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C*-algebras, positive scalar curvature and the Novikov conjecture, II

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0. Introduction and acknowledgements
In this lecture I hope to turn the tide of this Seminar around and discuss "operator-algebraic methods in geometry" rather than "geometric methods in operator algebras." The intention is to provide an introduction to some of the literature on topological obstructions to positive scalar curvature (including [27], [28], [29], [31], and [32]), with emphasis on the index-theoretic method of [30]. The first section of this paper will thus be expository, and biased toward topics likely to be of interest to those interested in applications of C*-algebras in differential geometry. While I was preparing this survey, I decided to attempt a deeper analysis of a conjecture of Gromov and Lawson that if true would provide a nice framework for the whole subject; this accounts for Sections 2 and 3 of this article. I suspect that much of the content of Section 2 may be known to the experts, but I have not seen any of this material written down, and the treatment here is my own. In any event, the results of this part are needed for Section 3, which I believe to be new. In fact, by way of advertisement for "non-commutative differential geometry," I might add that I can think of no technique that would yield the example of Theorem 3.1 without methods similar to those of [30].

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I would like to thank Professors Edward Zehres and Makhi Kassai for the opportunity to speak at this seminar, and for their excellent organizational work. In addition, I would like to thank Professor Evan Lass and Michael Groe for teaching me about this subject, and the members of the University of Pennsylvania geometry seminar for suggesting the problem treated in Section 3. (In a discussion at Penn in November, 1980, I naively suggested that if a manifold were to have a metric of positive scalar curvature, one ought to be able to "average" this metric under a freely acting finite group of diffeomorphisms, and so get a similar metric on any manifold (regularly) finitely covered by this original one. When Chris Atreus, Howard Gluck, Jerry Kass, and Wolfgang Krieger countered in unison that this certainly wasn't obvious, I slowly began to suspect that some topological obstruction might be involved.) I also thank

G. C. Barbot for sending me preprints of his recent work, which plays a vital role in the methods of [G] and thus in some of the results discussed here. Finally, I would like to thank Mr. Enol Kassai and Celia Sutherland for their hospitality at the University of New South Wales in July-August, 1983, during which time some of this work was completed.

Added in proof: After this paper was completed, it was pointed out to me that Y. Kiyumichi [on the existence of positive scalar curvature metrics on non-simply-connected manifolds, J. Fac. Sci. Univ. Tokyo, Sect. IA 22 (1984), 509-561], has done some work along the lines of Section 2 below (duplicating, for instance, our Theorems 2.5, 2.14, and 2.15), and that T. Ebert and W. T. Ziller have found a counterexample similar to that of Section 3 (with covering group \( H \)), based on the \( \mathbb{Z}_2 \)-valued obstruction of Milnor [15].

Meanwhile, I have been able to improve the results of [12] on the simply-connected spin case, and to rework the method of [30] using real \( K \)-theory so as to prove one direction of Conjecture 2.1 below for a large class of fundamental groups. These results will appear in a separate publication.

Finally, the dimension restriction in Proposition 2.3 and in Theorem 2.5 may be simplified to "\( n \geq 2 \)" using the following argument suggested by I. M. Singer. If \( n \geq 2 \) and \( x, y, z \) are any finitely presented group, one can construct a spin manifold \( \mathbb{S} \) with \( \pi_1 \mathbb{S} = x \) by starting with a connected sum of copies of \( \mathbb{S} \times \mathbb{S} \) and by doing surgeries (preserving the spin structure) to build the correct relations into the fundamental group. Since \( \mathbb{S} \times \mathbb{S} \) has positive scalar curvature and all the surgeries needed are in codimensions \( n \) and \( n-1 \), \( \mathbb{S} \) has positive scalar curvature by [12] or [30]. This \( \mathbb{S} \) may be substituted for the \( \mathbb{S} \times \mathbb{S} \) in 2.3.
Throughout this article, we shall be interested in the following geometric problem. Given a manifold $M^n$, which we shall always take to be orientable, smooth, compact, connected, and without boundary, we can choose a Riemannian metric $g$ on $M$ such that the associated scalar curvature function $k$ on $M$ is everywhere positive? More generally, what functions $k$ can arise from metrics on $M$?

When the dimension $n$ of $M$ is 2, information about this question is immediately provided by the Gauss-Bonet theorem, since in this case $n/2$ coincides with the Gaussian curvature. Since $\int_M \text{vol} - \text{Ker}$, where $K$ is the Euler characteristic, we see that $k > 0$ is impossible unless $M$ is a sphere, and that $k < 0$ is impossible unless $M$ has genus $> 1$.

Also note that every orientable closed surface except for the sphere is spherical, i.e., has contractible universal covering. A quick summary of what Gauss-Bonet says about positive scalar curvature is that no closed spherical 2-manifold admits a metric with $k > 0$.

In dimension $n \geq 3$, Aubin ([4], [5]) pointed out that the situation is fundamentally different: any closed manifold with $n \geq 3$ admits a metric with $k < 0$. In fact, Mabon and Nirenberg ([2]), Theorem 3.1 showed that given $M^n$ with $n \geq 3$ and any smooth real-valued function on $M$ that is negative somewhere, one can realize this function as the scalar curvature $k$ for some Riemannian metric on $M$. In particular, there can be no result like the Gauss-Bonet theorem relating the integral of $k$ to the topology of $M$. Nevertheless, we know that $k > 0$ is not always possible (a fact which for $n = 3$ seems to be relevant to relativistic cosmology). Indeed, there seem to be topological obstructions to positive scalar curvature of two very different sorts: some that apply even in the simply connected case, and others that depend on the size of the fundamental group. We shall discuss several of these obstructions and try to relate them to the common framework of the index of the Dirac operator.

The oldest, and in a sense the most basic, result saying that certain manifolds of dimension $\geq 3$ do not admit a metric with $k > 0$ is due to Lichnerowicz ([7]). He showed that if $M^n$ (as always closed, connected, and orientable) satisfies $\nu_2(M) > 0$ (where $\nu_2(M) \in H^2(M,\mathbb{Z})$ denotes the second Stiefel-Whitney class) and $n \geq 4$ (mod 8), and if $M$ admits a metric of positive scalar curvature, then one must have $\hat{A}(M) = 0$. Here $\hat{A}(M) = (\hat{A}(M), [M])$ and

$$\hat{A}(M) = 1 - \frac{D_4}{24} - \frac{1}{360}\left(p_2 - \frac{7}{2}p_1^2\right) - \cdots$$

is the "total $\hat{A}$-class", a certain polynomial in the rational Pontryagin classes $p_i \in H^4(M,\mathbb{Q})$ of the stable tangent bundle of $M$, and $[M] \in H_1(M,\mathbb{Q})$ is the fundamental homology class defined by the choice of an orientation. This may sound complicated but amounts to a definite restriction. For instance, a "K" surface $K^4$ (a smooth algebraic hypersurface in $\mathbb{CP}^3$ defined by an equation of degree 4) cannot admit positive scalar curvature since $\nu_2(K) > 0$ and $\hat{A}(K) = 0$. It is also worth pointing out that the invariants $\nu_2$ and $\hat{A}$ which come into the theorems depend only on the homeomorphism type of $M$, not on the differentiable structure. This is because the Stiefel-Whitney classes depend only on the homotopy type of $M$ (by the Wu formulas — see [3]), p. 10) and (by a deep theorem of Novikov (29)) the $p_i$'s are topological invariants. (In the case of a Riemannian, $p_2$ and so $\hat{A}$ is a homotopy invariant, but this fails to be true for manifolds of dimension $\geq 4$.) Also one may replace the condition $k > 0$ in the theorems by $k > 0$, $k \neq 0$, since as pointed out in [21], Proposition 3.5, a metric with $k \geq 0$ and $k \neq 0$ can be modified to achieve $k > 0$ everywhere.

Lichnerowicz's theorem was later extended by Hitchin in perhaps a surprising way. Hitchin ([15], §1.2) found additional obstructions to
positive scalar curvature for closed manifolds \( M \) with \( \kappa_g = 0 \), this
time for \( n = 1 \) or \( 2 \) (not 8). Unlike the obstruction of Lichnerowicz,
these do not just depend on the isometry class of \( M \), since they are
non-zero for some exotic sphere of dimension 9 but zero for others.
However, one can formulate the Hitchin and Lichnerowicz theorems in a uni-

ified way as follows: suppose \( M \) is an oriented closed manifold with
\( \omega(N) = 0 \). Then \( M \) admits a spin structure (see [25]), i.e., a lifting of
the oriented frame bundle, which is a principal bundle for the structure
group \( \text{SO}(n) \), to a principal bundle for the double cover of \( \text{SO}(n) \).
Choose such a spin structure, which is unique if \( M \) is simply connected
(or if \( H^1(M,\mathbb{Z}) = 0 \)). By definition, \( M \) together with its spin struc-
ture \( s \) determines a class in the spin bordism group \( \Omega_0^\text{spin} \), and then a
class \( [M,s] \) in the real bordism group \( K_0(M) \). This bordism class is
either the image of the spin bordism class of \( (M,s) \) under the natural
transformation \( \Gamma: K_0^\text{spin} \to K_0^0 \) or \( K_0(M) \), or else it is to be thought of in
terms of the geometric generator/relation presentation of \( K_0 \)-bordism
discussed in [5], p. 169, where we take for the real vector bundle \( E \)
over \( M \) just the trivial one-dimensional bundle. The class \( [M,s] \) may be
viewed as a "generalized index" of the Dirac operator on \( M \), as defined
using the spin structure \( s \). In fact, \( K_0(M) \cong \mathbb{R} \) for \( n = 9 \) (mod 4),
\( \cong \mathbb{Z} \) for \( n = 1 \) or \( 2 \) (mod 8), and \( = 0 \) for \( n = 3,5,6 \) or 7 (mod 8).
When \( n = 2 \) (mod 4), \( [M,s] \) (if \( n = 0 \) (mod 8)) or \( 2[M,s] \) (if \( n = 1 \)(mod 8))
may be identified with \( \hat{A}(M) \), which was computed in [1] to be
the index of the Dirac operator on \( M \) taking "positive" to "negative"
"half-spinors". When \( n = 1 \) or \( 2 \) (mod 8), \( [M,s] \) may again be viewed as
a \( 2 \) \( 2 \) index of the (real) Dirac operator. Thus the Hitchin and
Lichnerowicz theorems may be formulated as

**Theorem 1.1:** Let \( M \) be a closed, connected, oriented manifold with
\( \omega(N) = 0 \) and with a Riemannian metric for which \( \kappa > 0 \). Then for any
choice of a spin structure \( s \) on \( M \), \( [M,s] \neq 0 \) in \( K_0(M) \).

The proof of this theorem involves Lichnerowicz's observation that if \( D \)
is the Dirac operator on \( M \) defined by \( s \) and the Riemannian metric (an
elliptic first-order self-adjoint differential operator), \( D^2 = \bar{\nabla}^\ast \bar{\nabla} + \frac{n}{4} \),
where \( \bar{\nabla} \) is positive and self-adjoint. Thus if \( \kappa > 0 \), \( D^2 \) is strictly
positive, and all index invariants associated to \( D \) will vanish.

When \( M \) is simply connected, \( s \) is unique and we may write simply
\( [M] \in K_0(M) \). In this case, Gromov and Lawson conjecture in [13] that the
condition \( [M] = 0 \) is also sufficient for \( M \) to admit positive scalar
curvature. The status of this conjecture will be discussed in 9.2 below.

Now consider the case of non-simply connected manifolds. It seems that
the bigger the fundamental group of \( M \), the harder it is to achieve a
metric of positive scalar curvature on the manifold. In fact, Gromov and
Lawson have suggested the following, which meshes well with the corollary
of Gromov-Robert in dimension 2:

**Conjecture 1.2:** No closed spin manifold (of any dimension) admits
a metric of positive scalar curvature.

Our evidence for this is spotty and has accumulated piecemeal. This
conjecture was proved first by Schoen and Yau for the 3-torus [11], then
by the same authors for the \( n \)-torus and certain other manifolds with
\( n \leq 7 \) [9], then by Gromov and Lawson for a large class of spherical
closed manifolds, including compact manifolds of non-positive sectional
curvature (such as locally symmetric spaces of non-compact type) and
compact solvmanifolds (which of course include tori of all dimensions) [12]
and [14]. A feature of the Gromov-Lawson approach is that it gives
homotopy-type obstructions: a manifold homotopy-equivalent to a solvmanifold can't admit positive scalar curvature, whether or not it's diffeomorphic (or even isometric to) a solvmanifold.

It is at this point that one notices a relationship with the so-called "Novikov Conjecture" in differential topology. This conjecture, or rather class of conjectures, exists in various forms (see in particular [11] and [17]), all of which say roughly that the bigger the fundamental group of a closed manifold, the more the homotopy type determines the structure of the manifold. In the case of closed aspherical manifolds, it is possible that the fundamental group actually determines the manifold up to homeomorphism or at least stable homeomorphism (see, e.g., [12] and [13]).

More we say $N_1$ and $N_2$ are stable homeomorphic if $N_1 	imes R^k$ and $N_2 	imes R^k$ are homeomorphic for a suitably (usually small) value of $k$. Stable homeomorphism is sometimes easier to work with than homeomorphism. For instance, although for $n \geq 5$ not all contractible $n$-manifolds are homeomorphic to $R^n$, they are all stably homeomorphic to $R^n$.

To formulate things more precisely, we need one additional ingredient. Let $M$ be a closed, connected, oriented $n$-manifold with fundamental group $\pi_1$. Then the universal cover $\tilde{M}$ of $M$ is a principal $\pi_1$-bundle over $M$, hence is determined by a classifying map $f : M \to \tilde{M}$ which is an isomorphism on fundamental groups. Here $\tilde{M}$ is an Eilenberg-MacLane space with $\pi_i(\tilde{M}) \cong \pi_i$ and $\pi_j(\tilde{M}) = 0$ for $j > 0$, and $\tilde{M}$ and $f$ are well-defined up to homotopy. (Of course, the homotopy class of $f$ depends on the choice of a specific isomorphism $\pi_1(M) \cong \pi_1$.) The usual formulation of the Novikov Conjecture is the statement that the "higher signatures" $L_n^\pi(M) = \langle \lambda_n^\pi(M) \rangle$, where $\lambda_n^\pi(M)$ is a certain characteristic class of the spin structure $
u$.

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vector bundles not be trivial by Chern-Weil theory (which relates these characteristic classes to the curvature). However, invariance of the proof shows that the argument still works when \( E \) is a flat \( A \)-vector bundle in the case of \([91]\), where \( A \) is a \( C^* \)-algebra with unit, provided that we use an appropriate index theory for \( D_0 \). The index of \( D_0 \) (when \( N \) is even-dimensional) will take its values in \( K_0(\mathcal{A}) \) rather than in \( K_0(A) \). In applications, we always take \( A = C^*(\sigma) \) or \( \sigma^*(\tau) \) and \( E \) is the "universal" flat \( A \)-bundle over \( \mathcal{E} \). This \( A \)-bundle is of course pulled back from the flat \( A \)-bundle \( \mathcal{E} = \mathcal{E} \times \mathcal{A} \) over \( \mathcal{A} \). (Where \( \mathcal{A} \) is the universal cover of \( \mathcal{A} \), a contractible space on which \( \mathcal{A} \) acts freely.) The advantage of this is precisely that Chern-Weil theory breaks down for \( A \)-vector bundles, so that the flat bundle \( E \) may have non-zero rational characteristic classes. (This is due, roughly speaking, to the fact that the structure group for an \( A \)-vector bundle is infinite-dimensional and non-compact.)

In fact, Kasparov has used the \( A \)-bundle \( \mathcal{E} \) to define a homomorphism

\[ \beta : K_0(\mathcal{E}) \to K_0(A). \]

Here \( K_0(\mathcal{E}) \) denotes complex \( K \)-homology as extended to infinite CW-complexes.

If \( \mathcal{E} \) can be chosen compact, this map is easy to define. The bundle \( \mathcal{E} \) has a class \([\zeta]\) in

\[ K^0(\mathcal{E}) \cong K_0(C(\mathcal{E}) \otimes A) = K_0(C(\mathcal{E}) \otimes A), \]

and \( \beta \) is the Kasparov product \([\zeta] \otimes \cdot \) as defined in [12] (see also [7], [9]). When \( \mathcal{E} \) is an infinite complex, one defines \( \beta \) first this way on finite skeleta of \( \mathcal{E} \), then passes to the limit.

With \( A = C^*(\sigma) \) or \( \sigma^*(\tau) \) and \( \beta = \sigma^*(\tau) \), the index of \( D_0 \) (for \( N \) even-dimensional) in \( K_0(\mathcal{E}) \) is just \( \beta(\mathcal{E}, \pi, \mathcal{F}) \), where \( \mathcal{E}, \pi, \mathcal{F} \) denotes the image of the bordism class of \( (M, e) \) in \( K_0(\mathcal{E}) \) in the complex (hence the "c") \( K \)-homology group \( K_0(\mathcal{E}) \), as defined in [5]. The case of odd-dimensional \( N \) is reduced to the even-dimensional case by taking a product with \( \mathbb{R}^1 \), the net effect is to have \( \mathcal{E}, \pi, \mathcal{F} \) in \( K_0(\mathcal{E}) \) and to have an index of \( D_{\mathbb{R}} \) defined in \( K_0(\mathcal{A}) \). The conclusion of this analysis is the following result (a combination of Theorem 2.11 and Theorem 3.1) of [91]:

**Theorem 1.1:** Let \( (M, g, \sigma) \) be a closed, connected spin manifold admitting a metric of positive scalar curvature, and let \( f : M \to \mathbb{R} \) be the classifying map for its universal covering. If the Kasparov map \( \beta : K_0(\mathcal{E}) \to K_0(C^*(\sigma)) \) is injective, then \( [M, g, \sigma] \neq 0 \) in \( K_0(\mathcal{E}) \). If \( \beta \) is injective modulo torsion, then the higher \( \tilde{A} \)-genera \( \tilde{A}^*(M) \) vanish for all \( n \).

Kasparov has shown [26] that injectivity of \( \beta \) modulo torsion implies the Novikov Conjecture, and holds if \( \sigma \) can be embedded discretely in a connected Lie group. In fact, Milnor had claimed this earlier if \( \beta \) is the fundamental group of a complete Riemannian manifold of non-positive sectional curvatures. The proof seems to contain a gap when this manifold was non-compact (see [16]), but evidently this gap can be filled by using Kasparov's machinery (see sketches of the argument in [9] and [20]), so that one also has:

**Theorem 1.1:** If \( \beta \) is the fundamental group of a complete Riemannian manifold of non-positive sectional curvatures, then \( \beta : K_0(\mathcal{E}) \to K_0(C^*(\sigma)) \) is injective.

Some additional cases may be handled by [30], Theorem 2.6:

**Theorem 1.3:** If \( \pi \) is a countable soluble group having a composition series in which the composition factors are torsion-free abelian (but not necessarily finitely generated), then \( \beta : K_0(\mathcal{E}) \to K_0(C^*(\sigma)) \) is an isomorphism.
It is clear from the example of finite groups that $\beta$ cannot be injective for arbitrary groups (with torsion). However, if a group $\gamma$ contains a subgroup $\gamma_1$ of finite index for which $\beta$ is injective, then $\beta$ for $\gamma$ is at least an injection modulo torsion ([30], Proposition 3.7), which suffices for vanishing of the higher $\tilde{A}$-genus. It is conceivable (although perhaps overly optimistic) that $\beta$ is injective whenever $\gamma$ is finitely presented and torsion-free, and always injective modulo torsion when $\gamma$ is finitely presented. Whether or not this is true, it is rather striking that the classes of groups for which good results about the Novikov Conjecture or the positive scalar curvature problem have been obtained largely coincide with the classes of groups for which one can prove injectivity of $\beta$ modulo torsion (which we called in [30] the "Strong Novikov Conjecture").

2. Towards a conjecture of Gromov and Lawson

Michael Gromov and Simon Lafon have proposed a neat way of organizing all the results on topological obstructions to positive scalar curvature on closed manifolds, at least if we restrict attention to spin manifolds. Here is their conjecture:

Conjecture 1.1. Let $M$ be a closed, connected manifold with $w_2(M) = 0$, and let $\pi : \tilde{M} \to M$ be the classifying map for the universal covering of $M$. Then (if $\gamma$ is suitable and at least for $n$ sufficiently large) $\tilde{M}$ admits a metric of positive scalar curvature if and only if the following topological condition holds: for any spin structure $s$ on $M$, one has $[\tilde{M}, s, \tilde{f}] = 0$ in $\pi_1(\tilde{f})$, where $[\tilde{M}, s, \tilde{f}]$ is the image in real $K$-homology of the spin borromean class of $(M, s, \tilde{f})$.}

When $N$ is simply connected, $p_0$ is a single point and $\epsilon$ is unique, the conjecture thus reduces in this case to the conjecture of [31] that a simply connected spin manifold admits positive scalar curvature exactly when the "Ritchin obstruction" $\theta_0 \in K_0(\mathbb{Z})$ vanishes. In this case, substantial results were obtained in [31], Corollary 3.1, under the restriction $n \geq 3$. Of course, the conjecture is also clearly true for simply connected closed 1-manifolds (because there aren't any) or 3-manifolds.

It would be true for simply connected 3-manifolds if one knew the Novikov Conjecture. As for dimension 4, any simply connected spin manifold with vanishing Gromov's obstruction must have signature 0 and so by the Freedman classification theorem ([34], Theorem 1.5) be homeomorphic to $S^4$ or to a connected sum of $\#_2 S^2 	imes S^2$. If the homeomorphism were a diffeomorphism, the manifold would admit positive scalar curvature by [33], Theorem A, so the conjecture would again be correct. However, to the best of my knowledge, it is not yet known if the differentiable structure on $S^4$ (or a connected sum of $\#_2 S^2 	imes S^2$) is unique.

The non-simply connected case is of course much harder. As in the simply connected case, the available methods of proof for attacking the conjecture again impose mild restrictions on $\gamma$, which might turn out to be unnecessary. However, we will see that the conjecture cannot possibly hold for all groups $\gamma$, so that any "suitable" $\gamma$ should at least be torsion-free. An extra annoyance of the non-simply connected case is that $\gamma$ usually admits more than one spin structure, and $[\tilde{M}, s, \tilde{f}]$ may depend on $s$. For instance, if $M = \mathbb{R}^5$, $K_0(\mathbb{Z}) \cong \mathbb{Z}$, and $K_0(\mathbb{Z}) \cong \mathbb{Z}$, $\#_2 \cong \mathbb{Z}$.

In this case, $M$ has two spin structures, each with the same image in $\pi_1(\mathbb{Z})$, but with distinct images in $K_0(\mathbb{Z})$. However, if $[\tilde{M}, s, \tilde{f}]$ vanishes for one choice of the spin structure $s$, then it
\[
\begin{align*}
(\chi')^2 \rightarrow \chi'^2 \rightarrow (\chi')^2 \\
\end{align*}
\]
observe that $X_\lambda$ is obtained from $X_\mu$ by attaching handles of index $\leq n - 2$, i.e., by performing surgeries on embedded spheres of codimension $\geq 3$. Then by [13], Theorem A, or by [28], Corollary 4 to Theorem A, $X_\lambda$ admits a metric of positive scalar curvature.

**Proposition 2.5:** Let $(S^2, s)$ be a closed, connected spin manifold with $n \geq 5$, and let $f : X \to BF$ be the classifying map for its universal covering, where $n = \nu_1 (X)$. If $(S^2, s) \not\simeq BF$ represents the trivial element of $\pi_0^{spin}(BF)$, and if there exists a closed, connected spin manifold $\tilde{X}$ with $k \leq n - 2$ (this is automatic if $n \geq 6$), then $X$ admits a metric of positive scalar curvature.

**Proof:** As is well known, for any finitely presented group $\pi$, there is a closed stably parallelizable manifold $V^4$ with $\nu_1 (V^4) \simeq \pi$ ([28], p. 305), so the existence of the requisite $V$ is automatic if $n \geq 6$.

Anyway, given any $\tilde{V}^4$, there is always a metric of positive scalar curvature on $\tilde{V}^4 \times S^{n-k}$, provided $n - k \geq 2$, namely, the Riemannian product of any metric on $\tilde{V}$ with the constant-curvature metric on a Riemannian sphere of very small radius. Furthermore, if $V$ is a spin manifold, then $\tilde{V}^4 \times S^{n-k}$ is clearly the boundary of the spin manifold $V^4 \times S^{n-k}$ with the same fundamental group. Hence we may apply Theorem 2.2 with $X = X_\lambda$, $X_\mu = V^4 \times S^{n-k}$.

**Remark 2.6:** Although $X$ may admit more than one spin structure $s$, the condition that $(X, s) \not\simeq BF$ represent the trivial element of $\pi_0^{spin}(BF)$ is independent of $s$. Indeed, suppose $(X, s)$ is a spin manifold with $\nu_1 (X, s) = (X, s)$, and suppose $g: M \to BF$ extends $f$. If $s'$ is any other spin structure on $M$ associated to the same orientation of $X$ as determined by $s$, then $s'$ is obtained by modifying $s$ by an element $a + [a]^{\nu_2} (X, s_\mu) = \nu_2 (X, s_\mu)$ ([36]). But then if $b = a + [a]^{\nu_2} (W, s_\mu) = \nu_2 (W, s_\mu)$, the action of $b$ on $W$ defines a spin structure $\tilde{s}'$ on $W$ restricting to $s'$ on $X$. Hence $(X, s') \not\simeq BF$ also represents the trivial element of $\pi_0^{spin}(BF)$. Similarly, to reverse the orientation on $X$, we may reverse the orientation on $X$.

**Theorem 2.7:** Let $\pi$ be any finitely presented group such that there exists a closed, connected spin $n$-manifold $V^k$ with $\nu_1 (T) \simeq \pi$, and let $n \geq \max (5, k + 2)$. (Recall that $n \geq 6$ will always do.) Then there exists a subgroup $\pi_0 (N)$ of $\pi_0^{spin}(BF)$ with the following property: for any closed, connected $n$-manifold $M^0$ with $\nu_1 (M) = \nu_1 (N) = 0$, and $\nu_1 (M) \not\simeq \pi$, then $M$ admits a metric of positive scalar curvature if and only if the spin bordism class of $(M, s) \in BF$ lies in $N_0 (\pi)$.

**Proof:** Let $\pi_0 (N)$ be the set of classes in $\pi_0^{spin}(BF)$ of triples $(M, s, f)$, where $(M, s)$ is a closed, connected spin $n$-manifold admitting a metric of positive scalar curvature, and where $f : M \to BF$ is a classifying map for the universal covering. If we can show that $\pi_0 (N)$ is a group, then the conclusion will follow from Theorem 2.2.

To begin with, $\pi_0 (N)$ contains the 0-element of $\pi_0^{spin}(BF)$ by Proposition 2.1. Furthermore, it is clear that $\pi_0 (N)$ is closed under inversion (reversal of spin structure), so we must show that $\pi_0 (N)$ is closed under addition. This is non-trivial because of our restriction to connected manifolds $M$; the addition operation in $\pi_0^{spin}$ comes from the disjoint sum of manifolds, not the connected sum.

Thus suppose $(M_1, s_1) \not\simeq BF$ represent classes in $\pi_0 (N)$, where $M_1$ is a connected spin $n$-manifold with fundamental group $\pi$ and positive...
section curvature \((i = 1, 2)\). Choose generators \(a_1, \ldots, a_n\) for \(\pi\), and represent these by disjoint embedded oriented circles \(c_1, \ldots, c_k\) in \(M_i\).

Since \(M_i\) is oriented, \(c_i\) has trivial normal bundle, and hence a tubular neighborhood \(N_i\) diffeomorphic to \(\mathbb{R}^2 \times [0, 1]\). We form a new manifold \(M_{ij}\) from \((M_i \setminus \{c_i\}) \cup (M_j \setminus P_i^2)\) by gluing \(M_i\) to \(M_j\) so as to match \(\Sigma_{ij}\) with \(\Sigma_{ij}\). This may require, for some values of \(i\), changing a preliminary choice for the surgery by a generator of \(\pi_j(\mathbb{Z} - 1) \cong \mathbb{Z}\), see [29], §5. Then \(M_{ij}\) will be a spin manifold, and by repeated application of Van Kampen’s theorem, \(\pi_i(M_i)\) will be isomorphic to the unknotted free product of two copies of \(\pi\) in which corresponding copies of \(\pi\) are identified, which is just \(\pi\) again. The class of \(M_{ij}\) (together with its spin structure and classifying map) in \(\pi_i(\mathbb{H})\) will be the sum of the classes of the \((M_i, c_i, \nu_i)\), and since \(M\) is obtained from \(M_1\) and \(M_2\) by surgery along embedded spheres of codimension \(n - 1 \geq 4\), \(M\) admits positive scalar curvature by [13], Theorem 1, or [32], Theorem 4. This shows that \(\pi_2(M)\) is a group.

Remark 2.6: We needed to show that \(\pi_2(M)\) is not empty. Actually, under the assumptions of 2.3, there are "lumps" of spin manifolds with fundamental group \(\pi\). In the sense that any element of \(\pi_i(\mathbb{H})\) can be realized by a triple \((M, c, \nu)\) for which \(\nu: \pi(M) \to \pi_1(M) = \pi\) is an isomorphism. Then whether or not \(M\) admits positive scalar curvature is determined by the image of the given bordism class in \(\pi_i(\mathbb{H})/\pi(\mathbb{H})\). The proof of this fact is a similar surgery argument. Given \((M, c, \nu)\) representing a spin bordism class, glue a copy of \(\mathbb{H}^2 \times [0, 1]\) (notation of 2.3) onto \(M\) as in the above proof to obtain \((M', c')\) in \(\mathbb{H}\). Then \(\nu: \pi(M') \to \pi(M) = \pi\) surjective. Then kill the kernel of \(\pi_2\) by

additional surgeries to obtain \((M, a)\).

Now we are ready for some results regarding Conjecture 2.1 for specific groups. We begin with the following observation:

Proposition 2.7: Under the hypotheses and with the notation of Theorem 2.5, \(\pi_2(M)\) is contained in the kernel of the composition

\[\rho_n(\mathbb{H}) \to \rho^n(\mathbb{H}) \to \rho^n(\mathbb{H}),\]

which is the "complexification map" from real to complex K-theory.

Corollary 2.8: If \(\pi: \pi_n(M) \to \pi_2(M)\) is injective (i.e., \(\pi^n\) holds for \(n\), in the notation of [32]) and if \(\rho^n(\mathbb{H})\) has the \(\pi\)-primary torsion, then there is a direct proof of Conjecture 2.1 holds for spin manifolds with \(\pi_2(M) \cong \pi\). From \(\pi_2(M) \leq \ker \rho^n(\mathbb{H})\), \(\pi_2(M) \cong \ker \rho^n(\mathbb{H})\) for all \(n \geq 4\) (and also for \(n = 3\) if there exists a spin 3-manifold with positive scalar curvature and fundamental group \(\pi\)).

Proof of Proposition 2.7: The first statement follows from Hatcher’s result, Theorem 1.1 above. The second statement follows from Theorem 1.3 above.

Proof of Corollary 2.8: This follows from 2.7 and the fact that \(\pi: \pi_n(M) \to \pi_2(M)\) is an injection modulo \(\pi\)-torsion.

Theorem 2.9: Conjecture 2.1 holds for spin manifolds \(\mathbb{H}^n\) with \(\pi_2(M) = 0\), provided \(n \leq 4\).

Proof: Since \(\pi_2(M) = 0\), \(\pi_2(M) = 0\), \(\pi_2(M) = 0\), \(\pi_2(M) = 0\), \(\pi_2(M) = 0\).
for \( 5 \leq n \leq 9 \). For the other direction, we must show that if \((M^3, g)\) is a closed connected spin manifold with \(\pi_2(M) \neq 0\) and with \([\pi_2, c_1] = 0\) in \(\Omega^3_0(\mathbb{R} P^3)(n = 5, 6, 7, 8, 9)\), then we can represent the bordism class of \(M\) in \(\Omega^3_0(\mathbb{R} P^3)\) admitting positive scalar curvature. Since \(\sigma^0_0(x^3) = 0\) and \(\sigma^0_1(x^3) = 0\), the cases \(n = 6\) and \(7\) are trivial. Since \(\sigma^0_0(x^3) = \mathbb{Z}\) with generator \(x^3 \times x^3\), which has image of infinite order in \(\mathbb{K}(x^3)\), the case \(n = 5\) is also trivial. Next, \(\sigma^0_0(x^3) = \mathbb{Z}\), and a generator for the kernel of the map to \(\mathbb{K}(x)\) is \(x^3 \times x^3\). The manifold \(\mathbb{R} P^3 \times (x^3 \times x^3)\) is then a manifold in the same spin bordism class, admitting positive scalar curvature and having infinite cyclic fundamental group. Finally, \(\sigma^0_0(x^3) = \mathbb{Z}\), and a generator for the kernel of the map \(\sigma^0_0(x^3)\) to \(\mathbb{K}(x)\) are \(x^3 \times x^3\). These are identical as manifolds (they differ only in spin structure) and admit positive scalar curvature, as required.

**Theorem 2.10:** Conjecture 2.1 holds for spin manifolds \(N^n\) with \(\pi_2(M) \neq 0\) (the free group on \(k\) generators), provided \(5 \leq n \leq 9\).

**Proof:** The argument for this is almost exactly the same as the proof of 2.9, once we replace \(x^3\) with \(x^3 \times x^3\) and \(x^3 \times x^3\) with a connected sum of copies of \(x^3 \times x^3\). The requisite \(c^1\)-algebraic result follows from [30], Proposition 2.10 or else from Kasparov's Theorem, 1.1 above.

**Theorem 2.11:** Conjecture 2.1 holds for spin manifolds \(N^n\) with \(\pi_2(M) \neq 0\), \(g\) an oriented surface of genus \(g \geq 1\), provided \(5 \leq n \leq 9\).

**Proof:** This is again a situation where Kasparov's Theorem applies, since \(\pi_2\) has nonpositive curvature. Since \(\Omega^3_0(\mathbb{R} P^3)\) is concentrated in degrees 1 and 2, the Atiyah-Hirzebruch spectral sequences (see [11], §7) for calculation of \(\Omega^3_0(\mathbb{R} P^3), \Omega^3_0(M^4)\), and \(\Omega^3_0(\mathbb{R} P^3)\) collapse, and one can see that \(\pi_2(M)\) is torsion-free for \(5 \leq n \leq 9\), so the argument of 2.8 applies in one direction. As for the other direction, the kernel of \(\pi_2^0(\mathbb{R} P^3) \to \pi_1(\mathbb{R} P^3)\) for \(5 \leq n \leq 9\) comes from the kernel of \(\pi_2^0(\mathbb{R} P^3) \to \pi_1(\mathbb{R} P^3)\) except when \(n = 9\), so in all other cases we can argue as before. The kernel of \(\pi_2^0(\mathbb{R} P^3) \to \pi_1(\mathbb{R} P^3)\) is free abelian of rank \(2\), and is generated by \(\tilde{f}_1 = \mathbb{R} P^3 \times x^3 \to \mathbb{R} P^{n-3}\) for \(1 \leq 1 \leq 2\), where \(\tilde{f}_1\) is projection of \(\mathbb{R} P^3 \times x^3\) to \(\mathbb{R} P^{n-3}\), followed by the \(1\)-generator of \(\pi_2(\mathbb{R} P^3)\). Each such generator obviously corresponds to a manifold of positive scalar curvature obtained by taking the "connected sum along a circle" of \(\mathbb{R} P^3 \times x^3\) and \(x^3 \times x^3\) using \(\tilde{f}_1\). This completes the proof.

Similar calculations can be done for many other fundamental groups for which SMC 3 applies. However, the situation is somewhat different when \(\pi\) has torsion. In fact we have the following result.

**Theorem 2.12:** In general, Conjecture 2.1 fails if \(\pi\) is a cyclic group. For instance, every 5-dimensional spheres manifold \(M^5\) with \(\pi_2(M) = 3\) admits a metric of positive scalar curvature, even though \(\Omega^3_0(\mathbb{R} P^3) \subseteq \Omega^3_0(\mathbb{R} P^3)\) has order 9.

For \(n \geq 5\) and \(\pi\) a cyclic group \(\mathbb{Z}_q\) of order \(q\), so that \(\pi_2(M) = \mathbb{Z}_q\), we have the estimate

\[
|\Omega^3_0(\mathbb{R} P^3) \cap \Omega^3_0(\mathbb{R} P^3)| \leq q \cdot \Omega^3_0(\mathbb{R} P^3),
\]

which can be improved to \(q\) if Conjecture 2.1 holds for simply connected manifolds. (For comparison, \(|\Omega^3_0(\mathbb{R} P^3) \cap \Omega^3_0(\mathbb{R} P^3)| = q^2\), \(q\) for \(n = 5\) or...
7, and \( |\delta^3_{\text{spin}}(\mathbb{Z}_q)| = q^3 \). As indicated earlier, Conjecture 3.1 does hold for simply connected manifolds of dimension \( \leq 7 \), except possibly in dimensions 3 and 4, which is enough to guarantee that

\[
|\delta^0_{\text{spin}}(\mathbb{Z}_q)| = |\delta^0_{\text{spin}}(\mathbb{Z}_q)| \leq q,
\]

\[
|\delta^0_{\text{spin}}(\mathbb{Z}_q)| \leq q.
\]

Proof: The natural transformations of homology theories \( \delta^0_{\text{spin}} \rightarrow \mathcal{Q}_q \) and \( \delta^0_{\text{spin}} \rightarrow \mathcal{Q}_q \) ("forget the spin structure") induce maps of Atiyah-Hirzebruch spectral sequences (11, 77)

\[
\tilde{\mathbb{E}}_2(\mathbb{Z}_q) \rightarrow \tilde{\mathbb{E}}_2(\mathbb{Z}_q),
\]

\[
\tilde{\mathbb{E}}_2(\mathbb{Z}_q) \rightarrow \tilde{\mathbb{E}}_2(\mathbb{Z}_q),
\]

\[
\tilde{\mathbb{E}}_2(\mathbb{Z}_q) \rightarrow \tilde{\mathbb{E}}_2(\mathbb{Z}_q).
\]

When \( q \) is odd, since \( \mathcal{Q}_q \), \( \mathcal{Q} \), and \( \mathcal{Q}_q \) have only 0-torsion \( (33) \) and \( \mathbb{Q}_q(\mathbb{Z}_q) = 0 \), the only non-zero \( H^2 \)-terms of these spectral sequences are the term \( H^0(\mathbb{Z}_q) = 0 \) (and also \( t \geq 0 \) for \( \mathbb{Q}_q \) and \( \mathbb{Q} \)). Hence all the spectral sequences collapse and \( \tilde{\mathbb{E}}_2 = \tilde{\mathbb{E}}_2 \). Thus we can easily write down generators for \( \delta^0_{\text{spin}}(\mathbb{Z}_q) \), namely the manifolds \( S^{2n+1} \times \mathbb{Z}_q \), \( (n \geq 0, t \geq 0) \), where \( M \) runs over (torus-free) generators of \( \delta^0_{\text{spin}}(\mathbb{Z}_q) \), which we may take to be simply connected, and where \( S^{2n+1} \) is a circle for \( n = 0 \) and a lens space \( S^{2n+1}/\mathbb{Z}_q \), for \( n \geq 1 \). (The map \( 1 \times M \rightarrow \mathbb{Z}_q \) is the obvious one factoring through \( 1 \) and generating \( H_2(\mathbb{Z}_q) \)). Note that these manifolds have obvious metrics of positive scalar curvature when \( n \geq 1 \) (a Riemannian product of a metric of large constant curvature on \( 1 \), with any metric on \( M \)) or when \( n = 0 \) and \( M \) has positive scalar curvature (conjecturally, whenever \( \delta^0 = 0 \) in \( H^0(M) = \mathbb{Z}_q \)). Also note that since \( \delta^0_{\text{spin}}(\mathbb{Z}_q) \rightarrow \mathcal{Q}_q(\mathbb{Z}_q) \) is an isogeny, the maps

\[
\delta^0_{\text{spin}}(\mathbb{Z}_q) \rightarrow \mathcal{Q}_q(\mathbb{Z}_q)
\]

are isomorphisms, and by repeated applications of the 5-lemma, the map \( \delta^0_{\text{spin}}(\mathbb{Z}_q) \rightarrow \mathcal{Q}_q(\mathbb{Z}_q) \) is an isomorphism. This is useful since the detailed structure of \( \mathcal{Q}_q(\mathbb{Z}_q) \) is computed in [11, Ch. VII, VIII, and IX.

Now consider some special cases. The map \( \delta^0_{\text{spin}}(\mathbb{Z}_q) \rightarrow \mathcal{Q}_q(\mathbb{Z}_q) \) is an isomorphism and \( \delta^0_{\text{spin}} \rightarrow \mathcal{Q}_q \) is split surjective, so additional applications of the 5-lemma show that

\[
\delta^0_{\text{spin}}(\mathbb{Z}_q) \rightarrow \mathcal{Q}_q(\mathbb{Z}_q),
\]

\[
\delta^0_{\text{spin}}(\mathbb{Z}_q) \rightarrow \mathcal{Q}_q(\mathbb{Z}_q),
\]

\[
\delta^0_{\text{spin}}(\mathbb{Z}_q) \rightarrow \mathcal{Q}_q(\mathbb{Z}_q),
\]

have images of order \( q^2, q^2 \), and \( q^2 \), respectively. In the first two cases, we can even remove the torsion since \( \delta^0_{\text{spin}} \rightarrow \mathcal{Q}_q \). However, the only generators for \( \delta^0_{\text{spin}}(\mathbb{Z}_q) \) and \( \delta^0_{\text{spin}}(\mathbb{Z}_q) \) which don't obviously have positive scalar curvature are

\[
x^1 \times x^3 \rightarrow \mathbb{Z}_q
\]

and

\[
x^3 \times x^3 \rightarrow \mathbb{Z}_q.
\]

This gives the indicated estimates. If \( q = 3 \), we can do even better since by [11, Turaev 36.7], \( \delta^0_{\text{spin}}(\mathbb{Z}_q) \) (and thus \( \delta^0_{\text{spin}}(\mathbb{Z}_q) \)) is cyclic of order 9, with generator \( x^3 \rightarrow \mathbb{Z}_q \) that admits positive scalar curvature. Hence, by Theorem 2.5, every spin 3-manifold \( M \) with \( \delta^0(\mathbb{Z}_q) \geq \mathbb{Z}_q \) admits a metric of positive scalar curvature. This completes the proof for these special cases. In general, we have seen that

\[
\delta^0_{\text{spin}}(\mathbb{Z}_q) \rightarrow \mathcal{Q}_q(\mathbb{Z}_q) \rightarrow \mathbb{Z}_q,
\]

\[
\mathbb{Z}_q \rightarrow \mathbb{Z}_q
\]
is even-posed only for \( n \equiv 1 \pmod{4} \), and is generated by manifolds \( S^1 \times M^{n-1} \), where \( M^{n-1} \) runs over generators for \( (\pi_{n-1}/\pi_{n-1}(1)) \otimes \mathbb{Z}_2 \).

This gives the estimates in the remaining cases.

To conclude our discussion, we briefly say a few words about positive scalar curvature on non-simply connected manifolds. Following the argument of [13], Theorem C, one obtains the following analogue of Theorem 2.2 in the non-spin case:

**Theorem 2.13:** Let \( X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n \) be closed, oriented, connected \( n \)-manifolds with \( \pi_2(X_1) \cong \pi_2(X_2) = 0 \), with \( n \geq 5 \), and with \( \pi_2(X_i) \neq 0 \). Let \( \tau_1 : X_1 \rightarrow \mathbb{Z}_2 \) be the classifying map for the universal covering of \( X_1 \).

Assume \( X_2 \) has a metric of positive scalar curvature and that \( \tau_1 : X_1 \rightarrow X_2 \) lie in the same bordism class in \( \Omega_n \). Then \( X_1 \) admits a metric of positive scalar curvature.

**Proof:** One repeats the argument in the proof of Theorem 2.3, but substituting the idea of the proof of [13], Theorem C: since

\[
\pi_2(X_1) \xrightarrow{\tau_1} \pi_2(X_2) \xrightarrow{v_2} \pi_2(X_1)
\]

commutes and \( \pi_2(X_1) \neq 0 \), one may reduce to the case where \( \pi_2(X_1) \) is generated by elements of \( \pi_2(X_2) \) in the kernel of \( v_2 \), hence to the case where \( \pi_2(X_1) = 0 \). The argument is concluded as before.

One can prove a similar result for non-orientable manifolds, but this case is more complicated. It is not enough to look at non-orientable bordism; instead one must look at \( \mathbb{Z}_2 \)-equivariant bordism of the orientable double cover, and one must distinguish the cases where this does and does not have a spin structure. We hope to discuss this case in a future publication.

Meanwhile, we conclude with two simple applications of Theorem 2.13. Of course, certain other cases could be treated similarly.

**Theorem 2.15:** Let \( N \) be any connected closed manifold with dimension \( n \geq 5 \), with \( \pi_2(N) \neq 0 \), and with fundamental group \( \pi_1(N) \cong \mathbb{Z}_2 \) cyclic of odd order \( q \). Then \( N \) admits a metric of positive scalar curvature.

**Proof:** Since \( \mathcal{H}(N, \pi_2(N)) = 0 \), \( N \) is automatically orientable. By Theorem 2.13, it is enough to check that each generator of \( \mathcal{H}_n(N, \pi_2(N)) \) is represented by a manifold with positive scalar curvature with fundamental group \( \mathbb{Z}_2 \). As in Theorem 2.15, the bordism spectral sequence shows it is enough to look at the bordism classes of \( S^1 \times N^{k-1} \rightarrow N \) and of \( S^2 \times N^{k-1} \rightarrow N \), where \( L \) is a lens space and \( N \) is one of the torsion-free generators of \( \mathcal{H}_n(L) \). By [13], Corollary C, we may take \( N \) to have positive scalar curvature (this doesn't cover the cases \( t = 0 \) or \( t = 0 \), but of course when \( t = 0 \), we're left just with \( L \), and when \( t = 1 \), we may take \( N \) to be \( S^2 \)). This finishes the proof except for the case of \( S^2 \times S^1 \), where it is necessary first to make a surgery to reduce the fundamental group to \( \mathbb{Z}_2 \). By [13], Theorem A, this doesn't destroy the positive-scalar-curvature property.

**Theorem 2.16:** Let \( N \) be any connected closed orientable manifold with dimension \( n \geq 5 \), with \( \pi_2(N) \neq 0 \), and with infinite cyclic fundamental group. Then \( N \) admits a metric of positive scalar curvature.

**Proof:** \( \mathcal{H}_n(N) \) is generated by the bordism classes of \( N \) and of \( S^2 \times N^{n-2} \), where \( N \) runs over generators of \( \mathcal{H}_n \). Since \( N \) has sufficiently high dimension, we may take \( N \) to be simply connected, either \( \mathbb{R}^n \) or one of the manifolds covered by [13], Corollary C. In the case of \( S^2 \times N^{n-2} \), we already have a manifold with infinite cyclic funda-
mental group and with positive scalar curvature. In the other case, adjust the fundamental group by taking a connected sum with $S^1 \times S^{n-1}$. As before, we are done by Theorem 2.13.

3. Behavior under finite coverings and the transfer map in spin bordism

In this final section, we apply the results of §2 to study the following problem: given closed manifolds $M_1$ and $M_2$ and a finite covering $p : M_1 \to M_2$, what does existence or non-existence of a metric of positive scalar curvature on one manifold say about the other? One fact is obvious: any Riemannian metric on $M_2$ can be lifted to $M_1$, so if $M_2$ has a metric of positive scalar curvature, then so does $M_1$. However, the situation going the other way is not at all clear. The vague guess that a metric of positive scalar curvature on $M_1$ can be "averaged" in some way and then pushed down to $M_2$ is essentially correct in many cases, but not in all.

In fact, we shall produce an example in which $M_1$ has a metric of positive scalar curvature and $M_2$ does not.

To analyze the situation in greater detail, we shall restrict attention to the case where $M_2$ is a connected spin manifold, with spin structure $s_2$. Pulling back $s_2$ by $p$ then defines a spin structure $s_1$ on $M_1$. Of course, $M_2$ may not be connected, but the only interesting case is when it is. Let $\tilde{x}_1 : N_1 \to M_1$ be the classifying map for the universal covering of $M_1$; we may choose these so that the diagram

$$
\begin{array}{ccc}
M_1 & \longrightarrow & N_1 \\
p & \downarrow & \downarrow q \\
M_2 & \longrightarrow & N_2
\end{array}
$$

commutes and $p : C_2$ where $q : C_2 \to C_2$ is also a finite covering.

As we have seen in §2, the question of when $M_1$ admits a metric of positive scalar curvature seems to involve only the spin bordism and K-homology classes of $(\Omega s_1, s_1)$ and $[\Omega s_1, s_1]$. At least if $n \geq 5$. How the generalized bordism theories $\Omega s_2$, $K_0$, $(\Omega s_2)\text{Spin}$, etc., some equipped with transfer maps (see [2], Ch. 1 and (5), §4(6)). In our particular case, these may be simply defined geometrically so that

$$
q^*([\Omega s_2, s_2 \cup f_2]) = [\Omega s_1, s_1 \cup f_1]
$$

just as one defines $q^*$ in ordinary bordism by lifting a singular simplex $\Delta^n \to \Omega s_2$ to the pull-back $q^*\Delta^n \to \Omega s_1 \times \Delta^n \to \Omega s_1$. If Conjecture 2.1 were to hold in our situation, our question would reduce to the description of the kernel of the transfer map $q^* : \Omega s_2 / \Omega s_1 \to \Omega s_1 / \Omega s_1$.

To summarize, if this kernel were zero, then $M_2$ would have a metric of positive scalar curvature whenever $M_1$ did. If the kernel were not zero, it would classify examples where $M_1$ admits positive scalar curvature and $M_2$ does not. Even if Conjecture 2.1 didn't hold, we would (in situations where Theorem 2.5 applied to both $r_1$ and $r_2$) examine instead the kernel of the composite

$$
\gamma : \Omega s_1^\text{Spin} / \Omega s_1 \to \Omega s_1 / \Omega s_1
$$

the analogous obstruction group is then $\ker \gamma / \ker \gamma$. $(\gamma / \ker \gamma \subset \ker \gamma$ because we can lift positive-scalar-curvature metrics.)

Theorem 3.1: There exists an example of a regular 3-fold covering $p : M_1 \to M_2$ where $M_1$ and $M_2$ are closed 3-manifolds, $M_1$ admits a metric of positive scalar curvature, and $M_2$ does not.

Proof: Let $\gamma$ be the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts on $\mathbb{Z}$ by the matrix

$$
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
$$

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\[ A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \in \text{SL}(2, \mathbb{Z}). \]

Note that the eigenvalues of \( A \) are cube roots of 1, so that \( \lambda^3 = 1 \)
and \( \det(\lambda - 1) = 3 \). One can compute the homology of \( \mathbb{H}_2 \) from the
Motzkin-Stern spectral sequence with \( E_2 \)-term \( H_2(\mathbb{H}_2; H_6(\mathbb{H}, \mathbb{H}^2, \mathbb{Z})) \),
where \( H_6(\mathbb{H}, \mathbb{Z}) \) is just \( \mathbb{H}^2 \) and \( \mathbb{H}^2 \) acts by exterior powers of \( A \). Thus,
\[
H_2(\mathbb{H}_2; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{if } k = 0, 1, 2, 3, \\
\mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 4, \\
0 & \text{if } k > 3.
\end{cases}
\]

It is clear that \( \mathbb{H}_2 \) is a discrete cocompact subgroup of a Lie group of
the form \( \mathbb{H}^2 \times \mathbb{R} \), hence we may choose for \( \mathbb{H}_2 \) a closed submanifold
\( \mathbb{H}_3 \), which of course is parallelizable (and hence admits a spin structure).

Note that Theorem 3.5 (or 1.4) applies to \( \mathbb{H}_2 \). We may compute \( H_3(\mathbb{H}_2; \mathbb{Z}) \)
from the Atiyah-Hirzebruch spectral sequence with \( E_2 \)-terms
\( H_3(\mathbb{H}_2; H_2(\mathbb{H}, \mathbb{Z})) \), and in particular, one sees that \( H_3(\mathbb{H}_2; \mathbb{Z}) \)
must contain a summand \( \mathbb{Z} \), coming from \( H_3(\mathbb{H}_2; H_2(\mathbb{Z})) \). Also note that a generator
for this summand may be realized by the spin bordism class of a spin manifold \( K \)
covering \( \mathbb{H}_2 \) mapping to \( \mathbb{V}_1 \). By Remark 3.6, we may assume without loss of
generality that \( \mathbb{H}_2 \to \mathbb{V}_1 \) induces an isomorphism on fundamental groups.

Now let \( \mathbb{V}_1 = \mathbb{V}_2 \), \( \mathbb{W}_1 = \mathbb{W}_2 \). Since \( \lambda^3 = 1 \), \( \mathbb{W}_2 \) is a normal
subgroup of \( \mathbb{V}_2 \) isomorphic to \( \mathbb{V}_1 \), and we evidently have a triple covering
\( \varphi : \mathbb{V}_2 \to \mathbb{V}_1 \) induced by the homomorphism \( \varphi : \mathbb{H}_2 \to \mathbb{H}_2 \).
Let \( \mathbb{H}_2 \) be the corresponding triple cover of \( \mathbb{H}_2 \). Since \( \mathbb{H}_2 \to \mathbb{V}_1 \) defines an
odd torsion class in \( H_3(\mathbb{H}_2; \mathbb{Z}) \) and hence also in \( H_3(\mathbb{V}_2; \mathbb{Z}) \), \( \mathbb{H}_2 \)
cannot admit a metric of positive scalar curvature, by Theorem 3.7. On the other hand, we
claim \( \mathbb{H}_2 \) does have a metric of positive scalar curvature. In fact, \( \mathbb{H}_2 \to \mathbb{V}_2 \) also defines an odd torsion class in \( C_3^{\text{Spin}}(\mathbb{V}_2 ; \mathbb{C}) \), so the class
of \( \mathbb{H}_2 \to \mathbb{V}_2 \) in \( C_3^{\text{Spin}}(\mathbb{V}_2 ; \mathbb{C}) \), being the image of this class under \( \varphi \), is
also an odd torsion class. However, from the Atiyah-Hirzebruch spectral sequence with \( E_2 \)-term \( H_2(\mathbb{V}_2; C_3^{\text{Spin}}(\mathbb{V}_2 ; \mathbb{C})) \), \( C_3^{\text{Spin}}(\mathbb{V}_2 ; \mathbb{C}) \) can contain no odd torsion.
So the spin bordism class of \( \mathbb{H}_2 \to \mathbb{V}_2 \) is trivial and \( \mathbb{H}_2 \) admits a
metric of positive scalar curvature by Theorem 2.5.

Note that the above proof depends on our being able to find a (torsion-free) group \( \mathbb{H}_2 \)
for which \( H_3(\mathbb{H}_2; \mathbb{Z}) \) is injective, yet for which \( \mathbb{H}_2 \) has odd torsion in its integral homology. If \( \mathbb{H}_2 \) had started with \( \mathbb{V}_2 \) finite and \( \mathbb{V}_1 \) trivial, the situation would be quite different.

For instance, we may rewrite part of Theorem 2.12 as follows:

**Theorem 3.7**: Let \( \mathbb{H}_2 \) be a closed spin manifold with fundamental group
\( \mathbb{H}_3 \) cyclic of odd order \( n \), and let \( \mathbb{H}_2 \) be its universal covering. If \( n \leq 2 \) or \( n \neq 0, 1 \) (mod 4), or if \( n = 0, \) or if \( n = 1 \) and \( q = 3 \), then
\( \mathbb{H}_2 \) admits a metric of positive scalar curvature if and only if \( \mathbb{H}_1 \) does.

**Proof**: By Theorem 2.12, under the given hypotheses, \( \pi_1(\mathbb{H}_2) \to C_3^{\text{Spin}}(\mathbb{H}_2 ; \mathbb{C}) \)
\( \cong \mathbb{H}_3(\mathbb{H}_2; \mathbb{Z}) \). Hence the only obstruction to positive scalar curvature
on \( \mathbb{H}_2 \) comes from the spin bordism class of \( \mathbb{H}_2 \) in \( C_3^{\text{Spin}}(\mathbb{V}_2 ; \mathbb{C}) \).
Now as elements of \( \mathbb{H}_3(\mathbb{V}_2; \mathbb{Z}) \), \( \mathbb{H}_2 \) is a spin bordant to a
simply connected manifold, which under the transfer map will go to \( \eta \) disjunct copies of itself. So if \( C_3^{\text{Spin}}(\mathbb{V}_2 ; \mathbb{C}) \) has no \( q \)-torsion, which is
certainly the case if \( n = 0 \) or \( n \neq 0, 1 \) (mod 4), \( \mathbb{H}_2 \) is \( C_3^{\text{Spin}}(\mathbb{V}_2; \mathbb{C}) \) if and only
if \( \mathbb{H}_2 \to \mathbb{H}_3 \) has no \( q \)-torsion. The condition \( n \neq 0, 1 \) (mod 4) could be removed if we
knew Conjecture 3.1 for simply connected manifolds.
References


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Modular cohomology class from the viewpoint of characteristic class

1. Introduction

In the geometrical study of foliations topologists are familiar with secondary characteristic classes such as Godbillon-Vey classes (see, e.g., [11]) and leaf invariants (cf. [6]). They are closely related, and I will refer to the latter in the present note. In operator theory, remarkable progress has been made using primary characteristic classes through $K$-theory. But I am not aware of any works on secondary classes related to operator theory so far. This note is an attempt to link secondary characteristic classes to operator theory.

Let $X$ be a Hausdorff $C^\infty$-manifold with a countable open base and $\mathcal{F}$ a $C^\infty$-foliation of codimension $q$ on $X$. Sometimes we denote $\mathcal{F}$ by $(X,\mathcal{F})$. Let $p$ be the dimension of leaf of $\mathcal{F}$ so that $n = p + q$ is the dimension of $X$. We denote by $H^i_1, H^i_2, \ldots, H^i_n$ for $i = 2(n + 1)/3 - 1$ the graded exterior algebra generated by $h^i_1, h^i_2, \ldots, h^i_n$ where $\deg h^i_k = 2i - 1$. Let $L$ be a leaf of $\mathcal{F}$. Then a graded algebra map

$$\varphi^i_{L,H}: H^i_1, H^i_2, \ldots, H^i_n \rightarrow H^i_{L,H}(L)$$

depending only on $L$ and $H$ in $X$, is determined by virtue of Bott vanishing ([11]). $h^i_1(L, H) = \varphi^i_{L,H}(h^i_1)$ is the $i$-th leaf invariant of $\mathcal{F}$ with respect to $L$. $h^1_1(L, H)$ is natural with respect to transverse maps of foliated manifolds and hence it is regarded as a secondary characteristic class. R. Reinhardt [6] and R. Goldman [5] constructed these invariants for the foliations with trivial and nontrivial transverse vector bundles.