Chapter C

Calculus Background for Linear Differential Equations

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These notes present selected freshmen calculus and complex number background for the study of linear differential equations. The emphasis is how calculus may be used to understand the solutions to linear differential equations; as such the emphasis is different from calculus class. Many of the calculus problems are more difficult than the ones found in most calculus courses. These notes and problems may be used to enhance a calculus course or a differential equations course.

Sections

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1 Graphing $y = x^r$

Use freshmen calculus to do these graphing exercises.

Exercise 1.1 *Graph this function:* $y = f(x) = x^{\frac{-3}{10}}$, *when* $x > 0$.
What is $\lim_{x \downarrow 0} f(x) = \lim_{x \to 0^+} f(x)$*?*

Exercise 1.2 *Graph this function:* $y = f(x) = x^{\frac{3}{10}}$, *when* $x > 0$.
What is $\lim_{x \downarrow 0} f(x) = \lim_{x \to 0^+} f(x)$*?*

Exercise 1.3 *Graph these two pairs of equations:*

(a) $y = \pm x^2$.

(b) $y = \pm x^3$.

Observe that, as pairs, the two graphs have the same "shape"; they are of the same "type", – even though the graphs of $y = x^2$ *and* $y = x^3$ *have different shapes.*

The pairs of graphs for the equations $y = \pm x^r$, $r \in \mathbf{R}$ have five different "shapes" (depending on the number r). Examples of each of the shapes occur in the next exercise.

Exercise 1.4 *Graph these pairs of equations:*

(a) $y = \pm x^6$

(b) $y = \pm 3x^1$

(c) $\pm y^6 = x$, *or equivalently* $y = \pm(\pm x)^{\frac{1}{6}}$

(d) $y = \pm 7x^0$

(e) $y = \pm 3x^{-6}$

Exercise 1.5 *Make a chart or table which displays the answers to these questions about the graphs in the previous exercise:*

(i) *Which graphs are tangent to the x-axis at the origin? A graph is tangent to the x-axis at the origin when the graph has a horizontal tangent at the origin. This occurs when* $\frac{dy}{dx}|_{x=0} = 0$.

(ii) *Which graphs are tangent to the y-axis at the origin? A graph is tangent to the y-axis at the origin when the graph has a vertical tangent at the origin. This occurs when* $\lim_{x\to 0} \frac{dy}{dx} = \pm\infty$ *or when* $\frac{dx}{dy}|_{y=0} = 0$. .

(iii) *Which graph is concave upward? Which graph is concave downward?*

The graph of $y = f(x)$ is concave upward where $\frac{d^2y}{dx^2} > 0$. The graph of $y = f(x)$ is concave downward where $\frac{d^2y}{dx^2} < 0$.

(iv) *Which graphs are asymptotic to the x-axis? A graph is asymptotic to the x-axis when* $\lim_{x\to\pm\infty} = 0$

(v) *Which graphs are asymptotic to the y-axis? A graph is asymptotic to the y-axis when* $\lim_{x\to 0^+} = \pm\infty$ *or* $\lim_{x\to 0^-} = \pm\infty$

(vi) *Note that all five pictures of two graphs together are symmetric about the x-axis? Note that all five pictures of two graphs together are symmetric about the y-axis?*

Exercise 1.6 *Consider these three equations:*

(i)
$$y_1 = x^{1.1}.$$

(ii)
$$y_2 = x^1.$$

(iii)
$$y_3 = x^{0.9}.$$

(a) *Using your graphing calculator, graph these three equations together, on a single set of axes, for $0 \le x \le 1$. Can you distinguish between the 3 graphs?*

(b) *Using calculus, show that the graph of $y_1 = x^{1.1}$ is tangent to the x-axis at the origin. (Calculate by hand that $y_1(0) = 0$ and $\frac{dy_1}{dx}|_{x=0} = 0$.)*

(c) *Using calculus, show that the graph of $y_3 = x^{0.9}$ is tangent to the y-axis at the origin. (Calculate by hand that $y_3(0) = 0$ and $\lim_{x\to 0^+} \frac{dy_3}{dx} = \infty$.)*

(d) *Using your graphing calculator, graph these three equations together, on a single set of axes, using the "window"* $0 \leq x, y \leq 10^{-4}$. *Now is it clear that one graph is tangent to the* $x - axis$ *and another graph is tangent to the* $y - axis$ *at the origin?*

(e) *Using Mathematica, graph these three equations together, on a single set of axes, for* $0 \leq x \leq 1$. *Can you distinguish between the 3 graphs?*

Note that the graphing calculator has limited usefulness in graphing these equations.

Exercise 1.7 *Consider these three equations, when* $0 \leq x \leq 1$,*:*

(i)
$$y_4 = x^{0.1}.$$

(ii)
$$y_5 = x^0.$$

(iii)
$$y_6 = x^{-0.1}.$$

(a) *Using your graphing calculator, graph these three equations together, on a single set of axes, for* $0 \leq x \leq 1$. *Can you distinguish between the 3 graphs? If not, zoom in toward* $x = 0$ *until you can distinguish the 3 graphs. How close to* $x = 0$ *was it necessary to zoom in in order to distinguish the 3 graphs?*

(b) *Using calculus, show that the graph of* $y_4 = x^{0.1}$ *is tangent to the* y*-axis at the origin. (Calculate by hand that* $y_4(0) = 0$ *and* $\frac{dy_4}{dx}|_{x=0} = \infty$*.)*

(c) *Using calculus, show that the graph of* $y_6 = x^{-0.1}$ *blows up at the origin. (Calculate by hand that* $\lim_{x \to 0+} y_6(x) = \infty$.

(d) *Using Mathematica, graph these three equations together, on a single set of axes, for* $0 \leq x \leq 1$. *Can you distinguish between the 3 graphs?*

Exercise 1.8 *Graph the following pairs of equations. Determine whether each graph is tangent to an axis at the origin and whether each graph (or part thereof) is concave upward or downward? A consequence of the preceding exercise is that your graphing calculator will not be useful here. But calculus will be; Take, use and interpet the first and second derivatives of each function. Answering the above exercises should be useful.*

(a) $y^{100} = x^{101}$, *or equivalently* $y = x^{1.01}$. *Which graph in Exercise 1.4 is of the same "type" as this graph?*

(b) $y^{100} = x^{99}$, *or equivalently* $y = x^{.99}$. *Which graph in Exercise 1.4 is of the same "type" as this graph?*

(c) $y^{100} = x^1$, *or equivalently* $y = x^{.01}$. *Which graph in Exercise 1.4 is of the same "type" as this graph?*

(d) $y^{100} = x^{-1}$, *or equivalently* $y = x^{-.01}$. *Which graph in Exercise 1.4 is of the same "type" as this graph?*

Exercise 1.9 *Using the graphs of the preceding exercises as data, make conjectures about the various graphs of* $(y)^r = x^s$, *for various values of* r *and* s.

Exercise 1.10 (a) *Graph these equations together using a single set of axes:* $Ay = \pm Bx^6$ *when* $(A, B) = (4, 2), (9, 3), (1, 0), (7, 0), (0, 1)$.

(b) *Graph these equations together using a single set of axes:* $Ay = \pm Bx^0$ *when* $(A, B) = (2, 4), (3, 9), (1, 0), (7, 0)$.

Exercise 1.11 *When* $x \geq 0$, *show that all the graphs of* $y = Bx^q$, $\forall_{B>0}$ *and* $\forall_{q>1}$ *are tangent to the* x*-axis at the origin. The graphs are all concave upward.*

Exercise 1.12 *When* $x \geq 0$, *show that all the graphs of* $y^r = Bx^s$, $\forall_{\ r>s>0}$ *and* $\forall_{B>0}$ *are tangent to the* y*-axis at the origin. The graphs are all concave downward.*

Exercise 1.13 *When* $x > 0$, *show that all the graphs of* $x^s y^r = B$, $\forall_{\ r}$ *and* $_{s>0}$ *and* $\forall_{B>0}$ *are asymptotic to the* x *and* y*-axes. The graphs are all concave upward.*

The results of the graphs of these exercises are collected and generalized in the next proposition.

Proposition 1.1 (a) *All the graphs of* $A(\pm y)^r = B(\pm x)^s$, $\forall_{A,B\neq 0}$ *and* $\forall_{\ s>r>0}$ *(or equivalently* $y = C(\pm x)^q$, $\forall_{C\neq 0}$ *and* $\forall_{q>1}$ *) are tangent to the* x*-axis at the origin. The graphs are all concave upward in the upper-half plane (where* $y > 0$*); and the graphs are all concave downward in the lower-half plane (where* $y < 0$*).*

(b) *All the graphs of* $A(\pm y)^r = B(\pm x)^s$, $\forall_{\ r>s>0}$ *and* $\forall_{A,B\neq 0}$ *(or equivalently* $y = C(\pm x)^q$, $\forall_{\ 0<q<1}$ *and* $\forall_{C\neq 0}$*) are tangent to the* y*-axis at the origin. The graphs are all concave downward in the upper-half plane; and the graphs are all concave upward in the lower-half plane.*

(c) *All the graphs of* $A(\pm y)^r = B(\pm x)^s$, $\forall_{\ rs<0}$ *and* $\forall_{A,B\neq 0}$ *(or equivalently* $y = C(\pm x)^q$, $\forall_{\ q<0}$ *and* $\forall_{C\neq 0}$*) are asymptotic to the* x *and* y*-axes. The graphs are all concave upward in the upper-half plane; and the graphs are all concave downward in the lower-half plane.*

Remark. This proposition may be established by using freshmen calculus as in the preceding exercises.

2 Exponential growth and decay.

The exponential function. The exponential function is defined as the inverse function of the natural log function, $\ln x$. Hence

$$e^{\ln x} = x, \ \forall_{x>0}.$$

The basic law of exponents is:

$$e^{a+b} = e^a e^b, \ \forall_a \quad \text{and} \quad {}_b.$$

Other rules for exponents are:

$$e^{ab} = (e^a)^b; \ e^{-a} = \frac{1}{e^a}, \ e^0 = 1 \quad \text{and} \quad e^a > 0, \ \forall_{\text{real numbers } a}.$$

Example 2.1

$$e^{2 \ \ln x} = (e^{\ln x})^2 = x^2.$$

Exponential growth.

Proposition 2.2 *For each constant number,* r*, the general solution to* $\dot{x} = rx(t)$ *is* $x = Ae^{rt}, \ \forall_A$.

Remark. This is called "exponential" growth because the solution is the exponential function; there is no exponent in the defining equation. Memorize this proposition; "exponential" growth occurs a lot.

Calculations. Given: $\dot{x} = rx(t)$, divide both sides by x, now $\frac{1}{x}\frac{dx}{dt} = r, \ \forall_{x \neq 0}$. We exclude $x = 0$, since dividing by zero is not permitted. Integrating both sides with respect to t yields: $\int \frac{1}{x}\frac{dx}{dt} dt = \int r \ dt, \ \forall_{x\neq 0}$. The change of variable (cancelling the dt's) yields: $\int \frac{1}{x} dx = \int r \ dt, \ \forall_{x \neq 0}$. Evaluating both integrals:

$$\ln|x| + c_1, \ \forall_{x\neq 0} = \int \frac{1}{x} dx = \int r \ dt = rt + c_2 \quad , \ \forall_{x \neq 0}.$$

Setting $c_3 = c_2 - c_1$, simplifies this equation to: $\ln|x| = rt + c_3, \ \forall_{c_3}$ and $\forall_{x\neq 0}$. We can remove the log by taking its inverse function, the exponential: $e^{\ln|x|} = e^{rt+c_3}, \ \forall_{c_3}$ and $\forall_{x \neq 0}$, which simplifies to:

$$|x| = e^{\ln|x|} = e^{rt+c_3} = e^{rt}e^{c_3}, \forall_{c_3} \quad \text{and} \quad \forall_{x\neq 0}.$$

But $|x| = \pm x$, hence $\pm x = e^{rt}e^{c_3}$, and multiplying by ± 1 yields:

$$x = (\pm e^{c_3})e^{rt}, \ \forall_{c_3} \quad \text{and} \quad \forall_{x \neq 0}.$$

What is $\{(\pm e^{c_3}), \ \forall_{c_3}\}$? Looking at the graph of e^x, we remember that the range is all the positive numbers. Thus $\{e^{c_3}, \ \forall_{c_3}\}$ is the set of all the positive numbers, and hence $\{(\pm e^{c_3}), \ \forall_{c_3}\}$ is the set of all positive numbers together with all the negative numbers, that is, all the numbers except zero. Thus

$$\{\pm e^{c_3}, \ \forall_{c_3}\} = \{A, \ \forall_{A\neq 0}\}.$$

Plugging this in yields:

$$x = Ae^{rt}, \ \forall_{A \neq 0} \quad \text{and} \quad \forall_{x\neq 0}.$$

Finally, we observe that the excluded case $A = 0$ and $x = 0$ is also a solution, since: $x(t) = 0 = 0e^{rt}$. Hence the formula is valid for all numbers:

$$x = Ae^{rt}, \ \forall_A.$$

Check. What have we shown thus far? We started with the equation $\dot{x} = rx$ and concluded with $x = Ae^{rt}, \ \forall_A$. This shows that if the equation has a solution, it must be $x = Ae^{rt}, \ \forall_A$.

But maybe, there is no solution. We still need to check that $x = Ae^{rt}$, $\forall_A$ is actually a solution. We easily do this by plugging in and calculating: Differentiating $x = Ae^{rt}$, with respect to t, yields $\dot{x} = rAe^{rt} = rx$. $\surd$ YEA

Exercises

Exercise 2.1 *Let* $x(t) = 100e^{-7t}$, $\forall_t$; *let* $t_{\frac{1}{2}}$ *be the time it takes for* $x(t)$ *to decay/drop to half its original value, that is:* $x(t_{\frac{1}{2}}) = \frac{1}{2}x(0)$. *Show that* $x(2000 + t_{\frac{1}{2}}) = \frac{1}{2}x(2000))$.

Exercise 2.2 *Let* $x(t) = 100e^{-7t}$, $\forall_t$; *let* t_7 *be the time it takes for* $x(t)$ *to decay/drop to* 7% *its original value, that is:* $x(t_7) = .07x(0)$. *Show that* $x(2000 + t_7) = .07x(2000))$.

Exercise 2.3 *Let* $x(t) = x(0)e^{rt}$, $\forall_t$, *where* r *is a constant. Let* t_4, t_5 *and* $c > 0$ *be be three numbers such that:* $x(t_4) = cx(0)$. *Show that* $x(t_5 + t_4) = cx(t_5))$.

Definition. The *halflife* of exponential decay is the time $t_{\frac{1}{2}}$, it takes for the material x to be reduced to half its original amount, thus: $x(t_{\frac{1}{2}}) = \frac{1}{2}x(0)$. The *doubling time* of exponential growth is the time t_{double}, it takes for the material x to be increased to twice its original amount, thus: $x(t_{double}) = 2x(0)$.

Remark. What is special about *halflives* and *doubling times* is that they are the same, no matter at which time one starts counting or timing.

3 Parametric Equations

As background, we begin with a short description of "vector-valued" functions. This is a good time to review parametric equations in your calculus book.

If $x(t) = t$ and $y(t) = 2t$, then $v(t) = (x(t), y(t))$ means that $v(t) = (t, 2t)$ and v is a function of t, whose range is the set of coordinate vectors in the plane. This function $v(t)$ is called a *vector-valued function* of t. Sometimes, $x(t) = t$ and $y(t) = 2t$ are described as *parametric* equations with parameter t. Here, when $t = 10$, then $x = 10, y = 20$ and $v = (10, 20)$. Here $y = 2x$, $\forall\, t$; hence as t goes from 1 to 3, $v(t)$ travels along the straight line $y = 2x$ from (1,2) to (3,6).

Remark. Arrows should be placed on each curve to indicate in which direction the point determined by the parametric functions is moving as time increases.

Exercise 3.1 *Let* $x(t) = e^{2t}$ *and* $y(t) = e^t$, $-\infty < t < \infty$. *Draw the graph in the* $xy-$*plane.*

Exercise 3.2 *Let* $r(t) = t = \theta(t)$, $-\infty < t < \infty$, *for polar coordinates* (r, θ). *Draw the graph in the* $xy-$*plane.*

Exercise 3.3 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = 2x(t)$ *and* $\dot{y} = 4y(t)$. *Find (graph) the solution curve which goes through the point* $(x, y) = (3, 9)$.

Set-up: First find the general solutions for $x(t)$ and $y(t)$.

Exercise 3.4 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = 2x(t)$ *and* $\dot{y} = 3y(t)$. *Graph 5 different solution curves in the* $xy-$ *plane.*

Exercise 3.5 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = 4x(t)$ *and* $\dot{y} = 2y(t)$. *Graph 5 different solution curves in the* $xy-$ *plane.*

Exercise 3.6 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = -2x(t)$ *and* $\dot{y} = 4y(t)$. *Graph 5 different solution curves in the* $xy-$ *plane.*

Exercise 3.7 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = 2x(t)$ *and* $\dot{y} = -4y(t)$. *Graph 10 different solution curves in the* $xy-$ *plane.*

Exercise 3.8 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = -7x(t)$ *and* $\dot{y} = -7y(t)$. *Graph 5 different solution curves in the* $xy-$ *plane.*

Exercise 3.9 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = -7x(t)$ *and* $\dot{y} = 0$*. Graph 5 different solution curves in the* $xy-$ *plane.*

Exercise 3.10 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = 0$ *and* $\dot{y} = 7y(t)$*. Graph 5 different solution curves in the* $xy-$ *plane.*

Exercise 3.11 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time,* t*. Suppose that* $\dot{x} = 7y(t)$ *and* $\dot{y} = -7x(t), \ \forall_t$*. Let* $r(t)$ *and* $\theta(t)$ *be the corresponding (unknown) polar coordinate functions of time, that is* $r^2(t) = x^2(t) + y^2(t)$ *and* $\tan\,\theta(t) = \frac{y(t)}{x(t)}$*.*

Check that $\dot{r} = 0$ *and* $\dot{\theta} = -7, \ \forall_t$*.*

Exercise 3.12 (puzzle) *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Let* r *and* θ *be the corresponding polar coordinates. Suppose that* $\dot{r} = 0$ *and* $\dot{\theta} = -7, \ \forall_t$*. What can you say about the graph of these parametric functions?*

Exercise 3.13 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time and let* b *be a (constant unknown) real number. Suppose that* $\dot{x} = by(t)$ *and* $\dot{y} = -bx(t), \ \forall_t$*. Let* r *and* θ *be the corresponding polar coordinates.*

Show that $\dot{r} = 0$ *and* $\dot{\theta} = -b, \ \forall_t$*.*

Exercise 3.14 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Suppose that* $\dot{x} = 6x(t) + 7y(t)$ *and* $\dot{y} = -7x(t) + 6y(t), \ \forall_t$*. Let* r *and* θ *be the corresponding polar coordinates.*

Check that $\dot{r} = 3r$ *and* $\dot{\theta} = -7, \ \forall_t$*.*

Exercise 3.15 (puzzle) *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Let* r *and* θ *be the corresponding polar coordinates. Suppose that* $\dot{r} = 3r$ *and* $\dot{\theta} = -7, \ \forall_t$*. What can you say about the graph of these parametric functions?*

Exercise 3.16 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time and let* a *and* b *be (constant unknown) real numbers. Suppose that* $\dot{x} = ax(t) + by(t)$ *and* $\dot{y} = -bx(t) + ay(t), \ \forall_t$*. Let* r *and* θ *be the corresponding polar coordinates.*

Show that $\dot{r} = \frac{a}{2}r$ *and* $\dot{\theta} = -b, \ \forall_t$*.*

Exercise 3.17 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time for motion on the hyperbola* $x^2 - 9y^2 = 7$. *Show that:*

$$\frac{\dot{x}}{\dot{y}} = \frac{9y}{x}.$$

Conversely:

Exercise 3.18 *Given*

$$\frac{\dot{x}}{\dot{y}} = \frac{9y}{x}.$$

Show that this represents motion on a hyperbola $x^2 - 9y^2 = C$, *for some constant* C.

Exercise 3.19 *Given*

$$\dot{x} = 9y \quad \text{and} \quad \dot{y} = x.$$

Show that this represents motion on a hyperbola $x^2 - 9y^2 = C$, *for some constant* C.

Exercise 3.20 *Given*

$$\dot{x} = ay \quad \text{and} \quad \dot{y} = bx,$$

where a *and* b *are positive constants Show that this represents motion on some hyperbola* $Ax^2 - By^2 = C$, *for some constants* A, B *and* C, *with both* A *and* $B > 0$.

Exercise 3.21 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time for motion on the hyperbola* $x^2 + 9y^2 = 7$. *Show that:*

$$\frac{\dot{x}}{\dot{y}} = \frac{-9y}{x}.$$

Conversely:

Exercise 3.22 *Given*

$$\frac{\dot{x}}{\dot{y}} = \frac{-9y}{x}.$$

Show that this represents motion on an ellipse $x^2 + 9y^2 = C$, *for some constant* C.

Exercise 3.23 *Given*

$$\dot{x} = -9y \quad \text{and} \quad \dot{y} = x.$$

Show that this represents motion on an ellipse $x^2 + 9y^2 = C$, *for some constant* C.

Exercise 3.24 *Given*

$$\dot{x} = -ay \quad \text{and} \quad \dot{y} = bx,$$

where a *and* b *are positive constants Show that this represents motion on some ellipse* $Ax^2 + By^2 = C$, *for some constants* A, B *and* C, *with both* A *and* $B > 0$.

4 Energy of two springs.

Example 4.1 (of two horizontal springs) *Given a horizontal system consisting of a block attached by two springs to two walls.*

Figure C. .1 of two springs.

Let d *be the distance between the two walls minus the length of the block; let* k_1 *and* k_2 *be Hooke's spring constant for the two springs; let* l_1 *and* l_2 *be the natural (unstretched) length of the two springs.*

The stretched (and compressed) springs of this example are applying forces to the block that they are attached to. According to Hooke's Law, the two forces applied by the springs at time t are:

$$F_1 = -k_1(x - l_1) = -k_1 x + k_1 l_1 \quad \text{and} \quad F_2 = -k_2([d - x] - l_2).$$

Let $F = F_1 - F_2$ and let $k = k_1 + k_2$ and set $b = l_1 k_1 + (l_2 - d)k_2$. Since Hooke's spring constant are positive numbers, $k = k_1 + k_2$ is also a positive number. Then, the equations may be simplified to:

$$F = -kx + b.$$

Remark. Note that this equation has the same form as Hooke's Law for a single spring; here F is the sum of the spring forces acting on the block, and k is the sum of the spring constants.

A *rest position* occurs when the sum of the forces on the block is zero.

In this example, this occurs when the two spring forces on the opposite sides of the block are equal and thereby cancel each other. This occurs when: $0 = F = F_1 - F_2$, that is when $F_1 = F_2$. In this example, there is a unique rest position x_{rest}, it occurs at $x_{rest} = k^{-1}b$.

The formula for Potential Energy.

We will derive the formula for the potential energy of Example 4.1. Let us move this horizontal block by pushing on it, from the rest position $x_{rest} = x(0)$ at time $t = 0$ to another position $x = x(t)$ at time t. The change in potential energy is the negative of the work required to stretch them.

Let us calculate the work needed to stretch the springs. First a small review of the definite integral.

Small review of the definite integral.

The definite integral of a differentiable (or continuous) function can be defined [1] as the limit of Riemann sums, informally:

$$\int_a^b f(x)\ dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x)\Delta x \quad \textit{or simply} \quad \int_a^b f(x)\ dx = \lim_{\Delta x \to 0} \sum f(x)\Delta x.$$

It is a nusance to write out the index, i many times, more important the "n's" and the x keep changing as $\Delta x \to 0$; therefore we will omit them and simply understand that the appropriate indices are implied.

Remark. When the function $f(x)$ is differentiable (or continuous) on a closed interval $a \le x \le b$, then the limit *always* converges as $\Delta x \to 0$. Also $\int_a^b f(x)\ dx \approx \sum f(x)\Delta x.$ [2]

Area under a curve is the most popular example of the definite integral, but it is only one of many examples; it is not the definition. Energy and work are other important examples.

Writing work as a definite integral.

During a small time period Δt, let $\Delta x(t)$ be the change in position of the block.

Let ΔW be the work required to move the block the distance $\Delta x(t)$. Since the forces are not constant, each $\Delta W \approx F\Delta x$. Hence

$$\Delta W \approx \Delta x\ F = -\Delta x\ kx + \Delta x b.$$

Work is a "Whole $= \sum$ parts" quantity. Hence the total work is: $W = \sum \Delta W \approx \sum -\Delta x\ kx + \Delta x b.$ multiplying and dividing by Δt yields:

$$W \approx -\sum \frac{\Delta x}{\Delta t}\ k\ x\Delta t + \sum \frac{\Delta x}{\Delta t} b\ \Delta t.$$

Taking the limit as $\Delta t \to 0$, results in $\frac{\Delta x}{\Delta t} \to \frac{dx}{dt} = v$ and the $\sum$ becoming a definite integral:

$$W = \lim_{\Delta t \to 0} -\sum \frac{\Delta x}{\Delta t}\ k\ x\Delta t + \sum \frac{\Delta x}{\Delta t} b\ \Delta t = -\int_0^t vk\ x\ dt + \int_0^t \frac{dx}{dt}\ b\ dt.$$

The total change in potential energy for these springs is the negative of the work required to stretch them.

$$\textit{Change in P.E.} = -W = \int_0^t vk\ x\ dt - \int_0^t \frac{dx}{dt}\ b\ dt$$

$$\int_0^t vk\ x\ dt = \frac{1}{2} x\ k\ x \ |_{t=0}^{t=t} = \frac{1}{2}\ k\ x^2 - \frac{1}{2}\ k\ x_{rest}^2$$

[1] as noted in (7) and the paragraph that follows on Page 296 of the calculus textbook written by Professors Ellis and Gulick

[2] "$\approx$" means "approximately equal to" or "very close to".

since $x(0) = x_{rest}$.

$$\int_0^t \frac{dx}{dt}\ b\ dt = x\ b\ |_{t=0}^{t=t} = x\ b - x_{rest}\ b.$$

Combining these equations:

$$Change\ in\ P.E. = \frac{1}{2}\ k\ x^2 - \frac{1}{2}\ k\ x_{rest}^2 - x\ b + x_{rest}\ b.$$

Exercise 4.2 simplifies this to:

$$Change\ in\ P.E. = \frac{1}{2}\ k\ (x - x_{rest})^2.$$

Let E_0 be the potential energy of the rest position. Then the potential energy when the block is at position x is:

$$P.E. = \frac{1}{2}\ k\ (x - x_{rest})^2 + E_0.$$

Proposition 4.2 *The lowest potential energy, of such a system of one horizontal block and two linear springs, (connected to walls) always occurs at the rest position.*

Proof. We observed that $k = k_1 + k_2$ is a positive number. Squares are always positive; the sole exception is $(zero)^2 = 0$. The equation for Change in P.E. shows that the change is always *positive* except when $x = x_{rest}$. Hence all the other positions have higher potential energies than the rest position. $\surd$ YEA

Remark. The potential energy of the rest position is positive since work is required to stretch and /or compress the two springs into their (collective) rest position. This is *different* from the situation of a single (horizontal) spring.

The formulas for power and total energy.

Now let us look at the total energy of this example; it is the sum of the kinetic energy and the potential energy.

The kinetic energy of a particle is the work needed to bring the particle from rest to its current velocity. The familar formula (from physics and calculus) for kinetic energy is:

$$K.E. = \frac{1}{2}mv^2.$$

Let $E(t)$ be the energy of the system at time t and let E_0 be the potential energy of the rest position. Then

$$E(t) = K.E. + P.E. = \frac{1}{2}m\ v^2 + \frac{1}{2}\ k\ (x - x_{rest})^2 + E_0.$$

Definition. *Power* is defined to be the rate of change of energy, that is $P(t) = \frac{dE}{dt}$.

Combining Newton's Law and Hooke's Law, for these objects:

$$m\ddot{x} = F = -kx + b.$$

Proposition 4.3 *The equations of motion for this system of a block connected to two walls by two horizontal linear springs, with no external forces acting, is:*

$$m\ddot{x} + kx = b$$

The power being put out by such a ideal system is zero and the total energy of the ideal system is constant.

Proof. Your calculations for Exercise 4.3 prove that $\frac{dE}{dt} = 0$ and hence the total energy must be constsnt. This establishes the proposition. $\surd$ YEA

The formula for total energy of a shock absorber.

A shock absorber consists of a spring in a cyclinder filled with oil. The oil supplies a frictional force against the expansions and contractions of the spring. This frictional force opposes the motion and is proportional to the speed with which the spring is expanding or contracting. Let $v(t) = \dot{x}$ denote the velocity with which the (end of the) spring is moving at time t, and let $c > 0$ be the "frictional" constant of proportionality. Hence the frictional force at time t is $F_{friction} = -cv(t)$. Since the frictional force opposes the motion, it is in the opposite direction as the velocity. This is the reason for the minus sign before the c.

Combining Newton's Law, Hooke's Law, together with this frictional force, the equation for a shock absorber is

$$m\ddot{x} = \sum F = -kx - cv(t) + b.$$

Proposition 4.4 *The equations of motion for the basic shock absorber, with no external forces acting, is:*

$$m\ddot{x} + c\dot{x} + kx = b,$$

where $m, k, c \geq 0$. *The total energy of the system is monotone decreasing.*

Proof. Your calculations for Exercise 4.4 prove that $\frac{dE}{dt} \leq 0$ and hence the total energy must be a monotone decreasing function of time. This establishes the proposition.

Remark. The system described in Example 4.1 is an ideal system. Low level friction has been ignored, both the friction between the moving spring and the air it is moving through and the internal friction of the spring changing shape.

Exercises

Exercise 4.1 *Let* m *be a constant and let* $v(t)$ *be a function of time. Check that* $\frac{d(vmv)}{dt} = 2vm\,\dot{v}$ *and that* $\int vm\,\dot{v}\,dt = \frac{1}{2}vmv + c$.

Exercise 4.2 *Given that* $kx_{rest} = b$. *Show that*

$$\frac{1}{2}\,k\,x^2 - \frac{1}{2}\,k\,x_{rest}^2 - x\,b + x_{rest}\,b = \frac{1}{2}\,k\,(x - x_{rest})^2.$$

Exercise 4.3 *For constants* E_0, x_0, m *and* k, *it is given that:*

$$kx_0 = b \quad and \quad m\ddot{x} + kx = b.$$

$$E(t) = \frac{1}{2}m\,v^2 + \frac{1}{2}\,k\,(x - x_0)^2 + E_0.$$

Show that $\dot{E}(t) = 0$.

Exercise 4.4 *For constants* E_0, x_0, k, m *and* $c > 0$, *it is given that:*

$$kx_0 = b \quad and \quad m\ddot{x} + c\dot{x} + kx = b.$$

$$E(t) = \frac{1}{2}m\,v^2 + \frac{1}{2}\,k\,(x - x_0)^2 + E_0.$$

Show that $\dot{E}(t) \leq 0$.

Exercise 4.5 *Suppose that the system of Example 4.1 is turned so that it is vertical. Now gravity must be taken into consideration.*

(a) Calculate the potential energy.

(b) Show that the minimum potential energy occurs at the rest position.

(c) Calculate the equations of motion for this system of a block connected to two walls by two vertical linear springs, with no external forces acting.

(d) Calculate the total energy $E(t)$.

(e) By differentiating your answer to Part (c), show that $\dot{E}(t)$.

5 Complex conjugates.

COMPLEX NUMBERS: The concept of numbers is generalized to that of "complex numbers".

Definition. *Complex numbers* are numbers with the *standard form* $z = x + iy$, $\forall_{x,y \in \mathbf{R}}$ where $i^2 = -1$. The set of all complex numbers is the *complex plane* denoted by $\mathbf{C}$.

Complex numbers and polynomials are added and multiplied in the same way as real numbers and polynomials with the enhancement that $i^2 = -1$.

Complex numbers are not used for counting but then neither is the real number $\sqrt{2}$.

Complex numbers are mainly used in intermediary steps of problems with real numbers which have real solutions.

Complex conjugates

Definition. If $z = a+bi$ (a and b real numbers) then the *complex conjugate* of z is $\bar{z} = a-bi$.

Proposition 5.1 *A (possibly) complex number z is real if and only if $z = \bar{z}$.*

Remark. This proposition provides an equation for describing real numbers. It will be used to show that certain (possibly nonreal) complex numbers are real numbers.

Rules 5.2 (Arithmetic rules for complex conjugates)

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \ , \ \forall_{z_1, z_2 \in \mathbf{C}}$$

$$\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2} \ , \ \forall_{z_1, z_2 \in \mathbf{C}}.$$

Proposition 5.3 $z + \bar{z}$ *and* $i(z - \bar{z})$ $\forall_{z \in \mathbf{C}}$ *are real numbers.*

$$z \times \bar{z} \quad \textit{is a positive real number}, \ \forall_{z \in \mathbf{C}}, \ \textit{except}\ _{z=0)}.$$

Definition. Given a complex number $z = x + iy$ (x and y *real* numbers), then x is called the *real part* of z, and y is called the *imaginary part* of z. They are abbreviated as

$$Re\ z = x \quad \text{and} \quad Im\ z = y.$$

Observation 5.4 *Given a complex number $z = x + iy$ (x and y real numbers), then*

$$z + \bar{z} = 2\ Re\ z = 2x$$

$$z - \bar{z} = 2\ Im\ z = 2y$$

Observation 5.5 *Two complex numbers $z = a + bi$ and $w = c + di (a, b, c$ and $d \in \mathbf{R})$ are equal if and only if their real parts are equal and their imaginary parts are equal. That is*

$$a + bi = c + di \Leftrightarrow a = c \quad and \quad b = d \ (when\ a, b, c, d \in \mathbf{R}).$$

Since complex conjugation commutes with products, $\overline{z_1 \times z_2} = \bar{z}_1 \times \bar{z}_2$, it will also commute with positive integer powers:

$$\overline{z^2} = \overline{z \times z} = \bar{z} \times \bar{z} = (\bar{z})^2,$$

$$\overline{z^3} = \overline{z^2 \times z} = (\bar{z})^2\bar{z} = (\bar{z})^3,$$

etc.

Example 5.6 *If a (non-real) complex number z_0 is a solution to the real polynomial equation: $x^4 + x^3 + 2x^2 + 2x + 4 = 0$, then so is its complex conjugate.*

First, we translate this statement into equations.

The complex number z_0 being a solution means that it satisfies the given equation, that is: It is **given** that:

$$0 = z_0^4 + z_0^3 + 2z_0^2 + 2z_0 + 4.$$

That its complex conjugate is also a solution means: the **to show** is:

$$0 = \bar{z}_0^4 + \bar{z}_0^3 + 2\bar{z}_0^2 + 2\bar{z}_0 + 4.$$

Calculations. Start with the "given" equation, taking the complex conjugate of both sides yields:

$$0 = \bar{0} = z_0^4 + z_0^3 + 2z_0^2 + 2z_0 + 4.$$

Since the complex conjugate commutes with sums:

$$z_0^4 + z_0^3 + 2z_0^2 + 2z_0 + 4 = \overline{z_0^4} + \overline{z_0^3} + \overline{2z_0^2} + \overline{2z_0} + \bar{4}.$$

Since the complex conjugate commutes with products (and powers)

$$\overline{z_0^4} + \overline{z_0^3} + \overline{2z_0^2} + \overline{2z_0} + \bar{4} = \bar{z}_0^4 + \bar{z}_0^3 + 2\bar{z}_0^2 + 2\bar{z}_0 + 4.$$

Combining these equations:

$$0 = \bar{z}_0^4 + \bar{z}_0^3 + 2\bar{z}_0^2 + 2\bar{z}_0 + 4.$$

Thus $\bar{z}_0$ also satisfies the equation. $\surd$ YEA

Remark. This proof generalizes to any *real* polynomial equation.

Example 5.7 *Convert* $\frac{1}{4+3i}$ *into standard* $(a+bi)$ *form.*

Calculations:

$$\frac{1}{4+3i} = \frac{1}{4+3i} \times \frac{4-3i}{4-3i} = \frac{4-3i}{14+9} = \frac{4}{25} - \frac{3}{25}i.$$

Remark. Note that $4-3i = \overline{4+3i}$. We took advantage of the fact that $z\,\bar{z}$ is always a real number (Proposition 5.3).

Exercises

Puzzle C.5.1. Let $E(x)$ be an unknown function which satisfies the equation

$$E(x+y) = E(x)E(y), \ \forall_{\text{numbers } x \text{ and } y}. \tag{C.1}$$

It is also known that $E(9) = 10$.

(a) Find $E(18)$ and $(E(27)$.

(b) What other values of $E(x)$ can you find?

Puzzle C.5.2. Let $E(x)$ be an unknown function which satisfies the equation (C.1)

It is also known that $E(8) = 10$.

(a) Find $E(0)$.

Setup: Ask the question: What is zero?

Answer: 0 is the additive identity number, that is $8 + 0 = 8$.

Apply (C.1) to "$8 + 0$", that is set $x = 0$ and set $y = 8$. Then solve for $E(0)$.

(b) Find $E(-8)$.

Setup: Ask the question: What is -8?

Answer: -8 is the additive inverse of 8, that is $8 + (-8) = 0$.

Apply (C.1) to "$8 + (-8)$" and then solve for $E(-8)$.

(c) What other values of $E(x)$ can you find?

Puzzle C.5.3. Let $E(x)$ be an unknown function which satisfies the equation

$$E(x+y) = E(x)E(y), \ \forall_{\text{numbers } x \text{ and } y}.$$

It is also known that $E(1) = 3$. Show that $E(x) \neq 0, \forall_{\text{numbers } x}$, that is show that $E(x)$ is never zero.

Exercise C.5.4. Let $E(z)$ be any function which satisfies the equation

$$E(z+w) = E(z)E(w), \ \forall \text{ (complex) numbers } z \text{ and } w.$$

(a) Show that $E(2z) = (E(z))^2, \ \forall$ (complex) numbers z.

Setup: Ask the question: What is $2z$?

Answer: $2z = z + z$.

Apply (C.1) to "$z + z$". Then simplify.

(b) Show that $E(3z) = E(z)^3$, $\forall$ (complex) numbers z.

(c) Show that $E(7z) = E(z)^7$, $\forall$ (complex) numbers z.

(d) Explain why $E(nz) = E(z)^n$, $\forall$ (complex) numbers z and $\forall$ positive integers n.

Exercise C.5.5. Let $E(z)$ be any function which satisfies (C.1) and also $E(1) \neq 0$.

(a) Show that $E(0) = 1$.

Setup: Ask the question: What is zero?

Answer: 0 is the additive identity number, that is $z + 0 = z$.

Apply (C.1) to "$z + 0$", that is, set $w = 0$ and set $z = z_0$, where z_0 is a number such that $E(z_0) \neq 0$. Then simplify to $E(0) = 1$.

(b) Show that $E(-z) = \frac{1}{E(z)}$.

Setup: Ask the question: What is $-z$?

Answer: $-z$ is the additive inverse of z, that is $z + (-z) = 0$.

Apply (C.1) to "$z + (-z)$" and then simplifiy to $E(-z) = 1 \div E(z)$ using the fact that $E(0) = 1$ which you have just shown.

(c) Show that $E(z) \neq 0$.

Setup: Use the fact that $E(z)E(-z) = 1$ which appeared in your proof that $E(-z) = \frac{1}{E(z)}$.

Exercise 5.1 *Prove that*

$$\cos(\theta + \phi) + i\sin(\theta + \phi) = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi),\ \forall\ \theta \ \textit{ and } \ \phi \in \mathbf{R}.$$

Setup: (i) Use sum formulas for trig functions to expand $\cos(\theta + \phi) + i\sin(\theta + \phi)$.

(ii) Multiply out $(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$. Simplify.

(iii) Check that your answers to (i) and (ii) are equal.

Exercise 5.2 *Prove that* $\frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$, $\forall_{\theta \in \mathbf{R}}$.

Setup: Convert $\frac{1}{\cos\theta + i\sin\theta}$ to standard form.

Exercise 5.3 *Prove that* $\frac{d(\cos\theta + i\sin\theta)}{d\theta} = i(\cos\theta + i\sin\theta)$, $\forall_\theta$.

6 Complex Exponentials.

Definition. The function $e^{i\theta}$ is defined by $e^{i\theta} = \cos\theta + i\sin\theta, \forall$ real numbers θ, and e^{x+iy} is defined by $e^{(x+iy)} = e^x e^{iy}, \forall$ real numbers x and y.

Remark. There is *form* and there is *substance*. Defining $e^{i\theta}$ *merely* provides the *form* of an exponential; by itself, it does *not* provide the substance. We must check that it also has the substance and properties of an exponential. Yes! it would be foolish and misleading to use the exponential form for something that did *not* have the substance also.

The basic substance of an exponential is the rule: $x^{(a+b)} = x^a x^b$.

The important calculus rule for the exponential function is

$$\frac{\mathrm{d}e^x}{dx} = e^x$$

and hence (by the chain rule)

$$\frac{\mathrm{d}e^{cx}}{dx} = ce^{cx}, \forall \text{ numbers } c.$$

We will now check these rules for $e^{i\theta}$, when θ is a *real* number.

Rules 6.1 *(i)* $e^{i(\theta+\phi)} = e^{i\theta+i\phi} = e^{i\theta}e^{i\phi}, \forall$ *real numbers* θ *and* ϕ.

(ii) $\frac{de^{i\theta t}}{dt} = i\theta e^{i\theta t}, \forall$ *real numbers* θ.

Proof. (i) Is $e^{i(\theta+\phi)} = e^{i\theta+i\phi} \stackrel{?}{=} e^{i\theta}e^{i\phi}$? The definition says that:

$$e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$$

and

$$e^{i\theta}e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi).$$

As Exercise 5.1, you checked that these two expressions are equal. $\surd$ YEA

Proof. (ii) You checked this as Exercise 5.3, $\surd$ YEA

Rules 6.2 *(i)* $e^{z+w} = e^z e^w, \forall$ *complex numbers* z *and* w.

(ii) $\frac{de^{ct}}{dt} = ce^{ct}, \forall$ *complex numbers* c.

We will use Rules 6.1 and the definition of the complex exponenetial in order to establish Rules 6.2.

Proof. (i). Set $z = a + \theta i$ and $w = c + \phi i$ where $a,\ c,\ \theta$ and ϕ are all *real* numbers. Then

$$e^z e^w = e^{a+\theta i} e^{c+\phi i} = e^a e^{\theta i} e^c e^{\phi i}, \quad \forall \textit{ real numbers } a,\ c,\ \theta\ \phi$$

by the definitions. The associative law for multiplication permits us to rearrange the order of the factors:

$$e^a e^{\theta i} e^c e^{\phi i} = e^a e^c e^{\theta i} e^{\phi i}.$$

Using the basic rule for exponentials of real numbers and Rule 6.1 (i), one sees that:

$$e^a e^c e^{\theta i} e^{\phi i} = e^{a+c} e^{(\theta+\phi)i}$$

Finally, the definition of complex exponentials provides:

$$e^{a+c} e^{(\theta+\phi)i} = e^{a+c+(\theta+\phi)i}$$

The associative law for addition permits us to rearrange the order of the numbers in the exponent:

$$e^{a+c+(\theta+\phi)i} = e^{(a+i\theta)+(c+i\phi)} = e^{z+w} \quad \forall \textit{ real numbers } a,\ c,\ \theta\ \phi.$$

Combining all these equations yields the desired result: $e^{z+w} = e^z e^w, \forall$ complex numbers z and w.
√

Proof. (ii). Set $c = a + i\theta$. Then $e^{ct} = e^{(a+i\theta)t} = e^{at} e^{i\theta t}$. Hence $\frac{de^{ct}}{dt} = \frac{de^{at} e^{i\theta t}}{dt}$. Applying the product rule for derivatives and Rule 6.1 (i) yields:

$$\frac{de^{at} e^{i\theta t}}{dt} = ae^{at} e^{i\theta t} + i\theta e^{at} e^{i\theta t}$$

Factoring and using $c = a + i\theta$ yields

$$ae^{at} e^{i\theta t} + i\theta e^{at} e^{i\theta t} = (a + i\theta) e^{(a+i\theta)t} = ce^{ct}.$$

Combining all these equations yields the desired result: $\frac{de^{ct}}{dt} = ce^{ct}, \forall$ complex numbers c.
√

Complex exponential formulas for sine and cosine functions.

We note that:

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - \sin\theta.$$

Thus we have

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Adding:

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \tag{C.2}$$

Subtracting:

$$e^{-i\theta} - e^{-i\theta} = 2i\sin\theta. \tag{C.3}$$

Since these formulas are valid for all values of θ, they remain valid when θ is replaced by $n\theta$:

$$2\cos n\theta = e^{in\theta} + e^{-in\theta}$$

$$2i\sin n\theta = e^{in\theta} - e^{-in\theta}.$$

Thus

$$\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2} \tag{C.4}$$

and

$$\sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i} = -\frac{i}{2}\left(e^{in\theta} - e^{-in\theta}\right), \tag{C.5}$$

since $\frac{1}{i} = -i$.

Using complex exponentials to simplify and integrate trig functions.

Example 6.3 *Write* $f(\theta) = \cos\theta\sin 2\theta$ *as a linear combination of sines and/or cosines.*

Calculations. It is complex numbers, in the form of Equations (C.2) - (C.5), to the rescue:

$$\begin{aligned}
f(\theta) &= \cos\theta\sin 2\theta \\
4if(\theta) &= (2\cos\theta)(2i\sin 2\theta) \\
&= (e^{i\theta} + e^{-i\theta})(e^{2i\theta} - e^{-2i\theta}) \\
&= e^{3i\theta} - e^{-3i} + (e^{i\theta} - e^{-i\theta}) \\
4f(\theta) &= 2\sin 3\theta + 2\sin\theta.
\end{aligned}$$

Thus

$$\cos\theta\sin 2\theta = f(\theta) = \tfrac{1}{2}\left(\sin 3\theta + \sin\theta\right).$$

Query: How can we check that this (strange) formula is correct?

Answer: Graph $y_1 = \cos\theta\sin 2\theta$ *and* $y_2 = \frac{1}{2}(\sin 3\theta + \sin\theta)$ on your graphing calculator and check that the two graphs are identical.

Better Answer: Graph $\cos\theta\sin 2\theta - \frac{1}{2}(\sin 3\theta + \sin\theta)$ on your graphing calculator and check that the graph is identically zero.

Let us see what the steps in this procedure were.

Step#1 Convert a function of sines and/or cosines into complex exponentials using Equations (C.2) - (C.5).

Step#2 Multiply-out the complex exponentials.

Step#3 Regroup the terms as sine and/or cosine terms and then convert back into sine and/or cosine terms.

Step#4 Check answer on graphing calculator.

The result is a trigonometric identity which shows that the original sine/cosine function is equal to a linear combination of sine and cosine terms. This is useful because plain sine and cosine terms are easy to integrate.

Example 6.4 *Evaluate* $\int \cos\theta \sin 2\theta \, d\theta$. *(Suppose that no table of integrals or trig identities is handy).*

Calculations. It is complex exponentials to the rescue, in the form of Equations (C.2) - (C.5). One uses complex exponenetials to calculate that $\cos\theta\sin 2\theta = \frac{1}{2}(\sin 3\theta + \sin\theta)$. Then

$$\int \cos\theta \sin 2\theta \; d\theta = \frac{1}{2}\int (\sin 3\theta + \sin\theta) \; d\theta = -\frac{1}{6}\cos 3\theta - \frac{1}{2}\cos\theta + \; \text{c}.$$

Exercise

Exercise 6.1 *(a) Convert* $f(\theta) = \sin^2 3\theta \cos 4\theta$ *into a linear combination of sine and/or cosine terms. Check your answer with a graphing calculator or computer.*

(b) Evaluate $\int f(\theta) d\theta$.

7 Pascal's triangle and the binomial expansion.

We will present Pascal's triangle as an easy way to compute the coefficients of the binomial expansion of $(a+b)^n$, $n = 2, 3, 4, \cdots$.

Pascal's triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
$$

A number not on one of the sides is the sum of the two numbers above it. The numbers in this table form the beginning of *Pascal's triangle*, named after Blaise Pascal (1623-1662) because he wrote an influential treatise about them. Pascal's triangle and binomial coefficients were well known in Asia, for centuries before Pascal, but he had no way to know that.

Binomial coefficients get their name from the *binomial theorem*, which describes the powers of the binomial expression $a + b$. Let's look at the smallest cases of this theorem:

Binomial Triangle

$$
\begin{gathered}
(a+b)^0 = \mathbf{1} \\
(a+b)^1 = \mathbf{1}a^1 + \mathbf{1}b^1 \\
(a+b)^2 = \mathbf{1}a^2 + \mathbf{2}a^1b^1 + \mathbf{1}b^2 \\
(a+b)^3 = \mathbf{1}a^3 + \mathbf{3}a^2b^1 + \mathbf{3}a^1b^2 + \mathbf{1}b^3 \\
(a+b)^4 = \mathbf{1}a^4 + \mathbf{4}a^3b^1 + \mathbf{6}a^2b^2 + \mathbf{4}a^1b^3 + \mathbf{1}b^4. \\
(a+b)^5 = \mathbf{1}a^5 + \mathbf{5}a^4b^1 + \mathbf{10}a^3b^2 + \mathbf{10}a^2b^3 + \mathbf{5}a^1b^4 + \mathbf{1}b^5 \\
\vdots
\end{gathered}
$$

Note that each of these equations has the form:

$$(a+b)^n = 1a^n + na^{n-1}b^1 + \cdots + na^1b^{n-1} + 1b^n$$

and that the coefficients (in boldface type) form a Pascal triangle. Also, when reading from any term to its neighbor on the right, the exponent of a goes down by one and the exponent of b goes up by one; (this also occurs for the end terms $1a^n = 1a^nb^0$ and $1b^n = 1a^0b^n$).

Let us see why the coefficients follow Pascal's triangle. You may check the binomial triangle formulas for $(a+b)^0$ and $(a+b)^1$ by just looking at them. You may check the formulas for $(a+b)^2$ and $(a+b)^3$ by multiplying out (Exercise 7.1). Having checked out the binomial triangle formulas for $(a+b)^3$ (Exercise 7.1), we will now use it to check out the formula for $(a+b)^4$.

Example 7.1 *Assuming that* $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, *show that the above formula for* $(a+b)^4$ *is valid.*

Calculations. We note that

$$(a+b)^4 = (a+b)(a+b)^3 = a(a+b)^3 + b(a+b)^3.$$

Using this equation and the formula for $(a+b)^3$, we calculate:

$$\begin{array}{rl} a(a+b)^3 = & \mathbf{1}a^4 + \mathbf{3}a^3b^1 + \mathbf{3}a^2b^2 + \mathbf{1}ab^3 \\ +b(a+b)^3 = & \qquad\quad \mathbf{1}a^3b + \mathbf{3}a^2b^2 + \mathbf{3}a^1b^3 + \mathbf{1}b^4 \\ \hline (a+b)^4 = (a+b)(a+b)^3 = & \mathbf{1}a^4 + (\mathbf{3}+\mathbf{1})a^3b^1 + (\mathbf{3}+\mathbf{3})a^2b^2 + (\mathbf{1}+\mathbf{3})a^1b^3 + \mathbf{1}b^4. \end{array} \quad \surd$$

Notice how each coefficient of the expansion of $(a+b)^4$ is the sum of two coefficients of the expansion of $(a+b)^3$.

In Exercise 7.1, you will use the binomial triangle formula for $(a+b)^4$ while you check the formula for $(a+b)^5$. Continuing in this manner, one can establish the binomial triangle formulas.

Now let us modify the binomial triangle in order to obtain formulas for the powers $(a-b)^n$. Pascal's triangle enables us to find powers of a sum; so we must convert $a-b$ into a sum. We do this thus: $a-b = a+(-b)$, hence $(a-b)^n = (a+[-b])^n$. Applying the binomial triangle to $(a+[-b])^n$, one obtains:

$$\begin{gathered} (a-b)^0 = \mathbf{1} \\ (a-b)^1 = \mathbf{1}a^1 + \mathbf{1}(-b)^1 \\ (a-b)^2 = \mathbf{1}a^2 + \mathbf{2}a^1(-b)^1 + \mathbf{1}(-b)^2 \\ (a-b)^3 = \mathbf{1}a^3 + \mathbf{3}a^2(-b)^1 + \mathbf{3}a^1(-b)^2 + \mathbf{1}(-b)^3 \\ (a-b)^4 = \mathbf{1}a^4 + \mathbf{4}a^3(-b)^1 + \mathbf{6}a^2(-b)^2 + \mathbf{4}a^1(-b)^3 + \mathbf{1}(-b)^4. \\ (a-b)^5 = \mathbf{1}a^5 + \mathbf{5}a^4(-b)^1 + \mathbf{10}a^3(-b)^2 + \mathbf{10}a^2(-b)^3 + \mathbf{5}a^1(-b)^4 + \mathbf{1}(-b)^5. \\ \vdots \end{gathered}$$

We note that $(-b)^n = (-1)^n b^n = b^n$, when n is an *even* integer and that $(-b)^n = -b^n$, when n is an *odd* integer. This simplifies the triangle to:

Binomial Triangle for $(a-b)^n$

$$\begin{gathered} (a-b)^0 = \mathbf{1} \\ (a-b)^1 = \mathbf{1}a - \mathbf{1}b \\ (a-b)^2 = \mathbf{1}a^2 - \mathbf{2}ab + \mathbf{1}b^2 \\ (a-b)^3 = \mathbf{1}a^3 - \mathbf{3}a^2b + \mathbf{3}ab^2 - \mathbf{1}b^3 \\ (a-b)^4 = \mathbf{1}a^4 - \mathbf{4}a^3b + \mathbf{6}a^2b^2 - \mathbf{4}ab^3 + \mathbf{1}b^4. \\ (a-b)^5 = \mathbf{1}a^5 - \mathbf{5}a^4b + \mathbf{10}a^3b^2 - \mathbf{10}a^2b^3 + \mathbf{5}ab^4 - \mathbf{1}b^5 \\ \vdots \end{gathered}$$

Notice that the only differences between the Binomial Triangles for $(a+b)^n$ and for $(a-b)^n$ are the alternating minus signs.

We may use Pascal's Triangle, together with (C.2) - (C.5) to convert some complicated sine and cosine functions like $\sin^8\theta$ into a linear combination of sines and cosines. You will do this as Exercise 7.3.

Exercises

Exercise 7.1 *Check the binomial triangle formulas for* $(a+b)^n,\ n=0,1,2$ *and* 3.

Exercise 7.2 *Assuming the Pascal triangle formula for* $(a+b)^4$, *show that the Pascal triangle formula for* $(a+b)^5$ *is valid.*

You may use the method of Example 7.1.

Exercise 7.3 *(a) Convert* $f(\theta)=\sin^8\theta$ *into a linear combination of sine and/or cosine terms. Check your answer with a graphing calculator or computer.*

(b) Evaluate $\int f(\theta)d\theta$.

8 Absolute values of real and complexnumbers.

Exponential decay to steady state.

Proposition 4.9 (Exponential decay to steady state) *For constant numbers,* r *and* k*, the general solution to* $\dot{x} = rx(t) + k$ *is* $x = c + Ae^{rt}$*,* $\forall_A$*, where* $c = -k/r$ *is a constant solution.*

Remark. The constant solution is often called the *steady state solution* because its "state" is "steady", that is its "state" does not change as time changes. I named this "exponential decay to steady state" because the solution is a constant solution plus/minus an exponentially decaying term. For moderately negative or large negative values of r, many solutions quickly approach the constant steady state solution and stay close forever after.

Memorize this proposition; "exponential decay to steady state" occurs a lot. The Ae^{rt}, $\forall_A$-term is the standard exponential decay formula. Do not memorize the constant $c = -k/r$-term; calculate it quickly by guessing $x = c$ is a solution, plug into the dif. eq. and quickly solve for c.

Example 4.10 (Exponential decay to the ambient temperature) *An object is place in a space (for example, a refrigerator) with a constant temperature. Newton's Law of Cooling is applicable. After a while, the temperature of the placed object becomes close to that of the space and stays close forever after. In particular, if* $\theta(t)$ *is the temperature of the object at time* t*,* T *is the temperature of the ambient space and* $k > 0$ *is the "constant", then Newton's Law of Cooling is:* $\dot{\theta} = -k(\theta(t) - T)$*. A consequence is that*

$$\theta(t) = T + (\theta(0) - T)e^{-kt} \quad \textit{and} \quad \textit{hence} \quad \lim_{t\to\infty} \theta(t) = T$$

Example 4.11 (Exponential decay to the pollution fraction) *Liquid (mostly water) is flowing into a lake at a constant rate,* r*. The lake empties into a river at the same constant rate* r*. Since 1900, factories have added a pollutant to the lake at a constant rate,* r_p*. After a while, the pollution part of the liquid in the lake becomes close to that of the pollution fraction* r_p/r*, and stays close forever after.*

In particular, if $P(t)$ *is the fraction of pollutant in the lake at time* t*, then*

$$\lim_{t\to\infty} P(t) = r_p/r$$

Setup: Let $Q(t)$ be the amount of pollutant in the lake at time t, then $P(t) = Q(t)/\{Vol\ of\ lake\}$

Example 4.12 (Exponential decay to the voltage of the battery) *A single simple loop contains a battery (with constant voltage* V_0*), a resistor (with resistance* R*) and a capacitor (with capacitance* C*). Let* $q = q(t)$ *be the charge (amount of electrons) at time* t*, on the plates of the capacitor; let* $i = \dot{q} = dq/dt$ *denote the electrical current going through the loop. The voltage across the capacitor is* $V_c = q/C$*; the voltage across the resistor is* $V_R = iR$*. Kirchoff's Voltage Law says that the sum of the voltage drops around a loop is zero, hence:* $V_0 + V_c + V_r = 0$*. By calculation,* $\lim_{t\to\infty} V_c(t) = -V_0$*.*

Exercises

Exercise 4.5 *Prove Proposition II.4.9.*

Setup: Do a proof by calculation. Solve the equation: $\dot{x} = rx(t) + k$, when r and k are constant numbers.

Exercise 4.6 *Prove Example II.4.10.*

Setup: Do a proof by calculation. Use Proposition II.4.9 to quickly obtain the general solution to the equation: $\dot{\theta} = -k(\theta(t) - T)$, when T and k are constant numbers. Then find the constant "A" by using the initial condition $\theta = \theta(0)$ when $t = 0$.

Exercise 4.7 *Prove Example II.4.12.*

Setup: Do a proof by calculation. The conclusion (to show) is: $\lim_{t\to\infty} V_c(t) = -V_0$. The given is all the other equations stated in the example. Convert the equation $V_0 + V_c + V_R = 0$ into a differential equation about q. Use Proposition II.4.9 to solve for $q(t)$. Then prove (calculate) that $\lim_{t\to\infty} V_c(t) = -V_0$.