TOPOLOGICAL TWISTING
and geometric Langlands

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Structure of the Talk

1. Introduce the topological field theory
2. Examples of topological field
3. Mirror Symmetry
There are two types of topological field theory:

1. **The Schwarz type**, where the action is independent of the metric $g_{\mu\nu}$, i.e. For the action $S$ we have the stress-energy tensor $T^{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}} = 0$. Example: BF model, Chern-Simons theory.

2. **The Witten type** theory, where the action can be dependent on the metric, but we can modify so that the correlation functions are independent.
Topological Twist: Interesting Facts

1. Witten introduced the procedure in his paper "Topological Quantum Field Theory" in 1988.
2. Topological twists in \( N=(2,2) \) nonlinear sigma model leads to the definition of Gromov-Witten invariant and the mirror symmetry conjecture for it.
3. Topological twists in \( N=4 \ d=4 \) super Yang-Mills leads to the famous Kapustin-Witten geometric Langlands formulation.
Witten-type TQFTs arise if the following conditions are satisfied:

1. The action $S$ of the TQFT has a symmetry, i.e. if $Q$ denotes a symmetry transformation (e.g. a Lie derivative) then $QS = 0$ holds.
2. The symmetry transformation is exact, i.e. $Q^2 = 0$.
3. The observables $\mathcal{O}$ are satisfying the "closed" condition, namely $Q\mathcal{O} = 0$.
4. The energy stress-energy tensor $T^{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}} = QG^{\mu\nu}$, i.e. it is $Q$–exact.
"Proof" that topological twisting is topological

We have for any correlation function, for a Haar measure $\mu$ and correlation function $\langle O_i \rangle = \int d\mu O_i e^{iS}$

$$\frac{\delta}{\delta g_{\mu\nu}} \langle O_i \rangle = \int d\mu O_i \frac{\delta S}{\delta g_{\mu\nu}} e^{iS}$$

$$= \int d\mu O_i (Q G^{\mu\nu}) e^{iS}$$

$$= Q \int d\mu O_i G^{\mu\nu} e^{iS}$$

$$= 0$$

This is because $Q O_i = 0$ and $Q S = 0$. Moreover the last integral is a number and the lie derivative will be zero.
$Q$ forms a kind of "derivative" and topological field will calculate kind of "euler characteristic" for the $Q$-cohomology.

Indeed, the Euler characteristic of a Riemannian manifold can be calculated by the 1-dimensional supersymmetric sigma-model. Thus topological in some sense.
Let’s consider the 1-dimensional sigma model for a target space $\mathbb{R}$, so the action will be

$$S = \int \frac{1}{2} \dot{X}^2 dt$$

where $X(t)$ is a function of $t$.

We can generalize to the manifold $M$ with action

$$S = \frac{1}{2} \int g_{ij} \frac{dX_i}{dt} \frac{dX_j}{dt}$$
The supersymmetric (1, 1)-dimensional sigma model for a target space $\mathbb{R}$, so the action will be

$$S = \int \frac{1}{2} \dot{x}^2 dt - \frac{1}{2} ((h'(\dot{x}))^2 + \frac{i}{2} (\bar{\psi}\psi - \dot{\bar{\psi}}\dot{\psi}) - h''(x)\bar{\psi}\psi$$

where $x(t), \psi(t)$ is a function of $t$. The derivatives comes a choice of potential.

If $h = 0$ We can generalize to the manifold $M$ with action

$$S = \frac{1}{2} \int g_{ij} \dot{\phi}^i \dot{\phi}^j + \frac{i}{2} g_{ij} (\bar{\psi}^i D_t \psi^j - D_t \bar{\psi}^i \psi^j) - \frac{1}{2} R_{ijkl} \psi^i \bar{\psi}^j \psi^k \bar{\psi}^l$$

Of course with potential term this will be more complicated.
Supersymmetric Sigma Model

Euler characteristic

Let $Q = ig_{ij} \bar{\psi}^i \dot{\phi}^j, \bar{Q} = -ig_{ij} \psi^i \dot{\phi}^j$, and if we represents the observables in the differential form by

- $\phi^i = x^i \times$
- $p_i = -i \nabla_i$
- $\bar{\psi}^i = dx^i \wedge$
- $\psi^i = g^{ij} i_{\partial/\partial x^j}$

Then we can have the supercharges $Q = i \bar{\psi}^i p_i = dx^i \wedge \nabla_i = d$, and $\bar{Q} = d^\dagger$.

The $Q$-closed observables mod out by $Q$-exact are exactly the cohomology of $M$. Moreover one can calculate that $\langle Tr(-1)^F \rangle = \chi(M)$, where $F$ is the number of differentials/fermions.
\( \mathcal{N} = (2, 2) \) supersymmetry

Now we have \( x^0 = t, x^1 = s \) with Minkowski metric \( \eta_{00} = -1, \eta_{11} = 1 \) and other things zero. We have 4 fermionic coordinates \( \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^- \).

A Lorentz transformation acts on the bosonic and fermionic coordinates as

\[
\begin{bmatrix}
  x^0 \\
  x^1
\end{bmatrix} = \begin{bmatrix}
  \cosh(\gamma) & \sinh(\gamma) \\
  \sinh(\gamma) & \cosh(\gamma)
\end{bmatrix} \begin{bmatrix}
  x^0 \\
  x^1
\end{bmatrix}
\]

\[
\theta^\pm \rightarrow e^{\pm \gamma/2} \theta^\pm, \quad \bar{\theta}^\pm \rightarrow e^{\pm \gamma/2} \bar{\theta}^\pm
\]

And any field/function of these variables can be written as

\[
F(x^0, x^1, \theta^+, \theta^-, \ldots) = f_0(x^0, f^1) + \theta^+ f_+(x^0, x^1) + \text{taylor series}
\]
\( N = (2, 2) \) supersymmetry

We got 2 rotations named \( R \)-symmetry of the functions:
\[ e^{i\alpha F_V} : F(x_\mu, \theta^\pm, \bar{\theta}^\pm) \rightarrow F(x_\mu, e^{-i\alpha \theta^\pm}, e^{i\alpha \bar{\theta}^\pm}) \]
\[ e^{i\alpha F_A} : F(x_\mu, \theta^\pm, \bar{\theta}^\pm) \rightarrow F(x_\mu, e^{\mp i\alpha \theta^\pm}, e^{\pm i\alpha \bar{\theta}^\pm}) \]

These transformations are called vector and axial \( R \)-symmetry. What they do is that if we set differential operators
\[
Q_\pm = \frac{\partial}{\partial \theta^\pm} + i\bar{\theta}^\pm \partial_\pm,
\]
\[
\bar{Q}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} + i\theta^\pm \partial_\pm
\]

With Axial transformation \( Q_\pm \rightarrow e^{\mp i\alpha} Q_\pm, \bar{Q}_\pm \rightarrow e^{\pm i\alpha} \bar{Q}_\pm \). Simiparly we can write the vector transformation.
$\mathcal{N} = (2, 2)$ Sigma Model

Now in this case we need to consider the morphisms $f : \Sigma \to M$ where $M$ is a Kahler metric. The fermions are sections of the spinor bundle $\psi_\pm \in \Gamma(\Sigma, \phi^* T M^{(1,0)} \otimes S_\pm)$, $\bar{\psi}_\pm \in \Gamma(\Sigma, \phi^* T M^{(0,1)} \otimes S_\pm)$ and the familiar action

\[
\mathcal{L} = -g_{i\bar{j}} \partial^\mu \phi^i \partial_\mu \bar{\phi}^\bar{j} + ig_{ij} \bar{\psi}_-^j (D_0 + D_1) \psi_-^i \\
+ ig_{i\bar{j}} \bar{\psi}_-^\bar{j} (D_0 - D_1) \psi_+^i + R_{i\bar{j}kl} \psi_+^i \psi_-^k \bar{\psi}_-^\bar{j} \psi_+^\bar{l}
\]
Symmetry Broken

Though the twos are symmetris of the original action, the symmetry will be broken in the quantum level for $U(1)_A$.

Toy model: Let’s consider $\Sigma = T^2$ a Euclidean torus, $\phi = 0$ and we’re only left with $\psi$ terms.

$$S = \int_{T^2} d^2z (i\bar{\psi}_+ D_z \psi_+ + i\bar{\psi}_- D_{\bar{z}} \psi_-)$$

$\psi_\pm \in \Gamma(T^2, E \otimes S_\pm)$ and $\bar{\psi}_\pm \in \Gamma(T^2, E^* \otimes S_\pm)$.

The index theorem tell us

$$\dim \ker D_{\bar{z}} - \dim \ker D_z = \int_{T^2} c_1(E) = k.$$ 

In the quantum mechanics, it means we have exactly $k$ $D_{\bar{z}}$-zero modes and no $D_z$ zero modes to get a nonvanishing correlation function. So the function transforms as

$$\langle \psi_-(z_1) \ldots \psi_-(z_k) \bar{\psi}_+(z_1) \ldots \bar{\psi}_+(z_k) \rangle$$
Vector rotation:

$$\langle \psi_-(z_1) \ldots \psi_-(z_k) \bar{\psi}_+(z_1) \ldots \bar{\psi}_+(z_k) \rangle$$

$$\rightarrow \langle e^{-i\alpha} \psi_-(z_1) \ldots e^{-i\alpha} \psi_-(z_k) e^{i\alpha} \bar{\psi}_+(z_1) \ldots e^{i\alpha} \bar{\psi}_+(z_k) \rangle$$

Axial rotation

$$\langle \psi_-(z_1) \ldots \psi_-(z_k) \bar{\psi}_+(z_1) \ldots \bar{\psi}_+(z_k) \rangle$$

$$\rightarrow \langle e^{i\alpha} \psi_-(z_1) \ldots e^{i\alpha} \psi_-(z_k) e^{i\alpha} \bar{\psi}_+(z_1) \ldots e^{i\alpha} \bar{\psi}_+(z_k) \rangle$$

So the axial rotation is broken at quantum level if $k \neq 0$. 

Topological twist
In general symmetry broken index is given by

\[ \int_{\Sigma} c_1(\phi^* T^{(1,0)} M) = \langle c_1(M), \phi_*[\Sigma] \rangle \]

In general we have the following diagram:

<table>
<thead>
<tr>
<th>Model</th>
<th>Vector</th>
<th>Axial</th>
</tr>
</thead>
<tbody>
<tr>
<td>CY sigma model</td>
<td>good</td>
<td>good</td>
</tr>
<tr>
<td>sigma model with (c_1(M) \neq 0)</td>
<td>good</td>
<td>bad</td>
</tr>
<tr>
<td>LG model with generic (W)</td>
<td>bad</td>
<td>good</td>
</tr>
<tr>
<td>LG model with quasi-homogeneous (W)</td>
<td>good</td>
<td>good</td>
</tr>
</tbody>
</table>
If we have the Calabi-Yau target, then we may consider two ways of $Q$’s to do our topological twist:

1. $Q_A = \bar{Q}_+ + Q_-$, $Q_A^\dagger = Q_+ + \bar{Q}_-$
2. $Q_B = \bar{Q}_+ + \bar{Q}_-$, $Q_B^\dagger = Q_+ + Q_-$

A-twist and do the localization, it will leads to the moduli space of stable maps, i.e. the Gromov-Witten invariants.

The B-twist is still mysteries in math and I don’t quite understand it.
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For the N=4 Yang-Mills theory, we have a family of twists by

\[ Q = uQ_u + vQ_r \]

and we may construct a topological theory out of that. The idea is that we will get Hitchin’s equation from the equation of motion, so we are essentially studying the Hitchin moduli space.

The twisted theory for \( \Sigma \times C \) leads to mirror symmetry for A-model on \( \mathcal{M}_H(C, G) \) and B-model on \( \mathcal{M}_H(C, L G) \).