First steps towards a noncommutative theory of nonlinear elliptic equations

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Outline

1 Motivation
   - Review of Some Classical Examples
   - Transition to the Noncommutative World
   - Previous Work

2 Some New Results
   - The Noncommutative Laplace Equation
   - Harmonic Maps Between Noncommutative Tori

3 Conclusion
Many of the classical elliptic PDEs arise from variational problems in Riemannian geometry.

**Examples:**

- Harmonic map equation. Comes from looking for critical points of energy of a map $f : M^m \rightarrow N^n$,

  \[ E(f) = \int_M \|\nabla f\|^2 d\text{vol}, \quad (1) \]

  $M$ and $N$ Riemannian manifolds.

  **Special cases:**
  - $M = \mathbb{R}$. Geodesics.
  - $N = \mathbb{R}$. Laplace(-Beltrami) equation.
  - $m = 2$, $N = \mathbb{R}^3$. Minimal surfaces.
More Examples:

- Hilbert-Einstein equation. When $n = 4$ comes from looking for critical points of “total scalar curvature.”

- Yamabe equation. The nonlinear elliptic equation that comes from trying to deform a given metric within a given conformal class to achieve constant scalar curvature. Variational formulation using $\left( \int_M \tilde{R} \, d\tilde{\text{vol}} \right) / \tilde{\text{vol}}(M)^{2/p}$.

- Yang-Mills equation. Comes from looking for critical points of the energy of a connection $\nabla$ on a vector bundle $E \to M$,

$$\int_M \|\Theta\|^2 \, d\text{vol},$$

$\Theta$ the curvature 2-form.
Gelfand-Naimark Duality

Basic ideas of noncommutative geometry:
Recall: $X \rightsquigarrow C_0(X)$ is a contravariant equivalence of categories. This sets up a dictionary:

<table>
<thead>
<tr>
<th>Classical</th>
<th>Noncommutative</th>
</tr>
</thead>
<tbody>
<tr>
<td>locally compact space</td>
<td>$C^*$-algebra</td>
</tr>
<tr>
<td>compact space</td>
<td>unital $C^*$-algebra</td>
</tr>
<tr>
<td>vector bundle</td>
<td>f. g. projective module</td>
</tr>
<tr>
<td>smooth manifold</td>
<td>$C^*$-algebra with</td>
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<tr>
<td></td>
<td>“smooth subalgebra”</td>
</tr>
<tr>
<td>partial derivative</td>
<td>unbounded derivation</td>
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But it’s pointless to go to the noncommutative world just “because it’s there”—there should be a concrete motivation.
More concrete motivation comes from quantum physics.
Many of the classical elliptic PDEs are also the field equations of physical theories.
But the *uncertainty principle* forces quantum observables to be noncommutative.
There is also increasing evidence [Connes, Connes-Douglas-Schwarz, Seiberg-Witten, Mathai-Rosenberg] that quantum field theories should allow for the possibility of noncommutative space-times.
*Noncommutative sigma-models* will require the noncommutative harmonic map equation.
Connes’ Noncommutative Differential Geometry

Set-up:

- A unital $C^*$-algebra, $G$ a Lie group with action $\alpha$ on $A$, $\mathfrak{g}$ the Lie algebra of $G$, $\delta$ the differentiated action, $A^\infty = \{a \in A : t \mapsto \alpha_t(a) C^\infty\}$, $\Xi^\infty$ f. g. projective (right) $A^\infty$-module, $\Xi = \Xi^\infty \otimes_{A^\infty} A$, $\langle , \rangle$ a Hilbert $C^*$-inner product on $\Xi$.

- $\nabla$ a [unitary] connection on $\Xi^\infty$:

$$\nabla_X (\xi \cdot a) = \nabla_X (\xi) \cdot a + \xi \cdot \delta_X (a),$$

$$\delta_X (\langle \xi, \eta \rangle) = \langle \nabla_X \xi, \eta \rangle + \langle \xi, \nabla \eta \rangle.$$

- Curvature:

$$\Theta(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}.$$
The Connes-Rieffel Theory of Noncommutative Yang-Mills

Suppose $A$ has a $G$-invariant tracial state $\tau$, extended to $\text{End}_A(\Xi)$ as usual, and suppose $g$ has an invariant inner product (e.g., if $G$ abelian or compact). Define

$$E = -\tau(\langle \Theta, \Theta \rangle).$$

This is the Yang-Mills action. Critical points satisfy the noncommutative Yang-Mills equation.

**Example**

$A_\theta$ generated by two unitaries $U$, $V$ satisfying $UV = e^{2\pi i \theta} VU$. $A_\theta$ is simple with unique trace $\tau$ if $\theta \in \mathbb{R} \setminus \mathbb{Q}$. $G = \mathbb{T}^2$ acts by

$$(z_1, z_2) \cdot U = z_1 U, \quad (z_1, z_2) \cdot V = z_2 V, \quad |z_1| = |z_2| = 1.$$

$$A_\theta^\infty = \left\{ \sum_{m,n} c_{m,n} U^m V^n \mid c_{m,n} \text{ rapidly decreasing} \right\}.$$
The Connes-Rieffel Theory (cont’d)

Theorem (Pimsner-Voiculescu)

Assume \( \theta \in \mathbb{R} \setminus \mathbb{Q} \). Then \( \tau \) sets up an order isomorphism of \( K_0(A_\theta) \) with \( \mathbb{Z} + \theta \mathbb{Z} \subset \mathbb{R} \).

Theorem (Rieffel)

Finitely generated projective \( A_\theta \) modules are classified by \( K_0(A_\theta)_+ \).

Theorem (Connes-Rieffel)

Let \( A = A_\theta \) as above. Given a projective module \( \Xi^\infty \), the minima of \( E \) are precisely the connections of constant curvature, and if \( \Xi \) is not a multiple of another projective module, then the moduli space of Yang-Mills connections on \( \Xi^d \) may be identified with \( (T^2)^d / \Sigma_d \).
We move now to the noncommutative harmonic map equation. A map \( f : M^m \to N^n \), say with \( M \) and \( N \) compact, dualizes to a unital \( \ast \)-homomorphism \( \varphi : A \to B \), \( A = C(N) \) and \( B = C(M) \). Case of Dąbrowski, Krajewski, and Landi: \( N = S^0 \). A unital \( \ast \)-homomorphism \( C(S^0) = \mathbb{C} \oplus \mathbb{C} \to B \) is the same as a nonunital \( \ast \)-homomorphism \( \mathbb{C} \to B \), i.e., a choice of

\[
e = e^* = e^2 \in B.
\]

When \( A = A_\theta \), \( G = \mathbb{T}^2 \) as above, the natural “energy” analogous to (1) is

\[
E(e) = \tau((\delta_1(e))^\ast \delta_1(e) + (\delta_2(e))^\ast \delta_2(e)).
\]
The “Euler-Lagrange equation” for critical points of (2) is nonlinear second order. But absolute minima occur when $e$ satisfies the nonlinear first order equation for being self-dual or anti-self-dual. Dąbrowski, Krajewski, and Landi write down explicit solutions.
A (periodic) harmonic function on a compact Riemannian manifold \( M^n \) is a harmonic map \( f : M^n \rightarrow S^1 \). This is dual to a unital map \( C(S^1) \rightarrow C(M) \). **Noncommutative analogue:** \( \varphi : C(S^1) \rightarrow A \), or equivalently, a unitary \( u \in A \). Harmonicity amounts to looking for critical points of \( \tau((\nabla(u))^* \nabla(u)) \).

### Example

\( A = M_2(C(S^3)) \), \( u \in C(S^3, U(2)) \), want to minimize energy in homotopy class of the generator of \( K^1(S^3) \). Solution is

\[
u(z_1, z_2) = \begin{pmatrix} z_1/z_2 \\ -\bar{z}_2/\bar{z}_1 \end{pmatrix}, \quad |z_1|^2 + |z_2|^2 = 1.
\]
Let $A = A_{\theta}$ with action of $G = \mathbb{T}^2$ as before. $K_1(A_{\theta})$ is free abelian on the classes of $U$ and $V$.

**Theorem**

*The scalar multiples of $U^mV^n$ are critical points of the energy*

$$E(u) = \tau ((\delta_1(u))^*\delta_1(u) + (\delta_2(u))^*\delta_2(u)),$$

*and are local minima. Any critical point $u$ depending on $U$ alone is a power of $U$.**
Sketch of Proof.

Since $\delta_1$ and $\delta_2$ generate one-parameter groups of automorphisms, $\tau \circ \delta_j \equiv 0$. We start by deriving the “Euler-Lagrange equations” from the formula for $E$. If $u$ is unitary, then any nearby unitary is of the form $ue^{ith}$, $h = h^*$, and

$$
\left. \frac{d}{dt} \right|_{t=0} E(u e^{ith}) = \tau \left( -i \delta_1(h) u^* \delta_1(u) + i \delta_1(u)^* u \delta_1(h) \right)
+ \text{similar expression with } \delta_2
$$

So $u$ is a critical point iff $\forall h = h^*$,

$$
\tau \left( \delta_1(h) \text{Im} (\delta_1(u)^* u) + \delta_2(h) \text{Im} (\delta_2(u)^* u) \right) = 0.
$$

(3)
Sketch of Proof (cont’d).

In (3), the Im’s can be omitted since $u$ unitary $\Rightarrow \delta_j(u)^*u$ skew-adjoint. If $u = e^{i\lambda}U^mV^n$, then $\delta_1(u)^*u = -2\pi im$ and $\delta_2(u)^*u = -2\pi in$, so (3) becomes

$$\tau(m\delta_1(h) + n\delta_2(h)) = 0,$$

which is satisfied since $\tau \circ \delta_j \equiv 0$.

Furthermore, if $u$ depends on $U$ alone, then $\delta_2(u) = 0$. So if $u$ is a critical point, then $\tau(\delta_1(h) \cdot \delta_1(u)^*u) = 0 \ \forall h = h^*$. Since the range of $\delta_1$ contains $U^m$ unless $m = 0$ and $\tau$ induces a nonsingular pairing, $\delta_1(u)^*u$ is a scalar, and so $u = e^{i\lambda}U^m$ for some $m$. 
Sketch of Proof (cont’d).

Finally let’s show that \( u = e^{i\lambda}U^mV^n \) is a local minimum for \( E \). For simplicity take \( m = 1, \ n = 0 \). (The general case is similar.) Expanding shows that

\[
E(Ue^{iht}) = 4\pi^2 + t^2 \tau \left( \delta_1(h)^2 + \delta_2(h)^2 \right) + O(t^3).
\]

The term in \( t^2 \) vanishes exactly when \( \delta_1(h) = \delta_2(h) = 0 \), i.e., \( h \) is a constant, and in that case \( E(Ue^{iht}) = 4\pi^2 \) (exactly). Otherwise, the coefficient of \( t^2 \) is strictly positive and \( E(Ue^{iht}) \) has a strict local minimum at \( t = 0 \).
This section is joint work with Mathai Varghese, Adelaide.

**Theorem**

*Fix $\Theta$ and $\theta$ in $(0, 1)$, both irrational, and $n \in \mathbb{N}$, $n \geq 1$. There is a unital $\ast$-homomorphism $\varphi : A_\Theta \to M_n(A_\theta)$ if and only if $n\Theta = c\theta + d$ for some $c, d \in \mathbb{Z}$, $c \neq 0$. Such a $\ast$-homomorphism $\varphi$ can be chosen to be an isomorphism onto its image if and only if $n = 1$ and $c = \pm 1$.*

For simplicity let’s take $n = 1$. Denote the canonical generators of $A_\Theta$ and $A_\theta$ by $U$ and $V$, $u$ and $v$, respectively. The natural analogue of $E(f)$ in our situation is

$$E(\varphi) = \tau\left(\delta_1(\varphi(U))^*\delta_1(\varphi(U)) + \delta_2(\varphi(U))^*\delta_2(\varphi(U)) + \delta_1(\varphi(V))^*\delta_1(\varphi(V)) + \delta_2(\varphi(V))^*\delta_2(\varphi(V))\right).$$

(4)
Harmonic Maps Between Noncommutative Tori

For the automorphism \( \varphi_A: u \mapsto u^p v^q, v \mapsto u^r v^s \), with
\[
A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbb{Z}),
\]
we obtain
\[
E(\varphi_A) = \text{Tr} \left( \delta_1(u^p v^q)^* \delta_1(u^p v^q) + \delta_2(u^p v^q)^* \delta_2(u^p v^q) \right.
+ \left. \delta_1(u^r v^s)^* \delta_1(u^r v^s) + \delta_2(u^r v^s)^* \delta_2(u^r v^s) \right)
= 4\pi^2 \left( p^2 + q^2 + r^2 + s^2 \right). \tag{5}
\]

Conjecture

The value (5) of \( E(\varphi_A) \) is minimal among all \( E(\varphi), \varphi: A_\theta^\infty \odot \) a
\(-\)-endomorphism inducing the matrix \( A \in SL(2, \mathbb{Z}) \) on
\( K_1(A_\theta) \cong \mathbb{Z}^2 \).
The Conjecture is true if $\varphi: A_\theta^\infty \circlearrowright$ maps $u$ to a scalar multiple of itself. (In this case, $p = s = 1$ and $q = 0$.) The minimum is achieved precisely when $\varphi(v) = \lambda u^r v$, $\lambda \in \mathbb{T}$.

Each $\varphi_A$ is a critical point for $E$, and the Conjecture is “locally true” at the critical point $\varphi_A$. In other words, there is no continuous family of deformations of $\varphi_A: A_\theta^\infty \circlearrowright$ which decreases the energy functional $E$, and $E$ remains constant in a continuous family of deformations of $\varphi_A$ only in the case of gauge transformations (multiplication by the images of $u$ and $v$ each by a scalar of modulus 1).
The Conjecture is true for automorphisms, at least assuming $\theta$ satisfies a Diophantine condition (known to hold for almost all $\theta$). In other words, for generic $\theta$, if $\varphi$ is an automorphism of $A_{\theta}^\infty$ inducing the map given by $A \in SL(2,\mathbb{Z})$ on $K_1(A_{\theta})$, then

$$E(\varphi) \geq E(\varphi_A),$$

with equality if and only if $\varphi(U) = \lambda \varphi_A(U), \varphi(V) = \mu \varphi_A(V)$, for some $\lambda, \mu \in \mathbb{T}$.
Summary

- The important geometric elliptic PDE’s, like the harmonic map equation, have noncommutative analogues.
- The noncommutative Euler-Lagrange equations are usually very messy. Usually easier to work directly with variational problems.
- Even irrational rotation algebras provide lots of interesting examples.

**Unsolved problems for $A_\theta$:**

- Show the only minimizers for $E(u)$ are $e^{i\lambda}U^mV^n$.
- Complete study of energy of $*$-automorphisms.
- What about $*$-endomorphisms, especially when $\theta$ a quadratic irrational?
- What about variation of the metric?