fD(\alpha) + (e(df)\text{-}i(df))\alpha\,,$$

where e is exterior multiplication and i is interior multiplication, normalized so that for f real, $i(df) = e(df)^*$. Note that since df is an L^∞ 1-form, the operators e(df) and i(df) are bounded. We shall use this fact shortly.

The operator D can be used for doing index theory on M provided that we have an appropriate grading of our Hilbert space

$$\mathcal{H} = L^2(M,\Lambda^*),$$

i.e., we find a decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ of $\mathcal{H}$ such that $D\mathcal{H}^+ \to \mathcal{H}^-$ and $D\mathcal{H}^- \to \mathcal{H}^+$. (D cannot have a nonzero index on all of $\mathcal{H}$ since it is self-adjoint.) It is equivalent to find a grading operator $\varepsilon = \varepsilon^*$ with $\varepsilon^2 = 1$ and $D\varepsilon = -\varepsilon D$. There are two standard choices, the grading by parity of degree (i.e., $\varepsilon = (-1)^j$ on j forms), and the signature grading (when $n = 2\ell$ is even, this is defined by $\varepsilon = (i)^{j(j-1)+\ell}*$). Then we let $\mathcal{H}^{\pm}$ be the (± 1) eigenspace of ε and D viewed as an operator $\mathcal{H}^+ \to \mathcal{H}^-$ is called the *Euler* operator in the first case or signature operator in the second case. Exactly as in the smooth case (see [8, pp.575-576]), we have as an immediate consequence of the Hodge theorem:

PROPOSITION. For M^n a compact oriented Lipschitz manifold, the indices (i.e., (dimension of kernel)-(dimension of cokernel)) of the Euler and signature operators are the Euler characteristic and signature of M, respectively.

Here the signature is defined when $n = 2\ell$ as follows. The cup product

$$\cup : H^\ell(M,\mathbb{R}) \to H^{2\ell}(M,\mathbb{R}) \cong \mathbb{R}$$

gives (because of Poincaré duality) a non-degenerate bilinear form B on $H^\ell(M,\mathbb{R})$, which is symmetric when ℓ is even and antisymmetric when ℓ is odd. The signature of M is just defined to be the signature of this form, so that, when ℓ is even, this is the difference between the dimensions of maximal subspaces of $H^\ell(M,\mathbb{R})$ on which B is positive and negative definite.

In the smooth case, Atiyah and Singer were able to deduce from the above proposition the Chern-Gauss-Bonnet formula for the Euler characteristic and the Hirzebruch formula for the signature in terms of characteristic classes of the tangent bundle or (via Chern-Weil theory) integrals of certain polynomial functions of the curvature. Such formulae do not make sense in the Lipschitz case, since the "tangent bundle" is only a topological fibre bundle, not a vector bundle, and thus we have no theory of curvature and characteristic classes. Nevertheless, Teleman [15,16] was able to come up with a reasonable substitute. We shall follow Hilsum's simplification (and slight strengthening) [5] of his result.

Since we have no pseudodifferential calculus on a Lipschitz manifold, it is impossible to extract a cohomology class from the symbol of D. Therefore it seems essential to work with the formulation of the index theorem based on K-*homology*. It so happens that when n = dim M is even, the operator D, together with the parity or the signature grading τ on forms, is exactly what is needed to define a class in

$$K^0(C(M)) = K_0(M)$$

according to the "unbounded picture" of Kasparov theory as formulated by Baaj and Julg (see [3] and [4, §17.11]). The relevant axioms are that

(a) D is self-adjoint and $(1+D^2)^{-1}$ is compact;

(b) τ preserves the domain of D and $D\tau = -\tau D$;

(c) there is a dense subalgebra (namely, Lip(M)) of C(M) consisting of functions that preserve the domain of D and have bounded commutator with D.

Hilsum noticed that D only changes by a suitable notion of homotopy when the Lipschitz Riemannian metric is varied, and thus the class obtained from D depends only on the Lipschitz structure of M. Applying Sullivan's theorem gives the following:

THEOREM (Hilsum [5]). Let M^{2l} be a connected, closed (i.e., compact, without boundary) oriented topological manifold of even dimension $n = 2l \neq 4$. By putting a Lipschitz Riemannian structure on M, one can obtain classes

$$[D_{Euler}], [D_{sign}] \in K_0(M)$$

from the Euler and signature operators, and these classes are topological invariants. Furthermore,

$$\chi(M) = \mathrm{ind}(D_{Euler}) = c_*[D_{Euler}],$$

$$\mathrm{sign}(M) = \mathrm{ind}(D_{sign}) = c_*[D_{sign}],$$

where $c: M \to pt$ is the "collapse map".

The last statement is immediate from the definition of c_* in Kasparov theory, when we identify $K_0(pt)$ with $\mathbb{Z}$. The above theorem, though it may not look it, is the desired generalization for the Chern-Gauss-Bonnet and Hirzebruch formulae. In fact, when M is smooth, it is immediate from Atiyah-Singer theory that under the Chern character (a rational transformation of homology theories sending $K_0(\cdot)$ to even-dimensional ordinary homology $H_{even}(\cdot;\mathbb{Q})$), $[D_{Euler}]$ and $[D_{sign}]$ go to homology classes whose component in degree 0 is just the Poincaré dual of the cohomology class in degree n given by the Chern-Gauss-Bonnet or Hirzebruch formula.

At least in the signature case, however, the class $[D_{sign}] \in K_0(M)$ contains substantially more information than just the signature of M. When M is smooth, it is not too hard to show that

$$ch[D_{sign}] = 2^{[illegible]}[\mathcal{L}(M) \cap M],$$

where $\mathcal{L}$ is the Atiyah-Singer modification of the Hirzebruch L-polynomial, differing from Hirzebruch's polynomial only by suitable powers of 2. The form of this polynomial is such that one can recover from it all of the rational Pontrjagin classes p_i of the tangent bundle of M. Thus this reasoning gives us a new proof of the previous theorem:

THEOREM (originally due to Novikov, analytic proof in [illegible]). The rational Pontrjagin classes of a closed smooth manifold are topological invariants.

One can also invert the result to give a definition of Pontrjagin classes for topological or PL manifolds that doesn't depend on studying the homotopy type of classifying spaces such as BTop.

However, one should note that the above theorem of Novikov is more powerful than the corollary (Sullivan's theorem), since the class $[D_{sign}]$ also contains torsion information. In fact, one can prove

THEOREM ([illegible], [10]). The class $[D_{sign}]$ of the previous theorem is a homeomorphism class for $K_* \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ (hereafter denoted $K[\frac{1}{2}]_*$ for short). In other words, cap product with this class induces a Poincaré duality isomorphism $K[\frac{1}{2}]^*(M) \to K[\frac{1}{2}]_*(M)$.

Thus we've arrived (after treating the case of odd dimension or dimension 4 by "stabilizing", taking a product with S^1 or T^2 to jack up the dimension) at an analytical proof of the following celebrated result:

THEOREM (Sullivan - never published by him, but nicely written up in [8], Ch.5). Any closed oriented topological manifold has a canonical orientation for $K[\frac{1}{2}]$, related to the signature.

One might dismiss what we've done as a way of substituting one deep theorem of Sullivan (the one on existence of Lipschitz structures) for another (the one on K-orientations). However, this is not exactly so, since both theorems of Sullivan rely on the work of Kirby-Siebenmann, which shows that in dimensions ≥ 5, the difference between the PL and Top categories only involves 2-torsion. (The theorem on Lipschitz structures doesn't rely on this result explicitly but it does make essential use of one of the key ideas of the proof.) Thus the Sullivan theorem on K-orientations is always proved by using this to reduce to the PL case. And for PL manifolds, there is a canonical Lipschitz structure, and in fact the work of Teleman simplifies considerably. So the theorem about existence and uniqueness of Lipschitz structures is not needed to get $K[\frac{1}{2}]$-orientations.

Nevertheless, the proof of this last result involves a bit more than just the construction of the Teleman signature operator. In [10], we used the standard principle of algebraic topology that an orientation for a manifold (with respect to a certain homology theory) is equivalent to a Thom isomorphism for the tangent (micro-) bundle. Then we proved a result about Thom isomorphisms for Lipschitz bundles by a calculation of a Kasparov product together with a Mayer-Vietoris argument. The method of [6] is based on the related notion of Gysin maps and their functorial properties.

4. THE EQUIVARIANT CASE

One advantage of the analytic methods we have been discussing is that they carry over very directly to the equivariant setting (of compact groups acting on manifolds), whereas the more traditional methods of algebraic topology become substantially more complicated when made equivariant. In particular, the Baaj–Julg scheme will generalize to give us classes $[D_{sign}] \in K_0^G(M)$, the equivariant K-homology of the Lipschitz manifold M, provided we have the following additional conditions:

(a) the compact Lie group G acts on M by homeomorphisms and compatibly on the Hilbert space H by a unitary representation, and

(b) the unbounded operator D and the grading operator ε commute with the action of G.

It is clear that these will be satisfied provided that G acts on M by orientation-preserving Lipschitz homeomorphisms and that the Lipschitz Riemannian metric g is chosen to be G-invariant. Since the latter can always be arranged by "averaging" when the former is satisfied, it becomes necessary to deal with the following:

PROBLEM. Suppose a compact Lie group G acts on a closed topological manifold M^n by homeomorphisms. When does M have a G-invariant Lipschitz structure?

Though the complete answer to this is not known, evidence suggests that except for difficulties arising from peculiarities of dimension 4, the answer is "almost always". In any event, the following partial results are enough to deal with a wide variety of situations:

(1) If M is smooth and G acts by diffeomorphisms, or if M is PL, G is finite, and G acts by PL homeomorphisms, then the canonical Lipschitz structure on M is G-invariant. (This is trivial.)

(2) If G is finite and acts freely on M, and $\dim M \neq 4$, then M has an essentially unique G-invariant Lipschitz structure. (This follows from applying Sullivan's theorem to the topological manifold M/G.)

(3) (Rothenberg-Weinberger [11]). If G is finite and if for all subgroups $H \subset K$, the fixed set M^K is a topological submanifold which is locally flatly embedded in M^H, then for some torus T with trivial G-action, $M \times T$ with the product action has a G-invariant Lipschitz structure. Any two such strcutres become equivalent after taking a product with another suitably large torus. (The notion of G-invariant Lipschitz structure in this theorem is slightly different than in (2), though the distinction is technical and need not concern us here.)

(4) The situations of (1)-(3) are definitely not necessary for M to have a G-invariant Lipschitz structure. We constructed in [10] Lipschitz actions of finite cyclic groups on spheres, for which the fixed set M^G does not even have finitely generated homology (and thus is not even an ANR).

Thus in all of these situations, the machinery of §2 carries over. To get the most useful version of a G-index theorem, one wants to localize the K-homology element $[D_{Euler}]$ or $[D_{sign}]$ to fixed sets of subgroups. This requires a result dual to the Segal Localization Theorem [12, Prop. 4.1], which we formulate and prove in [10]. A finite generation assumption turns out to be necessary. Putting everything together gives the following Lipschitz analogue of the Atiyah-Singer G-Signature Theorem [2, Theorem 6.12]. The orientation is not needed for G-χ.

THEOREM ([10, theorem 4.9]). Let $M^{2\ell}$ be a connected, closed, oriented Lipschitz manifold on which a compact Lie group G acts by orientation-preserving Lipschitzeomorphisms. Assume that $K_G^*(M)$ is finitely generated over $R(G)$ — this is automatic if M is an equivariant ANR. Then the G-signature and G-Euler characteristic of M (the differences of the characters of the action of G on the positive and negative parts of $H^\ell(M,\mathbb{C})$, or on even and odd-degree real cohomology) are given by formulae

$$G\text{-Sign}(M)(s) = \sum_j \sigma(M_j^s),$$

$$G\text{-}\chi(M)(s) = \sum_j \rho(M_j^s), \quad s \in G,$$

where M_j^s runs over the components of the fixed set M^s. The terms on the right only depend on the local structure of M [illegible] with an action of the topologically cyclic group generated by g near M^g_i. In particular, if M^g_i is a smooth manifold with a local equivariant normal bundle, $a(M^g_i)$ and $\rho(M^g_i)$ are given by the formulae of Atiyah and Singer.

This result improves certain earlier non-smooth G-signature theorems ([3, Theorem 14.B.2] and [7, Theorem 9.2]) and has several useful applications in spite of a lack of an explicit formula for the local terms. Another version of our theorem (with slightly more restrictive hypotheses) may be found in [6, Theorem 7.3]. Here are a few immediate consequences. We didn't bother to deal with the Euler characteristic in [10], but of course (i) and (ii) below for the Euler characteristic have much easier topological proofs.

THEOREM ([10, Theorem 4.1]). Suppose M^{2k} is a connected, closed, oriented topological manifold and G is a finite group acting on M by orientation-preserving homeomorphisms.

(i) If G acts freely, the Euler characteristic and signature of M are divisible by $|G|$, and the signature vanishes if G acts trivially on $H^k(M;\mathbb{Q})$.

(ii) If $G = \mathbb{Z}_{p^r}$ (p any prime and $r \geq 1$) and $M^G = \emptyset$, [illegible] M and the action are PL, or if all fixed sets of subgroups are locally flatly embedded topological manifolds, then the Euler characteristic and signature of M are divisible by p, and the signature vanishes if G acts trivially on $H^k(M;\mathbb{Q})$.

Proof. We only have [illegible] to reduce to the case where M has a G-invariant [illegible] structure. But if this is the case, the previous theorem implies that G-Sign(M)(g), and $\chi(M^g)$ vanish for all $g \neq 1$ in case (i) and for all g not in the unique subgroup of index p in case (ii) (since $M^G = \emptyset$). In the first case, this means G-Sign and G-χ are characters supported on $\{1\}$, hence are multiples of the regular representation of G. In the second case, these characters are supported on a proper normal subgroup of index p, and thus are induced. The conclusion then follows easily.

To prove (ii) purely topologically, write M as a disjoint union of locally closed subsets:

$$M = M^{\mathbb{Z}_{p^r}} \cup (M^{\mathbb{Z}_{p^{r-1}}} - M^{\mathbb{Z}_{p^r}}) \cup \ldots \cup (M - M^{\mathbb{Z}_p}).$$

By assumption, $M^{\mathbb{Z}_{p^r}} = \phi$, and $\mathbb{Z}_p$ acts freely on the other pieces, so one can get the result from the Euler Poincaré principle and Mayer-Vietoris, using cohomology with compact supports. Nevertheless, it's also amusing to have an analytical proof.

Our G-signature theorem can also be used to study topological conjugacy of linear representations of finite groups, as explained in [7]. The idea is to build out of such a conjugacy a (topological) action of G on a sphere, and then to imitate the use of the G-signature formula in [1, Theorem 7.15]. More refined applications of the same idea may be found in [11] and [19].

We conclude by mentioning that in the equivariant case, our result from §2 about $K[\frac{1}{2}]$-orientations carries through for locally linear actions of finite groups of odd order. This was originally proved by Madsen and Rothenberg [9] by a rather complicated, purely topological, argument. Our method has the advantage that one can also get some information about locally linear actions of groups of even order or of connected compact Lie groups from further study of the G-signature formula.

THEOREM ([6, Prop. 7.6], [10, Cor. 4.14]). Let G be a group of odd order and M a closed, connected, oriented topological manifold on which G acts by a locally linear action. Then M is canonically oriented for $K_G[\frac{1}{2}]^*$. If M has a G-invariant Lipschitz structure and is even-dimensional, then $[D_{\mathrm{sign}}] \in K^G[\frac{1}{2}]_0$ is a fundamental class (i.e., cap product with this class defines a Poincaré duality $K_G[\frac{1}{2}]^*(M) \cong K^G[\frac{1}{2}]_*(M)$).

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