A new look at the Universal Coefficient Theorem for Ext

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Larry Brown's Contribution

Bull. Amer. Math. Soc. **81** (1975), no. 6, 1119–1121.
Operator algebras and group representations,
Vol. I (Neptun, 1980), 60–64, Monogr. Stud. Math.,
17, Pitman, Boston, MA, 1984.

Recall that $\mathrm{Ext}(A)$ is constructed from equivalence classes of extensions

$$0 \to \mathcal{K} \to E \to A \to 0.$$

Theorem 1 (Brown) *Let A be in the smallest class of nuclear C^*-algebras containing all type I algebras and closed under extensions and direct limits. There is a natural universal coefficient theorem (UCT) of the form*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) \to \mathrm{Ext}(A) \xrightarrow{\gamma} \mathrm{Hom}(K_1(A), \mathbb{Z}) \to 0,$$

γ induced by the connecting map in K-theory.

An easy proof

I. Madsen and JR, *Index theory of elliptic operators, foliations, and operator algebras* (New Orleans, LA/Indianapolis, IN, 1986), 145–173, Contemp. Math., 70, Amer. Math. Soc., Providence, RI, 1988.

Let $q\colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ be the quotient map, and let $K^{-*}(A)$ be the group of commutative diagrams

$$\begin{array}{ccc} K_*(A;\mathbb{Q}) & \xrightarrow{q_*} & K_*(A;\mathbb{Q}/\mathbb{Z}) \\ f\downarrow & & \downarrow g \\ \mathbb{Q} & \xrightarrow[q]{} & \mathbb{Q}/\mathbb{Z}. \end{array}$$

Then $A \mapsto K^{-*}(A)$ is a cohomology theory on Banach algebras. (The key observation is that it satisfies the exactness axiom, which follows from the fact that Hom(__, $\mathbb{Q}$) and Hom(__, $\mathbb{Q}/\mathbb{Z}$) are exact functors since $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible, hence injective.)

Proposition 2 *There is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_{*-1}(A), \mathbb{Z}) \to K^{-*}(A) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), \mathbb{Z}) \to 0.$$

Proof. For any abelian group B, we have an exact sequence

$$0 \to \mathrm{Hom}(B, \mathbb{Z}) \to \mathrm{Hom}(B, \mathbb{Q}) \xrightarrow{q_*} \mathrm{Hom}(B, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(B, \mathbb{Z}) \to 0.$$

So it's enough to construct an exact sequence

$$\mathrm{Hom}(K_{*-1}(A), \mathbb{Q}) \xrightarrow{q_*} \mathrm{Hom}(K_{*-1}(A), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\alpha} K^{-*}(A) \xrightarrow{\delta} \mathrm{Hom}(K_*(A), \mathbb{Q}) \xrightarrow{q_*} \mathrm{Hom}(K_*(A), \mathbb{Q}/\mathbb{Z}).$$

We define $\delta\colon K^{-*}(A) \to \mathrm{Hom}(K_*(A), \mathbb{Q})$ by

$$\left(\begin{array}{ccc} & \xrightarrow{q_*} & \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ & \xrightarrow[q]{} & \end{array}\right) \mapsto \left(\begin{array}{ccc} K_*(A) & \longrightarrow & K_*(A;\mathbb{Q}) \\ & & \big\downarrow{\scriptstyle f} \\ & & \mathbb{Q} \end{array}\right)$$

and $\alpha\colon \mathrm{Hom}(K_{*-1}(A), \mathbb{Q}/\mathbb{Z}) \to K^{-*}(A)$ by

$$\left(\begin{array}{c} K_{*-1}(A) \\ \big\downarrow{\scriptstyle \varphi} \\ \mathbb{Q}/\mathbb{Z} \end{array}\right) \mapsto \left(\begin{array}{ccc} & \xrightarrow{q_*} & \\ {\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle \varphi\circ\beta} \\ & \xrightarrow[q]{} & \end{array}\right),$$

where $\beta\colon K_*(A; \mathbb{Q}/\mathbb{Z}) \to K_{*-1}(A)$ is the Bockstein map or connecting homomorphism in

$$K_*(A) \to K_*(A; \mathbb{Q}) \xrightarrow{q_*} K_*(A; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} K_{*-1}(A).$$

Note that $\begin{array}{ccc} & \xrightarrow{q_*} & \\ {\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle \varphi\circ\beta} \\ & \xrightarrow[q]{} & \end{array}$ commutes since $\beta \circ q_* = 0$.

Also note that $q_* \circ \delta = 0$ and $\alpha \circ q_* = 0$.

Exactness at $\mathrm{Hom}(K_*(A), \mathbb{Q})$. If

$$f \in \mathrm{Hom}(K_*(A), \mathbb{Q}), \qquad q \circ f = 0,$$

then $f = \delta\left(\begin{array}{ccc} & \xrightarrow{q_*} & \\ \bar{f}\big\downarrow & & \big\downarrow 0 \\ & \xrightarrow[q]{} & \end{array} \right)$, where $\bar{f} = f \otimes 1_{\mathbb{Q}}$.

Exactness at $K^{-*}(A)$. If $\delta\left(\begin{array}{ccc} & \xrightarrow{q_*} & \\ f\big\downarrow & & \big\downarrow g \\ & \xrightarrow[q]{} & \end{array} \right) = 0$, then $f = 0$ and $g \circ q_* = 0$, so g factors through β and the element of $K^{-*}(A)$ lies in $\mathrm{im}\, \alpha$. □

Proof of Theorem 1. Observe that taking induced maps on K-theory gives a natural transformation of cohomology theories $\mathrm{Ext}^*(A) \to K^*(A)$ which is an isomorphism for $A = \mathbb{C}$. Thus it's an isomorphism for $A = C(X)$, X a finite CW-complex. Since both theories behave the same with respect to direct limits, it's an isomorphism for separable commutative C^*-algebras. Then by Morita invariance, etc., it's an isomorphism on separable type I C^*-algebras and on limits thereof. □

Two-Variable Ext

Recall that $\mathrm{Ext}(A, B)$ is constructed from equivalence classes of extensions

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0.$$

Theorem 3 (Kasparov) *If A and B are separable C^*-algebras, then the invertible elements in $\mathrm{Ext}(A, B)$ can be identified with $KK^1(A, B)$.*

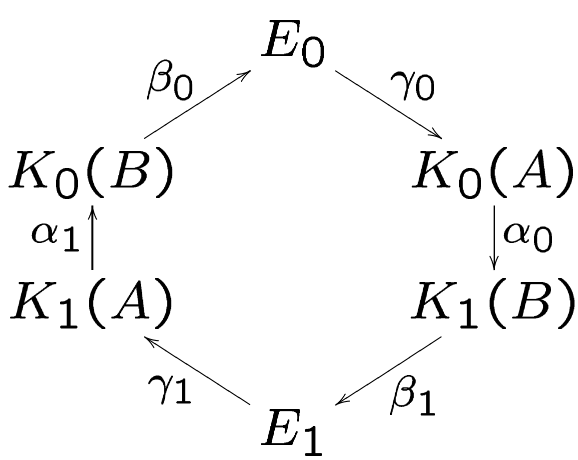
Theorem 4 (R.-Schochet) *Let A be in the smallest class of nuclear C^*-algebras containing all type I algebras and closed under extensions and direct limits. There is a natural universal coefficient theorem (UCT) of the form*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to \mathrm{Ext}(A, B) \xrightarrow{\gamma} \mathrm{Hom}(K_{*+1}(A), K_*(B)) \to 0,$$

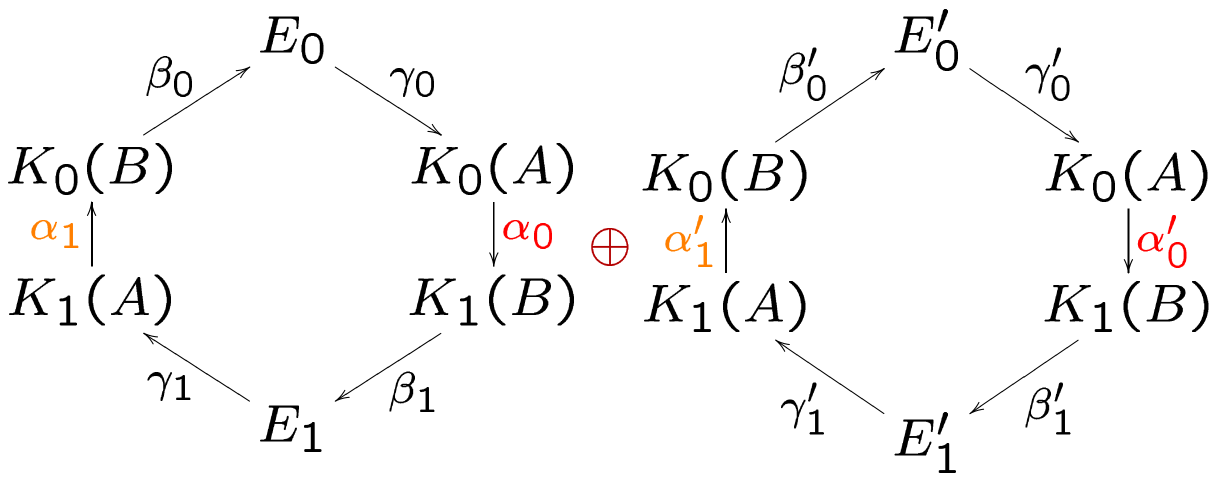
γ induced by the connecting map in K-theory.

Sketch of a direct proof

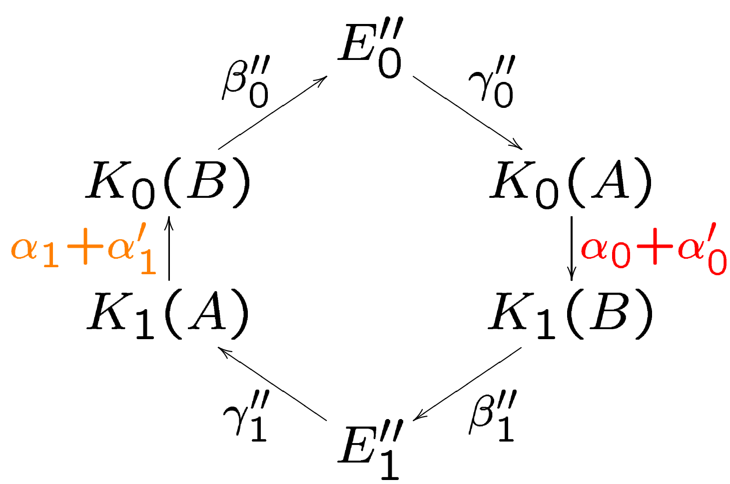
For complex Banach algebras A and B, let hext(A, B) ("hexagonal Ext") be the group of exact hexagons of abelian groups:



with addition given by the modified Baer sum:


E_0
β_0
γ_0
$K_0(B)$
$K_0(A)$
α_1
α_0
$K_1(A)$
$K_1(B)$
γ_1
β_1
E_1
$\oplus$
E'_0
β'_0
γ'_0
$K_0(B)$
$K_0(A)$
α'_1
α'_0
$K_1(A)$
$K_1(B)$
γ'_1
β'_1
E'_1

is


E''_0
β''_0
γ''_0
$K_0(B)$
$K_0(A)$
$\alpha_1+\alpha'_1$
$\alpha_0+\alpha'_0$
$K_1(A)$
$K_1(B)$
γ''_1
β''_1
E''_1

The addition operation is supposed to mimic what happens with addition in $\mathrm{Ext}(A, B)$. Given extensions E and E' of A by $B \otimes \mathcal{K}$, we add them by first taking the "fiber product" $E''' = E \oplus_A E'$:

$$0 \to B \otimes (\mathcal{K} \oplus \mathcal{K}) \to E \oplus_A E' \to A \to 0$$

and then "thickening" $\mathcal{K} \oplus \mathcal{K}$ to $M_2(\mathcal{K}) \cong \mathcal{K}$. From the Mayer-Vietoris sequence, one has an extension

$$0 \to K_{j+1}(A)/(\ker \alpha + \ker \alpha') \to K_j(E''') \\ \to K_j(E) \oplus_{K_j(A)} K_j(E') \to 0.$$

Then $K_*(E'')$ is related to $K_*(E''')$ by a diagram with exact rows:

$$\begin{array}{ccccccc}
K_{j+1}(A) & \xrightarrow{(\alpha,\alpha')} & K_j(B) \oplus K_j(B) & \xrightarrow{(\beta,\beta')} & K_j(E''') & \longrightarrow & K_j(A) \\
\| & & \downarrow (x,y) \mapsto x+y & & \downarrow \varphi & & \| \\
K_{j+1}(A) & \xrightarrow{\alpha+\alpha'} & K_j(B) & \longrightarrow & K_j(E'') & \longrightarrow & K_j(A)
\end{array}$$

Lemma 5 *There is a natural universal coefficient theorem (UCT) of the form*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to \mathrm{hext}(A, B) \xrightarrow{\gamma} \mathrm{Hom}(K_{*+1}(A), K_*(B)) \to 0.$$

Proof. Let γ take an exact hexagon to the connecting maps. If γ of a diagram vanishes, it splits into two short exact sequences, and the modified Baer sum reduces to the usual Baer sum, i.e., to the group operation on $\mathrm{Ext}_{\mathbb{Z}}$. □

Proof of Theorem 4. Observe that the long exact K-theory sequence of an extension gives a natural transformation $\mathrm{Ext}(A, B) \mapsto \mathrm{hext}(A, B)$. Also $\mathrm{hext}(A, B)$ satisfies the same basic properties of a cohomology theory as $\mathrm{Ext}(A, B)$ (for fixed B, separable nuclear A). The proof then proceeds just like the proof we gave for Theorem 1. □