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Groups with T_1 Primitive Ideal Spaces

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We give a simple necessary and sufficient condition for the group C^*-algebra of a connected locally compact group to have a T_1 primitive ideal space, i.e., to have the property that all primitive ideals are maximal. A companion result settles the same question almost entirely for almost connected groups. As a by-product of the method used, we show that every point in the primitive ideal space of the group C^*-algebra of a connected locally compact group is at least locally closed. Finally, we obtain an analog of these results for discrete finitely generated groups; in particular the primitive ideal space of the group C^*-algebra of a discrete finitely generated solvable group is T_1 if and only if the group is a finite extension of a nilpotent group.

1. Let G be a locally compact group and let $C^*(G)$ be its group C^*-algebra [6, 13.9.1]. We denote by $\mathrm{Prim}(G)$ the space of primitive ideals of $C^*(G)$, endowed with the hull–kernel topology, and refer to this simply as the primitive ideal space of the group G. If G is type I, the dual space $\hat{G}$ can be canonically identified with $\mathrm{Prim}(G)$; even when G fails to be type I so that (at least in the separable case) $\hat{G}$ is no longer smooth, $\mathrm{Prim}(G)$ is still a reasonable space, and there is increasing evidence that $\mathrm{Prim}(G)$, rather than $\hat{G}$, is a more interesting and appropriate object of study.

One problem in representation theory of considerable interest is to determine when G (or equivalently $C^*(G)$) is CCR or liminary, which means that for every irreducible unitary representation π of G and for every $f \in L^1(G)$,

$$\pi(f) = \int_G \pi(x) f(x)\, dx$$

is a compact operator. It is known that a group or a C^*-algebra

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is CCR if and only if it is type I and has a T_1 primitive ideal space. The T_1 separation axiom in this context means of course that every primitive ideal of the C^*-algebra is maximal. It is natural therefore to divide the above problem into two subproblems: (i) when is G type I?, and (ii) when is Prim(G) T_1? One of the principal results of this paper is a complete answer to (ii) when G is connected, generalizing the results in [2, Chap. V] for type I solvable Lie groups, and the results of L. Pukanszky [30] for general solvable Lie groups. Additionally we give a nearly complete solution for almost connected groups, that is groups for which G/G_0 is compact, where G_0 is the connected component of the identity in G. Since this work was completed we have learned [32] that Pukanszky has obtained many of the same results independently and has also made considerable progress on problem (i).

As an ingredient in the proof of our results, and as a result of some independent interest, we show that all points of Prim(G) are always locally closed, i.e., open in their closures, for connected G. In a rather different direction, we are able to characterize those finitely generated discrete solvable groups G for which Prim(G) is T_1, thus providing an exact analog for discrete groups of the results for solvable Lie groups. Let us turn to the precise statements of our results.

Let G be a Lie group with a finite number of components and with connected component G_0. Let $\mathfrak{g}$ be the Lie algebra of G (or G_0) and let $\mathfrak{r}$ be its radical. Recall that G is said to be type R if for every $g \in G$, the eigenvalues of Ad(g) on $\mathfrak{g}$ are of modulus 1. This is equivalent by [27] to saying that the adjoint action of G on $\mathfrak{g}$ is distal. We say that G is *type R on its radical* if the eigenvalues of $\mathrm{Ad}(g)|_{\mathfrak{r}}$ on $\mathfrak{r}$ are of absolute value one for all $g \in G$. This means that $\mathrm{Ad}(G)|_{\mathfrak{r}}$ is distal or equivalently that $\mathfrak{g}$ is the Lie algebra direct sum of a semisimple algebra and a Lie algebra of type R. (A Lie algebra $\mathfrak{h}$ is said to be of type R if it is the Lie algebra of a finitely connected Lie group of type R, or equivalently if ad(X), for $X \in \mathfrak{h}$, has only purely imaginary eigenvalues.) For connected groups we have the following result.

THEOREM 1. *If G is a connected locally compact group, all points of* Prim(G) *are locally closed. Moreover G has a T_1 primitive ideal space if and only if it is a projective limit of* (*connected*) *Lie groups each of which is type R on its radical.*

Remark. Note that this implies that for G a connected Lie group, whether or not Prim(G) is T_1 depends only on the Lie algebra of G. In contrast, Dixmier has shown [5] that there exist two connected groups with the same Lie algebra, one of which is type I and the other of which is not.

Since any connected group is a projective limit of Lie groups and since the T_1 property or local closure of points of Prim(G) is inherited by quotient groups of G, it is enough by [28, Proposition 2.2] (cf. [22] also) to prove the theorem for Lie groups. In the "only if" direction of the second part (which was obtained by the first-named author some years ago[1] and also independently by J. Dixmier) one passes to a quotient group of a given G not satisfying the condition where the structure is sufficiently simple so that one can produce a more or less specific nonmaximal primitive ideal. For the other direction, the idea is to reduce to the case of a locally algebraic group by using a consequence of the work of Pukanszky [31, Sect. 1]. Specifically, if G is a connected and simply connected Lie group and if $L = [G, G]$, Pukanszky defines a group $\tilde{G}$ which is a connected, simply connected Lie group whose Lie algebra is the same as that of the algebraic hull of G in some locally faithful matrix representation of G. We have that $L \subset G \subset \tilde{G}$ and $\tilde{G}/L$ is abelian; $\tilde{G}$ operates on the dual $\hat{L}$ of L. The following result will give the remainder of Theorem 1.

THEOREM 2. *If G is a connected and simply connected Lie group, any point of* Prim(G) *is locally closed* (*open in its closure*) *in* Prim(G); *moreover, any primitive ideal of $C^*(G)$ associated* (*in the sense of* [31]) *to a closed orbit of $\tilde{G}$ on $\hat{L}$ is maximal.*

This result has some interesting consequences of its own which will be discussed along with its proof.

In order to treat Prim(G) for disconnected Lie groups as well as for discrete groups, one needs what one might think of as "coming up" and "going down" theorems for the T_1 property, which enable one to go up or down by finite extensions. These are analogs of [27, Lemmas 4.1 and 4.2].

THEOREM 3. *Let G_1 be any open normal subgroup of finite index in the separable locally compact group G; then if* Prim(G) *is T_1,* Prim(G_1) *is also T_1. If further G/G_1 is solvable and* Prim(G_1) *is T_1,*

[1] A preliminary version of the "only if" part of Theorem 1 for type I Lie groups and the complete theorem for algebraic groups were announced in an invited hour address at the San Diego regional meeting of the AMS in November 1966.

then Prim(G) *is* T_1. *Under the same hypothesis, if all points of* Prim(G_1) *are locally closed, so are all points of* Prim(G).

The restriction of solvability in the second part is a nuisance and almost surely unnecessary, but a proof in general of this "almost obvious" result has eluded us. This result combined with Theorem 1 yields a result for almost connected groups.

THEOREM 4. *If G is almost connected and* Prim(G) *is* T_1, *then G is a projective limit of (finitely connected) Lie groups each of which is type R on its radical. If G/G_0 is prosolvable (i.e., a projective limit of finite solvable groups) the converse is valid, and all points of* Prim(G) *are locally closed.*

Our final result concerns discrete finitely generated groups. The correct analog of a type R solvable group is a finitely generated solvable group which has a nilpotent subgroup of finite index. Such groups are polycyclic, and are precisely the finitely generated solvable groups which have polynomial growth (cf. [24, 34]). The following is then a natural analog of the results for solvable Lie groups.

THEOREM 5. *Let G be a finitely generated discrete solvable group. Then* Prim(G) *is T_1 if and only if G has a nilpotent subgroup of finite index.*

A possible generalization of Theorem 1 to this context appears to be blocked by the fact that $SL_2(\mathbb{Z})$, the group of two-by-two integral unimodular matrices, has a non-T_1 primitive ideal space, as we shall demonstrate.

The outline of the paper is as follows: Section 2 is devoted to the proof of the "only if" part of Theorem 1, Section 3 to the proof of Theorem 2, Section 4 to the proof of the rest of Theorem 1, Section 5 to the proofs of Theorems 3 and 4, Section 6 to the proof of Theorem 5, and Section 7 to some concluding remarks.

2. To prove the "only if" part of Theorem 1, we may, as noted in Section 1, reduce to the case of a connected Lie group. We argue by contradiction and so assume that Prim(G) is T_1 but that G is not type R on its radical. Since Prim(G') is T_1 for any quotient group G' of G we are free to replace G by any quotient group (which of course must also fail to be type R on its radical if we are to get a contradiction). Let R be the radical of G, which is closed, let N be the nilradical of R (the maximal connected nilpotent

subgroup), and let K be a maximal compact subgroup of N; we can write K (which is necessarily abelian) as $\exp(\mathfrak{k})$ for a subalgebra $\mathfrak{k}$ of the Lie algebra $\mathfrak{n}$ of N. The kernel of the exponential map in $\mathfrak{k}$ is some lattice L, and by looking at the universal covering group of G, we see that L is centralized by the adjoint action of G. Thus $\mathfrak{k}$ is also centralized and so K is central in G. Since G/K is evidently not type R on its radical, as K is central, we can as noted above replace G by G/K (which has nilradical N/K) and hence assume that the nilradical of our group G is simply connected.

Now let $N_1 = [N, N] = \exp([\mathfrak{n}, \mathfrak{n}])$ be the commutator subgroup. Since any vector space complement V to $[\mathfrak{n}, \mathfrak{n}]$ in $\mathfrak{n}$ generates $\mathfrak{n}$ as Lie algebra [1], it follows that any eigenvalue of $\mathrm{Ad}(g)$ on $\mathfrak{n}$ is a product of some of its eigenvalues on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. Thus $G/[N, N]$ is not type R on its radical and by the same argument as above has nilradical $N/[N, N]$; thus we may and shall assume that the nilradical N of our group G is a vector group.

We consider the adjoint action of G on N, or equivalently on $\mathfrak{n}$. This is a linear representation in which N acts trivially so that G/N, which is a reductive group, acts. This representation is non-type R in that some eigenvalue of some group element is not on the unit circle. This is a general remark; being type R on the radical $\mathfrak{r}$ is the same as being type R on the nilradical $\mathfrak{n}$ since G operates trivially on the quotient $\mathfrak{r}/\mathfrak{n}$. The following proposition is now applicable.

Proposition 2.1. *If G is a connected reductive group and ρ a non-type R representation on a vector space V, then ρ has an irreducible quotient representation which is also not type R.*

Proof. We may take $G \subset GL(V)$. Let G^a be the algebraic hull of G, which is connected as an algebraic group. Our assumption on G is that its radical R is central in G, and this implies that the radical of G^a (which is the algebraic hull of R) is central in G^a, or equivalently the unipotent radical of G^a (as algebraic group) is central. Moreover G is type R when acting on V if and only if G^a is. The "if" part is clear, and for the converse we observe that [27, Theorem 2] displays explicitly any connected group which is type R on V as a subgroup of an algebraic group which is also type R on V. Also G and G^a have the same invariant subspaces so we may assume that G is in fact algebraic. Then $G = S \cdot U$, where S is a reductive algebraic group, U unipotent and central. By duality it will suffice to find an irreducible subspace which is not type R for G. Since U is unipotent, we know that G is type R

iff S is so; since S is reductive we may find an S-irreducible subspace $W \subset V$ so that the action of S is not type R on W. Now let V_0 be the common fixed point set of U, equivalently the common null space of the linear transformations in the Lie algebra $\mathfrak{u}$ of U. Now if $v \in V$, we claim that there is some t in the associative algebra A generated by $\mathfrak{u}$ so that $0 \neq t(v) \in V_0$ (since $A(v)$ is a U-invariant subspace which therefore contains nonzero fixed vectors). In particular let us pick $v \in W$; then as t commutes with S, and W is irreducible, we see that $t(W) \subset V_0$ and that t is an S-intertwining isomorphism of W onto an isomorphic irreducible S subspace W_0 of V_0. Then W_0 is the desired G-irreducible non-type R subspace.

Note. It is easy to construct counterexamples to the above if G is not required to be reductive.

We can now apply the proposition to the action of G/N on N and assume by going to a quotient that G operates irreducibly on N; however in this quotient the nilradical may have become larger. Let the new nilradical be N'; we know at least that G operates trivially on N'/N. We can repeat our first two steps of dividing out by the maximal compact subgroup of N' and then abelianizing N'; we assume this is done. We now pass to an irreducible quotient of the Lie algebra $\mathfrak{n}'$ of N' upon which the action of G is non-type R. By the above the image of $\mathfrak{n}$ in this quotient is the same as the image of $\mathfrak{n}'$ (and is isomorphic to $\mathfrak{n}$), and then this image is the (Lie algebra of the) nilradical. Thus we have reduced further to the case when G operates irreducibly on the Lie algebra of its nilradical.

It follows that the radical $\mathfrak{r}$ of $\mathfrak{g}$ acts semisimply on $\mathfrak{n}$ and has at most one nonzero root. If this root is real, R acts as scalars times the identity, and if not, there is a unique (up to conjugation) complex structure so that R acts by complex scalars. Since R/N is central in G/N it follows that in this case the action of G also preserves this complex structure.

We now need to know the following structural fact:

PROPOSITION 2.2. *The group G is globally the semidirect product of N and a reductive group H.*

Proof. We first show that the Lie algebra $\mathfrak{g}$ of G is the semidirect product of $\mathfrak{n}$ and a reductive algebra $\mathfrak{h}$. Let $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$ (semidirect) where $\mathfrak{s}$ is a semisimple Levi factor. We then choose an $\mathfrak{s}$-invariant vector space complement $\mathfrak{a}$ to $\mathfrak{n}$ in $\mathfrak{r}$. If $[\mathfrak{a}, \mathfrak{a}] = 0$, then we can take $\mathfrak{h} = \mathfrak{s} + \mathfrak{a}$ and we are done. If $[\mathfrak{a}, \mathfrak{a}] \neq 0$, it is necessarily $\mathfrak{s}$-invariant and contained in $\mathfrak{n}$. But the set of $\mathfrak{s}$-invariant vectors in $\mathfrak{n}$ is $\mathfrak{g}$-invariant and by irreducibility, $\mathfrak{s}$ acts trivially on $\mathfrak{n}$ and hence on $\mathfrak{r}$. Thus $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$ is a direct sum of ideals. Moreover if $[\mathfrak{a}, \mathfrak{a}] \neq 0$, $\mathfrak{a}$ is two dimensional and $\mathfrak{r}$ acts on $\mathfrak{n}$ by all complex scalars so that $\mathfrak{r}$ is in fact an algebraic Lie algebra and $\mathfrak{r} = \mathfrak{n} + \mathfrak{a}'$ (semidirect) for some $\mathfrak{a}'$. Then let $\mathfrak{h} = \mathfrak{a}' + \mathfrak{s}$ and the infinitesimal part of the result is proved.

Now let H be the analytic subgroup of G with Lie algebra $\mathfrak{h}$, and let $\bar{H}$ be its closure. Since $G = HN$, H is irreducible on $\mathfrak{n}$, but $\log(\bar{H} \cap N) \subset \mathfrak{n}$ is clearly H-invariant, and hence must be zero or all of N. If the latter is the case, it follows by standard Lie theory that $\mathfrak{n}$ is centralized by $\mathfrak{h}$, contrary to the assumption that G is not type R on $\mathfrak{n}$. Thus $\bar{H} \cap N = \{e\}$, and as $G = HN$ we conclude that $H = \bar{H}$ and that G is the semidirect product of N and H, and we are done.

The following allows us to complete the argument.

PROPOSITION 2.3. *There exists $f \in \mathfrak{n}^*$ (the dual of $\mathfrak{n}$) and $Y \in \mathfrak{h}$ (notation as above) such that $f(t) = \mathrm{Ad}^*(\exp(tY))(f) \to 0$ as $t \to \infty$ and such that the stabilizer of $f(t)$ in G is independent of t.*

Proof. If $\mathfrak{r}$ is not type R on $\mathfrak{n}$ or equivalently $\mathfrak{n}^*$, we select $Y \in \mathfrak{a}$ and let f be any vector in $\mathfrak{n}^*$. Then $\mathrm{Ad}^*(\exp(tY))f = \exp(-\alpha(Y)t)f = f(t)$, where α is not purely imaginary. Then $f(t) \to 0$ as $t \to \infty$ provided $\mathrm{Re}(\alpha(Y)) > 0$, which we can achieve by replacing Y by $-Y$ if necessary. In any case $f(t)$ is a complex multiple of f and the action of G is complex linear as noted so the stability group of $f(t)$ is independent of t.

If $\mathfrak{r}$ is type R, then $\mathfrak{s}$ (notation as above) acts noncompactly on $\mathfrak{n}$ and we choose a split Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{s}$ and $Y \in \mathfrak{b}$ so that $\mathrm{ad}(Y)$ is nonzero on $\mathfrak{n}$. The eigenvalues of $\mathrm{ad}(Y)$ on $\mathfrak{n}$ and $\mathfrak{n}^*$ are necessarily real and we choose f to be an eigenvector with negative eigenvalue. The argument proceeds just as before with $f(t)$ a real multiple of f, and we are done.

Now the element $f \in \mathfrak{n}^*$ above defines a one dimensional unitary representation λ of N and $f(t)$ defines a representation $\lambda(t)$ which is the result of conjugating $\lambda = \lambda(0)$ by the one parameter group $g(t) = \exp(tY)$ of the last proposition.

Let K be the isotropy group of $\lambda(0)$ in G, which by the last proposition is also the isotropy group of $\lambda(t)$. Since $K \supset N$ and G is the semidirect product of H and N, K is the semidirect product of N and $K_0 = K \cap H$. We know that $g(t)\, Kg(t)^{-1} = K$ and since $g(t) \in H$, it follows that $g(t)$ also normalizes K_0. Now any element of K can be written as $k = n \cdot k_0$ ($n \in N$, $k_0 \in K_0$) and we extend $\lambda = \lambda(0)$ and $\lambda(t)$ to one-dimensional representations $\sigma(t)$ of K by $\sigma(t)(n \cdot k_0) = \lambda(t)(n)$. It is clear that $\sigma(t) = g(t) \cdot \sigma(0)$, where this denotes the result of conjugating $\sigma(0)$ by $g(t)$. Now let $\pi(t)$ be the representation of G induced by $\sigma(t)$. According to the "Mackey machine" $\pi = \pi(t)$ is irreducible and independent of t.

Moreover since $\sigma(t) \to 1_K$, the identity representation of K, in the sense of the Fell topology, it follows by continuity of induction [8, Theorem 4.2] that π weakly contains the representation τ of G induced by 1_K of K. Finally τ and hence π weakly contains at least one irreducible representation τ_0, which must necessarily vanish on the normal subgroup N. Since π does not vanish on N, τ_0 cannot weakly contain π. Thus the kernel of π in $C^*(G)$ cannot be maximal. This contradiction completes the proof of the "only if" part of Theorem 1.

3. *Proof of Theorem* 2. Let G, $\tilde{G}$, and $L = [G, G]$ be as in Theorem 2. We use the notation of Pukanszky [31]. In particular, if H is a closed subgroup of the locally compact group K, and if σ is a unitary representation of H, $\mathrm{Ind}_{H \uparrow K}\, \sigma$ denotes the induced representation of K. This notation will also be used in the rest of this paper, except that we omit the subscripts when H and K are clear from the context. Also here and throughout this paper, the *kernel* of a unitary representation of a locally compact group is understood to mean the kernel of the associated representation of the group C^*-algebra. When confusion will not result, we sometimes identify a representation with its unitary equivalence class.

We must show that if $J_1 \in \mathrm{Prim}(G)$, then there is a closed ideal I_1 of $C^*(G)$ strictly containing J_1, such that any primitive ideal of $C^*(G)$ strictly containing J_1 contains I_1 as well. (If J_1 is maximal, we will have $I_1 = C^*(G)$.) We claim I_1 can be constructed as $J_1 +$ (the kernel of $\mathrm{Ind}_{L \uparrow G}\, \Pi$), where Π is a representation of L with kernel $I = \ker(\bar{E}_1 \backslash E_1)$, E_1 the $\tilde{G}$-orbit in $\hat{L}$ associated with J_1. (Here "ker" is to be taken in the hull–kernel sense. By convention, when E_1 is closed so that $\bar{E}_1 \backslash E_1 = \varnothing$, Π is the zero representation of L.)

To see this, let $J_2 \in \mathrm{Prim}(G)$ with $J_1 \subset J_2$. By [31, Sect. 1], we can choose factor representations $T_1 = T(\rho_1)$ and $T_2 = T(\rho_2)$ as in [31, Lemma 1.1.4] with $\ker T_1 = J_1$, $\ker T_2 = J_2$. Then T_1 weakly contains T_2 and by [8, Lemma 5.1], $\ker(T_1 \mid L) \subset \ker(T_2 \mid L)$. Let $\pi_j = \rho_j \mid L$ ($j = 1, 2$). Then $T_j \mid L$ lives on $\overline{G\pi_j} \subset \overline{\tilde{G}\pi_j} = \bar{E}_j$ and is weakly equivalent to $\overline{G\pi_j}$, so $\overline{G\pi_2} \subset \overline{G\pi_1} \subset \bar{E}_1$ and $E_2 \subset \bar{E}_1$ in the hull–kernel topology of $\hat{L}$.

Now the orbit space $\hat{L}/\tilde{G}$ is T_0, so either $E_1 = E_2$ or else $\bar{E}_2 \subsetneqq \bar{E}_1$. (No distinct orbits can have the same closure.) In the first case, $K_1 = K_2 = K$ (notation of [31]) and again by [8, Lemma 5.1], $\ker(T_1 \mid K) \subset \ker(T_2 \mid K)$. This means $\overline{G\rho_2} \subset \overline{G\rho_1}$, so by [31, Lemmas 1.1.6 and 1.1.11], $\rho_2 \underset{\Sigma}{\sim} \rho_1$ and $J_1 = \ker(\mathrm{Ind}_{K\uparrow G}\, \rho_1) = \ker(\mathrm{Ind}_{K\uparrow G}\, \rho_2) = J_2$.

Consider the second case where $\bar{E}_2 \subsetneqq \bar{E}_1$. E_2 and E_1 are distinct $\tilde{G}$-orbits in $\hat{L}$, hence $E_1 \cap E_2 = \varnothing$ and $E_2 \subset \bar{E}_1 \backslash E_1$. Since E_1 is locally closed in $\hat{L}$, $\bar{E}_1 \backslash E_1$ is closed in $\hat{L}$ and is the hull of some closed ideal I in $C^*(L)$, and $\bar{E}_2 \subset \bar{E}_1 \backslash E_1$. This means that with Π as above, Π weakly contains $(T_2 \mid L)$. We must show that $\mathrm{Ind}_{L\uparrow G}\, \Pi$ weakly contains T_2. But $T_2 = \mathrm{Ind}_{K_2\uparrow G}\, \rho_2$, where $\rho_2 \in \hat{K}_2$ extends some $\pi_2 \in E_2$, and clearly π_2 is weakly contained in Π. Since K_2/L is abelian, hence amenable, Corollary 1 to [8, Lemma 4.2] shows that ρ_2 is weakly contained in $\mathrm{Ind}_{L\uparrow K_2}(\rho_2 \mid L) = \mathrm{Ind}_{L\uparrow K_2}\, \pi_2$, hence a fortiori is weakly contained in $\mathrm{Ind}_{L\uparrow K_2}\, \Pi$. Since $\mathrm{Ind}_{L\uparrow G}\, \Pi \cong \mathrm{Ind}_{K_2\uparrow G}(\mathrm{Ind}_{L\uparrow K_2}\, \Pi)$, an application of [8, Theorem 4.2] shows that T_2 is weakly contained in $\mathrm{Ind}_{L\uparrow G}\, \Pi$, i.e., $J_2 \supset I_1$. The proof shows that if E_1 is closed, then J_1 is maximal, so to complete the argument, we need only check that $I_1 \neq J_1$. However, this is clear since $I_1 \subset J_1$ would imply (by continuity of restriction again) that $\bar{E}_1 \subset \bar{E}_1 \backslash E_1$, which is ridiculous. So the proof is complete.

COROLLARY 1. *In the situation of Theorem* 2, *if all the orbits of* $\tilde{G}$ *on* $\hat{L}$ *are closed, then* Prim(G) *is* T_1.

Proof. Immediate.

COROLLARY 2. *If G is any connected Lie group* (*not necessarily simply connected*), *the points of* Prim(G) *are locally closed.*

Proof. Let H be the universal covering group of G, $\varphi: H \to G$ the canonical map. As is well known, φ induces a map on representations that enables us to identify Prim(G) with a closed subspace of Prim(H). So the corollary is evident.

COROLLARY 3. *If G is any connected locally compact group, then points of* Prim(G) *are locally closed.* (*This is the first assertion of Theorem* 1.)

Proof. Any such G may be written as proj $\lim_\alpha G_\alpha$, with $G_\alpha = G/H_\alpha$ a connected Lie group and H_α a compact subgroup of G. By the lemma of Moore cited earlier [28, Proposition 2.2], Prim(G)

may be identified with $\bigcup_\alpha \mathrm{Prim}(G_\alpha)$, and by the same observation as in the proof of Corollary 2, $\mathrm{Prim}(G_\alpha)$ is closed in $\mathrm{Prim}(G)$. So the result follows from Corollary 2.

Corollary 3 shows that, in a certain sense, the C^*-algebras of connected groups are unusually "nice." The same is not valid even for countable solvable groups, as was demonstrated by Guichardet [14]. Specifically, Guichardet showed that a certain semidirect product of the integers and the additive group of dyadic rationals has the property that its group C^*-algebra is primitive but that the zero ideal is a non-locally closed point in the primitive ideal space. We shall return to discrete groups in Section 6.

Of course, the primitive ideal space of a type I group or C^*-algebra is almost Hausdorff in the sense that every nonempty closed subset contains a nonempty relatively open Hausdorff subset (see [6, théorème 4.4.5]). So for such a group or C^*-algebra, an easy exercise in general topology shows that the result of Corollary 3 is automatic.

4. In this section, we prove that if G is a connected Lie group which is type R on its radical, then $\mathrm{Prim}(G)$ is T_1. Since the condition on G depends only on its Lie algebra, it must hold for the universal covering group of G as well. Hence we may assume G is simply connected, since the property we are trying to prove is inherited by quotient groups.

At several points in the argument, we need to make use of a simple fact which is certainly well known. However, as there seems to be some confusion about this point in the literature, it seems best to state it as a lemma.

LEMMA 4.1. *Let (G, X) be a topological transformation group (with no separation axioms assumed), and suppose K is a compact subset of G and C is a closed subset of X. Then $K \cdot C$ is closed in X.*

Proof. Let $\{k_\alpha\}$ and $\{c_\alpha\}$ be nets in K and C, respectively, with $k_\alpha c_\alpha \to x$ in X. By compactness of K, we may pass to a subnet and assume that $k_\alpha \to k$ in K. Then by joint continuity of the map $G \times X \to X$, $c_\alpha \to k^{-1}x$. Since C is closed, $k^{-1}x \in C$ and $x = k \cdot k^{-1}x \in K \cdot C$. So $K \cdot C$ is closed in X.

COROLLARY. *Suppose further that X is a T_1-space and that G is compact. Then the G-orbits in X are closed.*

Proof. Let $C = \{x\}$ for some $x \in X$, and let $K = G$.

This will be applied in the situation where G is a locally compact group and $X = \hat{H}$ for some closed normal type I subgroup H of G —as is well known (see for instance [12], Lemma 1.3), (G, X) is a topological transformation group. We take this opportunity to note that this provides an easy proof of [7, Proposition 8(iii)]. (The argument given there has a gap in it since an orbit θ which is compact, hence separated in its relative topology, is not known a priori to consist of points separated in θ^-.)

To return to our main argument, assume G is connected, simply connected, and type R on its radical. Then we may easily construct $\tilde{G}$ as in Theorem 2 with the same properties. Let $L = [G, G] = [\tilde{G}, \tilde{G}]$. By Corollary 1 to Theorem 2, it is enough to show that the orbits of $\tilde{G}$ on $\hat{L}$ are closed. Hence we may replace G by $\tilde{G}$ and assume that G is locally algebraic. Since G is simply connected and type R on its radical, we may write $G = G_1 \times G_2$, where G_1 is noncompact semisimple (or else trivial) and G_2 is type R. Then $L = [G, G] = G_1 \times [G_2, G_2]$ and $\hat{L} = \hat{G}_1 \times [G_2, G_2]\hat{}$ as a topological space. So G will have closed orbits on $\hat{L}$ if and only if G_2 has closed orbits on $[G_2, G_2]\hat{}$. (The orbits in $\hat{G}_1$ are just single points, which are closed since G_1 is CCR, a fact which follows from [16, Theorem 5].) Without loss of generality, we may replace G by G_2 and assume G is type R. Furthermore, the action of G on $\hat{L}$ is the same as that of the algebraic group $H = \mathrm{Ad}_{\mathfrak{l}}(G)$, where $\mathfrak{l}$ is the Lie algebra of L. Let U be the unipotent radical of H. Since G is type R, $H = K \cdot U$ (semidirect product) with K compact. By the lemma, it is enough to prove that U has closed orbits on $\hat{L}$.

Suppose $\pi, \rho \in L$ and $\{u_n\}$ is a sequence in U with $u_n\pi \to \rho$ in the topology of $\hat{L}$. We must show that $\rho \in U \cdot \pi$.

Let N be the nilradical of L, so that $N \subset L$ and L/N is compact (by the type R condition). The restrictions of π and ρ to N must live on some L-orbits in $\hat{N}$ (N is regularly embedded in L by [7, Lemma 7], for instance), say $L \cdot \sigma$ and $L \cdot \lambda$ respectively, with $\sigma, \lambda \in \hat{N}$. Now U acts on $\hat{N}$ as well as on $\hat{L}$ (since $\mathrm{Ad}(G)$ acts on $\hat{N}$), and the actions of U and L on $\hat{N}$ commute. (To see this, recall that $[L, \mathrm{rad}(G)] \subset N$ and that N acts trivially on $\hat{N}$.) So $u_n\pi|_N$ lives on $L \cdot u_n\sigma$. By [8, Lemma 5.1], the operation of restricting representations of L to N preserves weak containment, and since $u_n\pi \to \rho$, $L \cdot \lambda \in (\bigcup_n L \cdot u_n\sigma)^-$. (The $^-$ denotes closure in $\hat{N}$.) Hence there is a sequence $\{l_n\}$ in L such that $l_n u_n \sigma \to \lambda$. By compactness of L/N and the corollary to the lemma (applicable since N is nilpotent, hence CCR, and thus $\hat{N}$ is T_1), we may pass to a subsequence and suppose that $l_n^{-1}\lambda \to l^{-1}\lambda$ for some $l \in L$. Replacing λ by $l\lambda$, we then have that $u_n\sigma \to \lambda$ (by joint continuity of the actions on $\hat{N}$). However, a repetition of the proof of [30, Lemma 31] shows that U has closed orbits on $\hat{N}$, and so $\lambda = u\sigma$ for some $u \in U$. Let U_σ be the stabilizer of σ in U. Since the natural map $U/U_\sigma \to U \cdot \sigma$ is a homeomorphism (see for instance [11, Theorem 1] or [33, Proposition 7.4]), $u_n \to u$ modulo U_σ. Replacing u_n by $u^{-1}u_n$, we may as well assume that $u_n\sigma \to \sigma$, and then $u_n v_n^{-1} \to e$ (the identity element of U) for suitable $v_n \in U_\sigma$. Since $u_n\pi = (u_n v_n^{-1})\, v_n\pi \to \rho$, we have $v_n\pi \to \rho$, and now $v_n\pi|_N$ lives on a constant L-orbit in $\hat{N}$. In the notation of [7], $v_n\pi \in \hat{L}_{L\cdot\sigma}$ for all n, and also $\rho \in \hat{L}_{L\cdot\sigma}$. By [7, Proposition 8(ii)], $\hat{L}_{L\cdot\sigma}$ is a discrete space; hence the sequence $\{v_n\pi\}$ must be eventually constant and $\rho = \lim v_n\pi \in U \cdot \pi$. This completes the proof of Theorem 1.

5. We turn now to the proof of Theorems 3 and 4. By, say, [25, p. 175], almost-connected groups (even if not separable) are projective limits of finitely connected Lie groups. By applying [28, Proposition 2.2] again, we see that Theorem 4 follows from Theorem 3 so we concentrate on the former. We assume, *for the rest of this section only*, that all groups are separable. We have an open normal subgroup G_1 of finite index in G, and we suppose first that Prim(G) is T_1. We want to show that Prim(G_1) is also. Let p_1 and p_2 be two primitive ideals in $C^*(G_1)$ with $p_1 \supset p_2$, and let σ_1 and σ_2 be irreducible representations of G_1 with kernels p_1 and p_2. Then by definition $\sigma_1 \prec \sigma_2$ where for rest of this section $\prec$ will mean (not necessarily proper) weak (or Fell) containment and $\sim$ will mean weak equivalence. Let $\pi_i = \mathrm{Ind}(\sigma_i)$ be the induced representations of G; by continuity of induction $\pi_1 \prec \pi_2$. By elementary results, π_1 and π_2 are finite sums of irreducible representations of G; let $\pi_1^{(k)}$ and $\pi_2^{(m)}$ be the distinct irreducible summands. But now any $\pi_1^{(i)}$ is weakly contained in the finite sum $\Sigma\pi_2^{(k)}$, and then since Prim(G) is a topological space $\pi_1^{(i)}$ is weakly contained in a single summand $\pi_2^{(j)}$. Since Prim(G) is T_1 it follows that $\pi_2^{(i)}$ is weakly equivalent to $\pi_2^{(j)}$, and hence their restrictions to G_1 are weakly equivalent [8, Lemma 5.1]. However the restriction of $\pi_1^{(i)}$ or $\pi_2^{(j)}$ to G_1 is a direct sum with multiplicities of the finite number of conjugates of σ_1 and σ_2 respectively. In particular it follows that σ_2 is weakly contained in a finite direct sum of the conjugates $g \cdot \sigma_1$ ($g \in G/G_1$) of σ_1. Again as Prim(G_1) is a topological space, σ_2 is then weakly contained in one such summand $g \cdot \sigma_1$. Thus we have $\sigma_1 \prec \sigma_2 \prec g \cdot \sigma_1$, and if n is the order of g in G/G_1, we obtain by iteration weak containments of the form $\sigma_1 \prec \sigma_2 \prec g \cdot \sigma_1 \prec g^2 \cdot \sigma_1 \prec$

$\cdots \prec g^n \cdot \sigma_1 = \sigma_1$. Thus σ_1 is weakly equivalent to σ_2 and $p_1 = p_2$ so that $\text{Prim}(G_1)$ is T_1.

For the other direction of Theorem 3 we shall need three elementary propositions.

PROPOSITION 5.1. *Let X be a T_0 space and let A and B be finite subsets which are each T_1 in their relative topologies. Then if $\bar{A} = \bar{B}$, we have $A = B$.*

The simple proof is omitted. Now let G_1 be an open normal subgroup in G of finite index.

PROPOSITION 5.2. *If π is an irreducible representation of G, its restriction to G_1 is a finite sum of irreducible representations.*

Proof. This can be seen in several different ways. One method is to combine the theorem of [19] with [6, 18.9.9]. Alternatively, this follows directly from [13, Theorem 2.3], which asserts that π is strictly contained in $\text{Ind}_{G_1 \uparrow G}\, \sigma$ for some irreducible representation σ of G_1.

It now follows from the above proposition, just as in the Mackey little group method, that the restriction of any $\pi \in \hat{G}$ to G_1 is the direct sum, with multiplicities, of the (finite number) of irreducible representations of G_1 belonging to some G-orbit O in $\hat{G}_1$. We say then that π is associated to the orbit O. It follows from the little-group method again that there are finitely many $\pi \in \hat{G}$ associated to a fixed orbit $O \subset \hat{G}_1$. Let $P(O)$ be the finite set of primitive ideals in $C^*(G)$ which are kernels of some $\pi \in \hat{G}$ associated to O. We have the following fact:

PROPOSITION 5.3. *Suppose that for each G-orbit O in $\hat{G}_1$, $P(O)$ is T_1 in its relative topology. Then* (i) *if* $\text{Prim}(G_1)$ *is* T_1, *so is* $\text{Prim}(G)$, (ii) *if every point of* $\text{Prim}(G_1)$ *is locally closed, the same is true for* $\text{Prim}(G)$.

Proof. (i) Let p_1 and p_2 be primitive ideals in $C^*(G)$ with $p_1 \supset p_2$ and let π_i be irreducible representations with kernels p_i. Let $\sigma_i \in \hat{G}_1$ be elements of the G-orbits O_i in $\hat{G}_1$ to which π_i is associated. We are given that $\pi_1 \prec \pi_2$ and hence the same is true of their restrictions. It follows just as in the argument of the first part of the theorem that σ_1 is weakly contained in some conjugate $g \cdot \sigma_2$ of σ_2, and then by assumption we know that σ_1 is weakly equivalent to $g \cdot \sigma_2$. Let λ_1 and λ_2 be the representations of G induced by σ_1 and σ_2 (equivalently

$g \cdot \sigma_2$). Then λ_1 is weakly equivalent to λ_2 by continuity of induction. But now by Frobenius reciprocity [26], λ_i is the sum with multiplicities of all the representations associated to the orbit O_i for $i = 1, 2$. The assertion that $\lambda_1 \sim \lambda_2$ then translates into the assertion that $\overline{P(O_1)} = \overline{P(O_2)}$. Now by our assumption and Proposition 5.1, it follows that $P(O_1) = P(O_2)$. But $p_i \in P(O_i)$ so $p_i \in P(O_1)$ which is T_1 in the relative topology. Thus $p_1 \supset p_2$ implies that $p_1 = p_2$ and so Prim(G) is T_1 as desired.

(ii) Suppose now that every point in Prim(G_1) is locally closed. As in (i), let $\pi_1, \pi_2 \in \hat{G}$ have kernels $p_1 \supset p_2$ in Prim(G) and choose $\sigma_i \in O_i$, O_i the G-orbit is $\hat{G}_1$ associated to π_i. As before, we get $\sigma_1 < g \cdot \sigma_2$ for some $g \in G$. If in fact $\sigma_1 \sim g \cdot \sigma_2$, the argument of (i) shows that $p_1 = p_2$. So assume $\sigma \cdot g_2 \not< \sigma_1$. We proceed more or less as in the proof of Theorem 2. Let q_1, q_2 be the kernels of σ_1, $g \cdot \sigma_2$, respectively, in Prim(G_1). Then $q_1 \in \{q_2\}^- \backslash \{q_2\}$, which is closed since $\{q_2\}$ is locally closed. Applying the action of G (on Prim(G_1), where G acts as a topological transformation group), it is easy to see that $G \cdot q_1 \subset (G \cdot q_2)^- \backslash G \cdot q_2$. By hypothesis, continuity of induction, and Proposition 5.1, the set $P(O)$, O a G-orbit in $\hat{G}_1$, depends only on the image of O in Prim(G_1). Let I_1 be the intersection in $C^*(G)$ of the elements of $P(O)$, as O runs over G-orbits in $\hat{G}_1$ whose image in Prim(G_1) is contained in the closed set $(G \cdot q_2)^- \backslash G \cdot q_2$, and let $I = p_2 + I_1$. We have seen that $p_1 \supsetneq p_2$ implies that $p_1 \supset I_1$ and hence that $p_1 \supset I$. So I is a minimal closed 2-sided ideal of $C^*(G)$ containing p_2. To see that $\{p_2\}$ is locally closed in Prim(G), it is enough to show that $I \neq p_2$ under the assumption that $G \cdot q_2$ is not closed. (When $G \cdot q_2$ is closed, we saw in (i) that $\{p_2\}$ is closed.) But if this is false, we have $I \subset p_2$, i.e., $\{p_2\}$ is in the closure of the set of primitive ideals lying over $(G \cdot q_2)^- \backslash G \cdot q_2$. Using continuity of the operation of restricting representations to G_1 (this is [8, Lemma 5.1] again), we get $G \cdot q_2 \subset (G \cdot q_2)^- \backslash G \cdot q_2$, which is absurd. So $\{p_2\}$ is locally closed.

We proceed to the proof of the other direction in Theorem 3. Specifically we assume that Prim(G_1) is T_1 or that all its points are locally closed and that G/G_1 is solvable. We wish to show that Prim(G) is T_1 or that all its points are locally closed. Our argument will go by induction on the order of G/G_1 where we assume that the result is true for all G and G_1 when $|G/G_1| < n$. If G/G_1 is not cyclic of prime order, we can find a normal subgroup H with $G_1 \subsetneq H \subsetneq G$. By induction Prim$(H)$ has the desired property and again by induction Prim(G) has the desired property.

Therefore we assume that G/G_1 is cyclic of prime order. By Proposition 5.3 it suffices to show that $P(O)$ is T_1 in the relative topology for every G-orbit O in $\hat{G}_1$. There are two possibilities; either $O = \{\sigma_0\}$ is a single point or O has cardinality equal to the order of G/G_1 and G/G_1 acts freely. In the latter case, the Mackey method tells us that there is only one irreducible representation of G associated to O, namely $\mathrm{Ind}(\sigma)$ for any $\sigma \in O$. Thus $P(O)$ consists of one point and is T_1 in its relative topology.

In the former case, we again apply the Mackey method. Since G/G_1 is cyclic, the Mackey obstruction vanishes and σ_0 extends to a representation σ of G. The irreducible representations of G associated to the orbit 0 are exactly those of the form $\sigma \otimes \lambda$ where λ is a one-dimensional representation of G/G_1 lifted up to G. We have to check that there are no proper weak containment relations between these representations. If we had one, we could change the extension σ so that we had a weak containment of the form $\sigma \otimes \lambda \prec \sigma$ for some $\lambda \neq 1$. Since tensoring with λ is weakly continuous [9, Theorem 2], we have, as before, $\sigma = \sigma \otimes \lambda^n \prec \sigma \otimes \lambda^{n-1} \prec \cdots \prec \sigma \otimes \lambda \prec \sigma$, where n is the order of λ. Thus $\sigma \sim \sigma \otimes \lambda$ and the proof of Theorem 3 is complete.

Theorem 4 follows immediately from Theorems 1 and 3.

Remarks. One would hope that the hypothesis of solvability in Theorem 3 could be eliminated but this has eluded us. One might ask moreover if Theorem 3 is true when we simply assume that G/G_1 is compact instead of finite. We know of no counterexamples, and in fact if G_1 were type I and had a T_1 primitive ideal space, then G_1 would be CCR, and then it is standard (see [7, Sect. 10]) that G would also be CCR and hence have a T_1 primitive ideal space. Thus the going up part of the result is true for compact extensions when the group G_1 is type I. The going down theorem is open even for type I groups in this context. However it is somewhat suspect because a natural extension of it is (somewhat surprisingly) false. Namely we could consider instead of the group G_1, a separable C^*-algebra A_1 together with a compact group H of automorphisms of A_1. We form the C^*-crossed product of A_1 and H which we call A. This plays the role of G and H the role of G/G_1. If A_1 is CCR it is easy to see that A is also. For going down type I-ness may fail, but even so if A is CCR one might ask if A_1 has necessarily a T_1 primitive ideal space. This is false even with A_1 type I. Specifically let $H = \mathbb{T}$ be the circle group and let A_1 be C^*-algebra consisting of compact operators plus multiples of the identity on the Hilbert space $L^2(\mathbb{T})$. We let $H = \mathbb{T}$ act on A_1 via conjugation of operators by the regular representation of $\mathbb{T}$. Then A_1 is type I but not CCR, and an elementary calculation shows that the crossed product of A_1 by $\mathbb{T}$ with this action *is* CCR.

6. We turn now to the proof of Theorem 5. First suppose that G is a discrete solvable finitely generated group which is a finite extension of a nilpotent subgroup. We must show that Prim(G) is T_1. By the "going up" part of Theorem 3 it suffices to show this for some subgroup of finite index so in particular we may take G nilpotent. Moreover G is polycyclic and has a faithful matrix representation (cf. [34]) and so by Selberg's theorem (cf. [34]) G has a torsion-free subgroup of finite index which will still be finitely generated; hence we may take G to be torsion free. R. Howe [17] has singled out among such groups those which he calls *elementary exponential* (e.e.). In order to define these recall that G may be embedded as a discrete cocompact subgroup in a (unique) connected and simply connected nilpotent Lie group N with Lie algebra $\mathfrak{n}$. Then $L = \log(G) \subset \mathfrak{n}$ is a discrete set whose span over the rational numbers $\mathbb{Q}$ is a Lie algebra which is a rational form for $\mathfrak{n}$ [1, Chaps. I and IV]. One says that G is e.e. if first L is a group under addition (in which case one says that G is a *lattice group*), and moreover if L is a Lie ring over the integers such that $[L, L] \subset k!L$, where k is the nilpotent length of G. Howe [17, p. 29] shows that any G contains an e.e. subgroup of finite index and so we may assume that G is e.e. But now one of the major results in [17, Theorem 2] is that (passing again perhaps to a subgroup of finite index) Prim(G) is T_1 in this case (in fact Howe gives a kind of Kirillov description for Prim(G)) and this completes the proof in one direction.

For the other direction one could mimic the argument we gave in the continuous case, at least for polycyclic groups, but there is in fact a simpler technique available making use of results in [23]. So we let G be a finitely generated solvable (or merely amenable) group with Prim(G) T_1. We let F_1 be the normal subgroup consisting of *FC-elements*, elements with finite conjugacy classes, and we define by induction the (transfinite) upper FC-series $\{F_\alpha\}$. $F_{\alpha+1}$ is specified by the condition that $F_{\alpha+1}/F_\alpha$ is the subgroup of FC-elements in G/F_α; for a limit ordinal β, $F_\beta = \bigcup_{\alpha<\beta} F_\alpha$. This series eventually stabilizes and $F = \lim_\alpha F_\alpha$ is called the *FC-hypercenter* of G. It is normal and $G/F = G_0$ has no elements other than the identity with finite conjugacy classes. But now the regular representation of G_0 is a factor representation and since G_0 is amenable, the regular representation is faithful on $C^*(G_0)$. Thus the zero ideal in $C^*(G_0)$ is the kernel of a factor representation, and so [6, 3.9.1c] is primitive since G_0 is countable.

Since Prim(G) and hence Prim(G_0) are given to be T_1, $C^*(G_0)$ is simple. However the one-dimensional trivial representation of G_0 has kernel an ideal of codimension one. Thus $G_0 = \{e\}$; that is $G = F$ is FC-hypercentral. Now [23, Theorem 2] says that G is a finite extension of a nilpotent group since it is finitely generated. This completes the proof of Theorem 5. We note from the proof that if the solvability hypothesis could be deleted from Theorem 3, then we could replace "solvable" by "amenable" in the statement of Theorem 5.

Theorem 5 is indeed the exact analog of the results in [2, Chap. V] and in [30] since a finite solvable extension of a nilpotent group is the discrete analog of a type R solvable Lie group. (We shall discuss this further in Sect. 7.) In view of this we might ask if there are discrete analogs of Theorem 1. The first case to consider would be a discrete subgroup Γ of a semisimple Lie group G such that G/Γ has finite volume, or is compact. If the analogy held, Prim(Γ) would be T_1 but we shall show that this is false for $\Gamma = SL_2(\mathbb{Z})$, the group of 2×2 integral unimodular matrices.

We first prove the following.

PROPOSITION 6.1. *Let $\Gamma = \mathbb{F}_n$ be the free group on n generators (n an integer ≥ 2) and $\Gamma^{(2)}$ its second derived group. Then* Prim($\Gamma/\Gamma^{(2)}$) *and hence* Prim(Γ) *are not T_1.*

Proof. The first derived group $\Gamma^{(1)}$ of $\mathbb{F}_n$ is known to be a free group on infinitely many generators [20, Ch. IX]. Thus $\Gamma/\Gamma^{(2)}$ is a finitely generated solvable group whose first derived group is not finitely generated. So $\Gamma/\Gamma^{(2)}$ is not polycyclic [34, Proposition 4.1] and by Theorem 5 Prim($\Gamma/\Gamma^{(2)}$) is not T_1.

Remark. In fact, Prim($\mathbb{F}_n$) does not even have locally closed points. One can easily see this by noting that Guichardet's group (mentioned above at the end of Sect. 3) is generated by two elements, hence is a quotient of $\mathbb{F}_n$ ($n \geq 2$).

PROPOSITION 6.2. *No subgroup of $SL_2(\mathbb{R})$ commensurable with $SL_2(\mathbb{Z})$ has a T_1 primitive ideal space.*

Proof. Let Γ be such a subgroup; by definition $\Gamma \cap SL_2(\mathbb{Z})$ is of finite index in Γ and in $SL_2(\mathbb{Z})$. By the coming down part of

Theorem 3 we may assume that $\Gamma \subset SL_2(\mathbb{Z})$. But $\mathbb{F}_2$ is isomorphic to a subgroup of finite index in $SL_2(\mathbb{Z})$ and again we may assume that $\Gamma \subset \mathbb{F}_2$. Now $\Gamma/(\Gamma \cap \mathbb{F}_2^{(2)})$ is a finitely generated two-step solvable group of finite index in $\mathbb{F}_2/\mathbb{F}_2^{(2)}$ which by Theorem 3 and the last proposition cannot have a T_1 primitive ideal space.

7. To conclude, we would like to rephrase some of our results and compare them with others in the literature. We should mention first of all that it is immediate from Theorem 4 that any almost-connected locally compact group G with a T_1 primitive ideal space (or which in particular is CCR) is necessarily unimodular. Since a quotient of G by some compact normal subgroup is a Lie group, it is enough to observe this for Lie groups. But unimodularity of a finitely connected Lie group G depends only on the universal covering group of the connected component of the identity in G, and when Prim(G) is T_1, this is a direct product of a semisimple group and of a type R group (with either factor possibly missing). But as is well known, semisimple and type R groups are unimodular, hence so is a product of two such.

Secondly, we would like to point out that the type R condition for Lie groups has arisen in other contexts which seem somewhat connected with representation theory in ways that so far are not totally understood. For instance, Guivarc'h [15] and Jenkins [18] have shown that an almost-connected Lie group has polynomial growth (that is, for any compact set K in the group, the Haar measure of K^n grows like a polynomial function of n) if and only if it is type R. It is in this sense (recall [24, 34]) that type R solvable groups are the analogs of finitely generated discrete solvable groups which are nilpotent-by-finite. Furthermore, Azencott [3], following work of Furstenberg [10], has defined a *property* T for locally compact groups (transitivity of the group on the Poisson spaces of all its étalé probability measures) which has now been shown ([4, 29]) for connected Lie groups to be equivalent to the "type R on the radical" condition, provided that the quotient of the group by its radical has finite center. However, the condition on the center is necessary [29] so property T and the condition of being type R on the radical are not quite the same. Yet we see that for a connected Lie group, Prim(G) is T_1 if and only if the adjoint group of G has property T; this fact still seems rather mysterious.

Finally, we note that our characterization of general connected groups and of certain almost connected groups with T_1 primitive ideal spaces can be put in perhaps more elegant form if we make use of the Lie algebra of such a group as defined by Lashof [21]. Recall that if $G = \text{proj lim } G_\alpha$, where $\{G_\alpha\}$ is an inverse system of finitely connected Lie groups, then the Lie algebra $\mathfrak{g}$ of G is the induced projective limit of the Lie algebras $\mathfrak{g}_\alpha$ of the G_α. Furthermore, $\mathfrak{g}$ is independent (up to isomorphism of topological Lie algebras) of the choice of the approximating Lie groups G_α, and depends only on G. If we let $\mathfrak{r}_\alpha = \text{rad}(\mathfrak{g}_\alpha)$, it is easy to see that $\text{proj lim } \mathfrak{r}_\alpha$ exists and defines a closed Lie subalgebra of $\mathfrak{g}$, also independent of the choice of the G_α's, which we call the radical $\mathfrak{r}$ of $\mathfrak{g}$. By [21, Corollary 4.24], $\mathfrak{g}$ is of the form $\mathfrak{h} \times \mathfrak{a} \times \mathfrak{s}$, where $\mathfrak{h}$ is a finite dimensional Lie algebra, $\mathfrak{a}$ is a (possibly infinite) direct product of one dimensional Lie algebras, and $\mathfrak{s}$ is a product (again possibly infinite) of compact semisimple Lie algebras. Hence if $\mathfrak{z}$ is the center of $\mathfrak{g}$, $\mathfrak{z} \subset \mathfrak{r}$ and $\mathfrak{r}/\mathfrak{z}$ is finite dimensional. The adjoint action of $\mathfrak{g}$ on $\mathfrak{r}/\mathfrak{z}$ is defined as usual, and we can define $\mathfrak{g}$ (or G) to be type R on its radical if for every $X \in \mathfrak{g}$, the eigenvalues of $\text{ad}(X)$ on $\mathfrak{r}/\mathfrak{z}$ are purely imaginary. It is easy to see that this will be the case if and only if each $\mathfrak{g}_\alpha$ is type R on $\mathfrak{r}_\alpha$. In addition, an easy application of [21, Lemma 3.12 and Theorem 3.5] shows that the adjoint actions of G on $\mathfrak{g}$ and on $\mathfrak{r}/\mathfrak{z}$ can be defined as for Lie groups, and that $\mathfrak{g}$ is type R on its radical if and only if the eigenvalues of $\text{Ad}(g)$ on $\mathfrak{r}/\mathfrak{z}$ are of modulus one for all $g \in G$. Putting all this together with [27] again, we get the following restatement of Theorem 4:

THEOREM 4′. *Let G be an almost connected locally compact group, and let* $\mathfrak{g}$, $\mathfrak{r}$, *and* $\mathfrak{z}$ *be as above. Consider the following conditions*:

(i) $\text{Prim}(G)$ *is* T_1,

(ii) *g is type R on its radical, and*

(iii) *G acts distally* (*via the adjoint representation*) *on* $\mathfrak{r}/\mathfrak{z}$.

Then (ii) *and* (iii) *are equivalent and* (i) *implies* (ii) *and* (iii). *If G/G_0 is prosolvable, then* (ii) *and* (iii) *imply* (i).

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