Exercises for Oberwolfach Seminar
on Topological $K$-Theory
of Noncommutative Algebras

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Note: My slides for the Seminar are available at:
http://www.math.umd.edu/users/jmr/OberwolfachS05.pdf
The file is a work in progress, but I am updating it as I go along.

1 Lecture I: A Survey of Bivariant $K$-Theories

1. Verify that $K_*(\__; \mathbb{Q})$ and $K_*(\__; \mathbb{Q}/\mathbb{Z})$, as we defined them for local Banach algebras as $K_*(\__; \hat{\otimes} D)$ for a suitable tensor product and suitable auxiliary algebras $D$, are indeed homology theories (homotopy invariant, half-exact, with long exact sequences).

2. Show that the definition of $K_*(\__; \mathbb{Z}/n)$ defined using the mapping cone of the unital map $\mathbb{C} \to M_n(\mathbb{C})$ agrees with the "classical choice," where we take for $D$ the commutative algebra $C_0(X \setminus \{x_0\})$, where $X$ is the mod-$n$ "Moore space," a CW complex with 3 cells defined by attaching a 2-cell to $S^1$ by a map of degree $n$, and $x_0$ is a basepoint (the 0-cell in $X$).

3. Check the details of the theorem that $DK^*$, as defined in the slides, is a cohomology theory on algebras.
2 Lecture II: Twisted $K$-Theory

1. Check the calculation of the twisted $K$-theory

$$K_{\delta_n}^{-*}(S^3) = K_*(CT(S^3, \delta_n)),$$

where $\delta_n$ has as Dixmier-Douady invariant $n$ times the fundamental class on $S^3$.

2. (harder) Use the last exercise and the Atiyah-Hirzebruch spectral sequence (the spectral sequence induced by the skeletal filtration) to show that if $X$ is a finite CW complex and $\delta \in H^3(X, \mathbb{Z})$, there is a spectral sequence

$$H^p(X, K^q) \Rightarrow K^{p+q \mod 2}(X),$$

in which the first non-trivial differential is $d_3 = \cup \delta + Sq^3$.

3 Lecture III: Connes’ Thom Isomorphism

1. Deduce from Connes’ Thom isomorphism theorem that for a connected, simply connected solvable Lie group $G$ of dimension $n$, $K_*(C^*(G))$ depends only on $n \mod 2$ and not on anything else. (Hint: $G$ has a closed connected normal subgroup of codimension 1.)

2. Let $\alpha$ and $\alpha'$ be exterior equivalent actions of a locally compact group $G$ on a $C^*$-algebra $A$. Prove that $A \rtimes_\alpha G$ and $A \rtimes_{\alpha'} G$ are $*$-isomorphic. Show in fact that one can choose the isomorphism to be the identity on the natural copies of $A$ in the multiplier algebras.

3. Let $\mathbb{R}$ act on $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ by flow along lines of slope $\theta$, i.e., by

$$\alpha_t(x, y) = (x + t, y + \theta t) \mod \mathbb{Z} \times \mathbb{Z}.$$

Compute the $K$-theory of the crossed product $T^2 \rtimes_\alpha \mathbb{R}$ (as a group). **Harder:** Find specific generators for $K_*(T^2 \rtimes_\alpha \mathbb{R})$. **Note:** This is an example of an induced action. Thus the $K$-theory can also be computed by the Pimsner-Voiculescu sequence for the action of $\mathbb{Z}$ on $T$ by rotation by $2\pi \theta$. 
4 Lecture IV: Applications to Physics

1. Suppose a compact group \( T \) acts freely on a (reasonably nice) space \( X \), with the quotient map \( X \to Z \) a principal \( T \)-bundle, and suppose \( E \xrightarrow{p} X \) is a principal \( G \)-bundle over \( X \), for \( G \) some other group (in our applications \( PU \)). Show that the \( T \)-action on \( X \) lifts to an action on \( E \) by bundle automorphisms if and only if \( p \) is pulled back from a \( G \)-bundle over \( Z \).

2. Let \( p : T \to Z \) be a principal \( T \)-bundle, with \( T \) and \( Z \) locally compact. Let \( T \) act on \( C_0(T) \) in the obvious way. Show that \( C_0(T) \rtimes \mathbb{T} \cong C_0(X, \mathcal{K}) \), and that the dual action \( \theta \) of \( Z \) on \( C_0(Z, \mathcal{K}) \) has the property that \( C_0(Z, \mathcal{K}) \rtimes_\theta Z \cong C_0(T, \mathcal{K}) \).

3. With notation as in the last exercise, verify that

\[
\text{Ind}_Z^X C_0(Z, \mathcal{K}) \cong CT(S^1 \times Z, \delta),
\]

where \( \delta = a \times [p] \), \( a \in H^1(S^1) \) the usual generator and \( [p] \in H^2(Z) \) the characteristic class of the \( T \)-bundle \( p : T \to Z \).