TOPOLOGICAL TWISTING

and geometric Langlands

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Structure of the Talk

1. Introduce the topolgical field theory
2. Examples of topological field
3. Mirror Symmetry

Topological Field Theory

There are two types of topological field theory:

1. The Schwarz type, where the action is independent of the metric $g_{\mu\nu}$, i.e. For the action S we have the stress-energy tensor $T^{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}} = 0$.
   Example: BF model, Chern-Simons theory.
2. The Witten type theory, where the action can be dependent on the metric, but we can modify so that the correlation function are independent.

Topological Twist: Interesting Facts

1. Witten introduced the procedure in his paper "Topological Quantum Field Theory" in 1988.
2. Topological twists in N=(2,2) nonlinear sigma model leads to the definition of Gromov-Witten invariant and the mirror symmetry conjecture for it.
3. Topological twits in N=4 d=4 super Yang-Mills leads to the famous Kapustin-Witten geometric Langlands formulation.

Topological Twist: The procedure

Witten-type TQFTs arise if the following conditions are satisfied:

1. The action S of the TQFT has a symmetry, i.e. if Q denotes a symmetry transformation (e.g. a Lie derivative) then $QS = 0$ holds.
2. The symmetry transformation is exact, i.e. $Q^2 = 0$.
3. The observables $\mathcal{O}$ are satisfying the "closed" condition, namely $Q\mathcal{O} = 0$.
4. The energy stress-energy tensor $T^{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}} = QG^{\mu\nu}$, i.e. it is $Q-$exact.

"Proof" that topological twisting is topological

We have for any correlation function, for a Haar measure μ and correlation function $\langle \mathcal{O}_i \rangle = \int d\mu \mathcal{O}_i e^{iS}$

$$\begin{aligned}\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_i \rangle &= \int d\mu \mathcal{O}_i \frac{\delta S}{\delta g_{\mu\nu}} e^{iS} \\ &= \int d\mu \mathcal{O}_i (Q G^{\mu\nu}) e^{iS} \\ &= Q \int d\mu \mathcal{O}_i G^{\mu\nu} e^{iS} \\ &= 0\end{aligned}$$

This is because $Q\mathcal{O}_i = 0$ and $QS = 0$. Moreover the last integral is a number and the lie derivative will be zero.

Instant reaction from math

Q forms a kind of "derivative" and topological field will calculate kind of "euler characteristic" for the Q-cohomology.

Indeed, the Euler characteristic of a Riemannian manifold can be calculated by the 1-dimensional supersymmetric sigma-model. Thus topological in some sense.

Sigma Model

Let's consider the 1-dimensional sigma model for a target space $\mathbb{R}$. so the action will be

$$S = \int \frac{1}{2}\dot{X}^2 dt$$

,where $X(t)$ is a function of t.

We can generalize to the manifold M with action

$$S = \frac{1}{2}\int g_{ij}\frac{dX_i}{dt}\frac{dX_j}{dt}$$

The supersymmetric $(1, 1)$-dimensional sigma model for a target space $\mathbb{R}$. so the action will be

$$S = \int \frac{1}{2}\dot{x}^2 dt - \frac{1}{2}((h'(\dot{x}))^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - h''(x)\bar{\psi}\psi$$

,where $x(t), \psi(t)$ is a function of t. The derivatives comes a choice of potential.

If $h = 0$ We can generalize to the manifold M with action

$$S = \frac{1}{2}\int g_{ij}\dot{\phi}^i\dot{\phi}^j + \frac{i}{2}g_{ij}(\bar{\psi}^i D_t\psi^j - D_t\bar{\psi}^i\psi^j) - \frac{1}{2}R_{ijkl}\psi^i\bar{\psi}^j\psi^k\bar{\psi}^l$$

Of course with potential term this will be more complicated.

Euler characteristic

Let $Q = ig_{ij}\bar{\psi}^i\dot{\phi}^j, \overline{Q} = -ig_{ij}\psi^i\dot{\phi}^j$, and if we represents the observables in the differential form by

- $\phi^i = x^i \times$
- $p_i = -i\nabla_i$
- $\bar{\psi}^i = dx^i \wedge$
- $\psi^i = g^{ij} i_{\partial/\partial x^j}$

Then we can have the supercharges $Q = i\bar{\psi}^i p_i = dx^i \wedge \nabla_i = d$, and $\overline{Q} = d^\dagger$.

The Q-closed observables mod out by Q-exact are exactly the cohomology of M. Moreover one can calculate that $\langle Tr(-1)^F \rangle = \chi(M)$, where F is the number of differentials/ fermions.

$N = (2, 2)$ supersymmetry

Now we have $x^0 = t, x^1 = s$ with Minkowski metric $\eta_{00} = -1, \eta_{11} = 1$ and other things zero. We have 4 fermionic coordinates $\theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-$.

A Lorentz transformation acts on the bosonic and fermionic coordinates as

$$\begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} \cosh(\gamma) & \sinh(\gamma) \\ \sinh(\gamma) & \cosh(\gamma) \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}$$

$$\theta^\pm \to e^{\pm\gamma/2}\theta^\pm, \bar{\theta}^\pm \to e^{\pm\gamma/2}\theta^\pm$$

And any field/function of these variables can be written as $F(x^0, x^1, \theta^+, \theta^-, \ldots) = f_0(x^0, f^1) + \theta^+ f_+(x^0, x^1) +$ taylor series

$N = (2,2)$ supersymmetry

We got 2 rotations named R-symmetry of the functions:
$e^{i\alpha F_V} : F(x_\mu, \theta^\pm, \bar{\theta}^\pm) \to F(x_\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm)$
$e^{i\alpha F_A} : F(x_\mu, \theta^\pm, \bar{\theta}^\pm) \to F(x_\mu, e^{\mp i\alpha}\theta^\pm, e^{\pm i\alpha}\bar{\theta}^\pm)$

These transfomations are called vector and axial R-symmetry.
What they do is that if we set differential operators

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\partial_\pm,$$
$$\bar{Q}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm$$

With Axial transformation $Q_\pm \to e^{\mp i\alpha}Q_\pm$, $\bar{Q}_\pm \to e^{\pm i\alpha}\bar{Q}_\pm$.
Simiparly we can write the vector transformation.

Now in this case we need to consider the morphisms $f : \Sigma \to M$ where M is a Kahler metric. The fermions are sections of the spinor bundle $\psi_\pm \in \Gamma(\Sigma, \phi^* TM^{(1,0)} \otimes S_\pm), \bar{\psi}_\pm \in \Gamma(\Sigma, \phi^* TM^{(0,1)} \otimes S_\pm)$ and the familiar action

$$\mathcal{L} = -g_{i\bar{\jmath}} \partial^\mu \phi^i \partial_\mu \overline{\phi^{\bar{\jmath}}} + i g_{i\bar{\jmath}} \overline{\psi}^{\bar{\jmath}}_- (D_0 + D_1) \psi^i_- \\ + i g_{i\bar{\jmath}} \overline{\psi}^{\bar{\jmath}}_+ (D_0 - D_1) \psi^i_+ + R_{i\bar{\jmath}k\bar{l}} \psi^i_+ \psi^k_- \psi^{\bar{\jmath}}_- \psi^{\bar{l}}_+$$

Symmetry Broken

Though the twos are symmetris of the original action, the symmetry will be broken in the quantum level for $U(1)_A$.

Toy model: Let's consider $\Sigma = T^2$ a Euclidean torus, $\phi = 0$ and we're only left with ψ terms.

$$S = \int_{T^2} d^2 z (i\bar{\psi}_+ D_z \psi_+ + i\bar{\psi}_- D_{\bar{z}} \psi_-)$$

$\psi_\pm \in \Gamma(T^2, E \otimes S_\pm)$ and $\bar{\psi}_\pm \in \Gamma(T^2, E^* \otimes S_\pm)$.

The index theorem tell us

$$\dim \ker D_{\bar{z}} - \dim \ker D_z = \int_{T^2} c_1(E) = k.$$

In the quantum mechanics, it means we have exactly k $D_{\bar{z}}$-zero modes and no D_z zero modes to get a nonvanishing correlation function. So the function transforms as

$$\langle \psi_-(z_1) \dots \psi_-(z_k) \bar{\psi}_+(z_1) \dots \bar{\psi}_+(z_k) \rangle$$

Symmetry Broken, II

Vector rotation:

$$\begin{aligned}&\langle \psi_-(z_1)\dots\psi_-(z_k)\bar\psi_+(z_1)\dots\bar\psi_+(z_k)\rangle\\ \to\ &\langle e^{-i\alpha}\psi_-(z_1)\dots e^{-i\alpha}\psi_-(z_k)e^{i\alpha}\bar\psi_+(z_1)\dots e^{i\alpha}\bar\psi_+(z_k)\rangle\end{aligned}$$

Axial rotation

$$\begin{aligned}&\langle \psi_-(z_1)\dots\psi_-(z_k)\bar\psi_+(z_1)\dots\bar\psi_+(z_k)\rangle\\ \to\ &\langle e^{i\alpha}\psi_-(z_1)\dots e^{i\alpha}\psi_-(z_k)e^{i\alpha}\bar\psi_+(z_1)\dots e^{i\alpha}\bar\psi_+(z_k)\rangle\end{aligned}$$

So the axial rotation is broken at quantum level if $k \neq 0$.

Symmetry Broken, II

In general symmetry broken index is given by

$$\int_\Sigma c_1(\phi^* T^{(1,0)} M) = \langle c_1(M), \phi_*[\Sigma] \rangle$$

In general we have the following diagram:

	Vector	Axial
CY sigma model	good	good
sigma model with $c_1(M) \neq 0$	good	bad
LG model with generic W	bad	good
LG model with quasi-homogeneous W	good	good

Back to Witten's topological field

If we have the Calabi-Yau target, then we may consider two ways of Q's to do our topological twist:

1. $Q_A = \bar{Q}_+ + Q_-, Q_A^\dagger = Q_+ + \bar{Q}_-$
2. $Q_B = \bar{Q}_+ + \bar{Q}_-, Q_B^\dagger = Q_+ + Q_-$

A-twist and do the localization, it will leads to the moduli space of stable maps, i.e. the Gromov-Witten invariants.

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Back to Witten's topological field

For the N=4 Yang-Mills theory, we have a family of twists by

$$Q = uQ_u + vQ_r$$

and we may construct a topological theory out of that. The idea is that we will get Hitchin's equation from the equation of motion, so we are essentially studying the Hitchin moduli space.

The twitsted theory for $\Sigma \times C$ leads to mirror symmetry for A-model on $\mathcal{M}_H(C, G)$ and B-model on $\mathcal{M}_H(C, ^L G)$.