Noncommutative Geometry and Applications to Physics

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I. The Philosophy of Noncommutative Geometry

A basic notion in mathematics, going all the way back to Descartes, is that we study a space by means of *functions* on the space. In fact, the algebra of functions "determines" the space.

Examples of this principle:

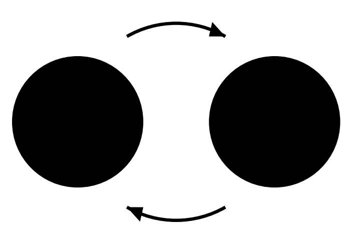
- Algebraic Geometry: R a commutative ring, Spec R a scheme.

- Gelfand-Naimark correspondence: X a locally compact Hausdorff space, $C_0(X)$ a commutative C^*-algebra (a Banach algebra that can be realized as a norm-closed $*$-algebra of bounded operators on a Hilbert space).

Quantum mechanics, however, suggests that some physical systems should be modeled by "spaces" on which "functions" are not commutative. C^*-algebras are natural models for the function algebras, since they have a good structure theory and since quantum mechanics demands that observables be self-adjoint operators on some Hilbert space.

Example: a spinning electron, with two [pure] states: "up" and "down."

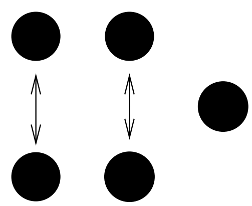
- Semiclassical model: space S^0 of two points, possibilities of transitions between them.



- Quantum model: space with "functions" $M_2(\mathbb{C})$. Generators are matrix units e_{ij} with relations $e_{ij}e_{kl} = \delta_{jk}e_{il}$.

This is a special case of the C^*-algebra of a (locally compact) *groupoid*. Other examples:

- An equivalence relation on a finite set. The C^*-algebra is a finite-dimensional semisimple algebra (over $\mathbb{C}$). Example:



$$M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus \mathbb{C}$$

- A locally compact group G. The C^*-algebra is $C^*(G)$, the universal object for unitary representations of G.

- A locally compact group G acting on a locally compact space X. The C^*-algebra $C^*(G, X) = C_0(X) \rtimes G$ is generated by formal products $f\varphi$, $f \in C_0(X)$, $\varphi \in C^*(G)$, multiplied using the rule that

$$ufu^{-1} = u{\cdot}f, \quad u{\cdot}f(x) = f(u^{-1}{\cdot}x), \quad u \in G.$$

Geometric examples:

- (Connes, Connes-Skandalis) A foliated manifold $(M, \mathcal{F})$. The C^*-algebra of the holonomy groupoid (roughly speaking, the equivalence relation of being on the same leaf) is denoted $C^*(M, \mathcal{F})$. If $\mathcal{F}$ is a fibration

  $$F \to M \to X,$$

  $C^*(M, \mathcal{F})$ is up to Morita equivalence just $C_0(X)$, the functions on the quotient space $M/\mathcal{F}$. But if $\mathcal{F}$ is minimal (every leaf dense), $C^*(M, \mathcal{F})$ is a simple algebra (Fack-Skandalis), even though it may have fairly complicated structure and K-theory.

- (Farsi) An orbifold M. The associated C^*-algebra is $C(P) \rtimes O(n)$, P the principal frame bundle, which is Morita equivalent to $C_0(M)$ if M is actually a manifold.

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