Linear Differential Equations

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These notes are a concise understanding-based presentation of the basic linear-operator aspects of solving linear differential equations.

We will be solving the equations of motion of a shock absorber $x''(t)+11x'(t)+28x(t) = 0$ and similar linear differential equations.

Notation. We will use just "f" as a shorthand for "$f(x)$" or "$f(t)$" and often $x = x(t)$.

Dots indicate differentiation with respect to *time*; for example $\dot{y} = \frac{dy}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$.

The set of all functions with continuous second derivatives is denoted by $\mathbf{C}^2$. Writing $f \in \mathbf{C}^2$ is mathematical shorthand for writing that the function f has second derivatives and that f'' is a continuous function.

Operators and Linear Combinations

Example 1 . *The function* $5\sin x + 7e^x$ *is called a "linear combination" of the two functions* $\sin x$ *and* e^x*.*

Definition. A *linear combination* of two functions f and g is any function with the form $Af + Bg$, $\forall_{numbers\ A}$ and $_B$.

The function $\sin x = 1\sin x + 0e^x$ is considered a linear combination of the two functions $\sin x$ and e^x. [2] So is the zero function, since $0 = 0\sin x + 0e^x$.

The function $5(\sin x)e^x$ is a "combination" of the two functions $\sin x$ and e^x, but it is *not* a linear combination.

Remark. Now for mind stretching and abstracting time. The concept of a function will be generalized to permit the inputs and outputs to be functions instead of just numbers.

Definition. An *operator* is a function whose inputs and outputs are functions. Its domain and range are sets of functions.

Example 2 . **(a)** *The derivative operator is defined by the formula:* $D(f) = f'(x)$*. Thus* $D(\sin x) = \cos x$*. (The domain is the set of differentiable functions.)*

(b) *Given* $L(f) = f'' + 11f' + 28f$*. For* $f(x) = \sin 2x$*;*

$$L(\sin 2x) = (\sin 2x)'' + 11(\sin 2x)' + 28(\sin 2x) = -4\sin 2x + 22\cos 2x + 28\sin 2x.$$

[2] In standard (non-mathematical) English, the function $\sin x$ is not a combination (linear or otherwise) of the functions $\sin x$ and e^x. This is one of the situations, in which mathematical jargon uses words differently than standard English. Otherwise, we would have to have use the ackward phrase: "a linear combination of the two functions $\sin x$ and e^x or a multiple of $\sin x$ or e^x or just zero" instead.

Remark. That is, just substitute/plug-in $\sin 2x$ for f in the formula that defines $L(f)$.

EXERCISES

Exercise 1 . *Let* $L(f) = f'' + 11f' + 28f$. *Calculate* $L(e^x)$, $L(5e^x)$, $L(\sin x)$, *and* $L(e^x + \sin x)$.

Find a formula which connects $L(5e^x)$ *and* $L(e^x)$.

Find a formula which connects $L(e^x + \sin x)$ *with* $L(e^x)$ *and* $L(\sin x)$.

What is special about the functions e^x *and* $\sin x$*? Can you replace* e^x *and* $\sin x$ *by other functions and still have the two formulas, you just found, remain valid? List some pairs of other functions for which the two formulas remain valid.*

What is special about the operator $L(f) = \{f'' + 11f' + 28f\}$*? Can you replace* $L(f) = \{f'' + 11f' + 28f\}$ *by another operator and have the two formulas remain valid? Guess some other operators for which the two formulas remain valid.*

What is special about the number 5 *? Guess some other numbers for which an analogous* [3] *formula is valid.*

Guess a general rule which contains and generalizes your answers to these questions.

Exercise 2 *(a) Let* $L(f) = f''(x) + 7f'(x) - 11f(x)$, $\forall_{f \in \mathbf{C}^2}$. *Prove that* $L(e^{7x})$ *is a multiple of* e^{7x}. *Prove that* $L(\sin 7x)$ *is a linear combination of* $\sin 7x$ *and* $\cos 7x$. *Prove that* $L(\cos 7x)$ *is also a linear combination of* $\sin 7x$ *and* $\cos 7x$.

(b) Let $L(f) = 7f''(x) + 11f(x)$, $\forall_{f \in \mathbf{C}^2}$. *Prove that* $L(\sin 7x)$ *is a multiple of* $\sin 7x$.

(c) Let $L(f) = f''(x) + 7f'(x) - 11f(x)$. *Prove that* $L(constant)$ *is always a constant. That is, if* $f(x) = c$, *for any unspecified constant* c, *then* $L(f)$ *is also a constant, perhaps a different one.*

Set-up: *Calculate* $L(e^{7x})$ *and* $L(\sin 7x)$.

Exercise 3 . *Using the results from the previous exercise as data, guess more general situations where similar things happen. Check your guesses.*

Exercise 4 *Discover some functions* $y = f(x)$ *which satisfy this equation:*

$$y'' + 11y' + 28y = 0.$$

(a) Is $y = e^{-4x}$ *a solution ? Substitute in and find out.*

(b) Which of these functions are also solutions: (i) $y = e^{-7x}$ *, (ii)* $y = e^{4x}$ *, (iii)* $y = \sqrt{2}e^{-4x} + 97e^{-7x}$*?*

[3] which have the same form

Exercise 5 . *(a) For (iii) of part (b) of the previous exercise, is there anything special about the numbers* $\sqrt{2}$ *and* 97*? Can you find other numbers which will also yield solutions? Guess. Check your guess. (b) State a general rule which describes many solutions? Check your rule.*

Exercise 6 *(a) For Exercise 4, is there anything special about the numbers -4 and -7?*

(b) Using your answers to Exercise 4 as a guide, find two solutions to

$$y'' + 6y' + 5y = 0.$$

(c) Using your answers to Exercise 4 as a guide, find other solutions? Guess. Check your guess.

Exercise 7 *Find* $\int te^{2t}dt$*, without using integration by parts.*

Set-up: Use a hit-and-miss educated-guessing strategy to find a function, whose derivative is te^{2t}.

Linear Operators

Your results for Exercises 1 and 2 motivate the following definition:

Definition. An operator L is *linear* if

(i) $L(f+g) = L(f) + L(g)$, $\forall$ functions $f, g \epsilon$ domain of L.

(ii) $L(Af) = AL(f)$, $\forall$ functions $f \epsilon$ domain of L and $\forall$ numbers A.

Remark. Often, the phrase "ϵ domain of L" will be understood and not stated.

Example 3 *The derivative operator* $D(f) = \frac{df(x)}{dx}$ *and the second derivative operator* $D^2 = D \circ D$ *are linear operators.*

Proof. As you know

$$D(f+g) = (f+g)' = f' + g' = Df + Dg, \quad \forall \text{differentiable functions } f \text{ and } g.$$

$$D(cf) = (cf)' = cf' = cDf, \quad \forall \text{numbers } c \text{ and } \forall \text{differentiable functions } f.$$

$\checkmark$

For second derivatives (which are first derivatives of first derivatives):

$$D^2(f+g) = D(D(f+g)) = D(f'+g') = f'' + g'' = D^2f + D^2g,$$

$\forall$ twice differentiable functions f and g.

Example 4 *All operators of the form*

$$L(x) = \ddot{x} + a\dot{x} + bx \quad and \quad L(y(x)) = y'' + a(x)y' + b(x)y,$$

(where a *and* b *are numbers and* $a(x)$ *and* $b(x)$ *are functions) are linear operators.*

Outline of Proof. Let $L(y) = y'' + a(x)y' + b(x)y$ be an operator. We check rule (i) for linear operators:

$$L(f+g) = (f+g)'' + a(x)(f+g)' + b(x)(f+g)$$
$$L(f+g) = f'' + g'' + a(x)f' + a(x)g' + b(x)f + b(x)g$$
$$L(f+g) = [f'' + a(x)f' + b(x)f] + [g'' + a(x)g' + b(x)g]$$
$$L(f+g) = L(f) + L(g).$$

$\surd$

Similarly, one can check that $L(cy) = cL(y)$, which is rule (ii) for linear operators. $\surd$

Having checked the two rules for a linear operator, we conclude that $L(y) = y'' + a(x)y' + b(x)y$ is always a linear operator. $\surd$

Similarly, (from Math 241), partial derivatives, the gradient, the divergence and the curl are all linear operators, since, for example, $\overline{\nabla}(f+g) = \overline{\nabla}f + \overline{\nabla}g$.

Proposition 5 *If* L *is a linear operator, then*

$$L(Af + Bg) = AL(f) + BL(g),$$

$\forall_{functions\ f,g\epsilon\ domain\ of\ L}$ *and* $\forall_{numbers\ A\ and\ B}$.

Proof. (We will be setting $f_1 = Af$ and $g_1 = Bg$.)

$$\begin{aligned} L(Af+Bg) &= L(f_1+g_1) \overset{\text{by (i)}}{=} L(f_1) + L(g_1) \\ &= L(Af) + L(Bg) \overset{\text{by (ii) twice}}{=} AL(f) + BL(g). \end{aligned}$$

Remark. As usual mathematicians state rules in terms of two numbers or functions when they apply to any finite number:

Proposition 6 *If* L *is a* linear operator *then*

$$L\left(\sum_i A_i f_i\right) = \sum_i A_i L(f_i),$$

$$\forall_{finite\ sums} \quad and \quad \forall_{numbers A_i} \quad and \quad \forall_{functions f_i}.$$

Example 7 *The* only *linear functions of a single variable are* $f(x) = ax, \forall_{\ numbers\ a}$.

Remark. This is in sharp contrast to the definition of linear functions in calculus class; therein the "affine" functions $f(x) = ax + b$ are called "linear" functions.

Example 8 *The following functions are* not *linear:*

$$
\begin{aligned}
f(x) = x + 1 &\ \textit{since}\ 2 = f(1+0) \neq f(1) + f(0) = 3 \\
\sqrt{x} &\ \textit{since}\ \sqrt{a+b} \neq \sqrt{a} + \sqrt{b} \\
\tfrac{1}{x} &\ \textit{since}\ \tfrac{1}{a+b} \neq \tfrac{1}{a} + \tfrac{1}{b} \\
\log x &\ \textit{since}\ \log(a+b) \neq \log a + \log b \\
e^x &\ \textit{since}\ e^{a+b} \neq e^a + e^b \\
x^2 &\ \textit{since}\ (a+b)^2 \neq a^2 + b^2 \\
\sin x &\ \textit{since}\ \sin(\theta + \phi) \neq \sin\theta + \sin\phi.
\end{aligned}
$$

The linear operator way to integrate $\int te^{2t}dt$ – or what to do if you have forgotten integration by parts.

Example 9 (linear operator way to integrate) *Evaluate:* $\int te^{2t}dt$.

We need a function whose derivative contains te^{2t}. Try

$$
\begin{aligned}
x_1 &= te^{2t} \\
\dot{x}_1 &= 2\underbrace{te^{2t}}_{\text{needed}} + \underbrace{e^{2t}}_{\text{not needed}}
\end{aligned}
$$

Try $x_2 = e^{2t}$. Then $\dot{x}_2 = 2e^{2t}$.

Set

$$
\begin{aligned}
x_3 &= 2x_1 - x_2 \\
\dot{x}_3 &= 2\dot{x}_1 - \dot{x}_2 = 4te^{2t}
\end{aligned}
$$

Try

$$x_4 = \frac{1}{4}x_3 \Rightarrow \dot{x}_4 = \frac{1}{4}\dot{x}_3 = te^{2t}, \text{ as needed.}$$

Hence

$$x_4 = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + c = \int te^{2t}dt.$$

√ YEA

EXERCISES

Exercise 8 *State the defining equations for a linear operator.*

Exercise 9 *Quickly, explain why the function* e^x *is not a linear function. State which of the defining equations for a linear operator, is not satisfied.*

Exercise 10 *The "LaPlace transform" is the operator defined by* $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$*, when this integral converges. Prove that the "LaPlace transform" is a linear operator.*

Exercise 11 *Let* $L(f)$ *be a linear operator. Given two special functions,* $f(x)$ *and* $g(x)$*, such that* $L(f) = 0 = L(g)$ *and given* $h(x) = 3f(x) + 7g(x)$*, prove that* $L(h) = 0$*.*

Exercise 12 *Generalize the result of the last exercise. That is, using the calculations as data, predict something useful.*

Exercise 13 *Restate the result of the preceding two exercises in "verbal" form, that is state it using as few math symbols or equations as possible.*

Exercise 14 *Let* $L(x) = 4\ddot{x} + 8\dot{x} + 9x$*. Check that* $L(e^{-t}\cos\sqrt{\frac{5}{2}}t) = 0$*. This checks that* $x(t) = e^{-t}\cos\sqrt{\frac{5}{2}}t$ *is a solution to the equation:* $4\ddot{x} + 8\dot{x} + 9x = 0$*.*

Remark. Your results from Exercises 1, 2 and 3 are useful data for the next 5 exercises.

Exercise 15 *Let* $L(x) = \ddot{x} + 11\dot{x} + 28x$*. Find a function* $x_1(t)$ *such that* $L(x_1) = 108e^{2t}$*.*

Exercise 16 *Let* $L(x) = \ddot{x} + 11\dot{x} + 28x$*.*

> *(a) Find a function* $x_4(t)$ *such that* $L(x_4) = 108,000e^{2t}$*.*
>
> *(b) Find a function* $x_2(t)$ *such that* $L(x_2) = 10e^{-2t}$*.*
>
> *(c) Find a function* $x_3(t)$ *such that* $L(x_3) = 108e^{2t} + 10e^{-2t}$*.*

Exercise 17 *Let* $L_2(x) = \ddot{x} + 9x$*. Find a function* $x_2(t)$ *such that* $L_2(x_2) = 10\sin 2t$

Exercise 18 *. Let* $L(x) = \ddot{x} + 11\dot{x} + 28x$*. Find a function* $x_3(t)$ *such that* $L(x_3) = 162te^{2t}$

Exercise 19 *Let* $L_4(x) = \ddot{x} + 11\dot{x} + 28x$*. Find a function* $x_4(t)$ *such that* $L_4(x_4) = 10\sin 2t$

Exercise 20 *Let* $L(f)$ *be a linear operator. Given three special functions,* $f(x)$*,* $u(x)$ *and* $h(x)$*, such that* $L(f(x)) = h(x)$ *and* $L(u(x)) = 0$*, and* $g(x) = f(x) + u(x)$*. Prove that* $L(g(x)) = h(x)$*.*

Exercise 21 *Restate the result of the preceding exercise in "verbal" form, that is state it using as few math symbols or equations as possible.*

Exercise 22 *Let* $L(f)$ *be a linear transformation. Given three special functions,* $f(x)$, $g(x)$ *and* $h(x)$, *such that* $L(f(x)) = h(x) = L(g(x))$, *and* $u(x) = f(x) - g(x)$. *Prove that* $L(u(x)) = 0$.

Exercise 23 *Restate the result of the preceding exercise in "verbal" form, that is state it using as few math symbols or equations as possible.*

Homogeneous linear equations

In this section, you will learn to solve equations like: $\ddot{x} + 11\dot{x} + 28x(t) = 0$. Wise to review the results of Exercises 11, 12 and 13.

Definition: Given a linear operator $L(f)$, then equations with the form $L(f) = 0$ are called *homogeneous linear equations.*

Example 10 *Check that the equation:* $f'' - 11f' + 28f = 0$ *is a homogeneous linear equation.*

Proof. Set $L(f) = f'' - 11f' + 28f$ and recognize (using Example 4) that $L(f)$ is a linear operator. Hence $L(f) = f'' - 11f' + 28f = 0$ is a homogeneous linear equation. $\surd$ YEA

Using this new vocabulary (of homogeneous linear equation), the results of Exercises 11 and 12 may be generalize (for two solutions) as:

Given: a linear operator L (and functions y_1 and y_2 and numbers A and B). Also given: $L(y_1) = 0 = L(y_2)$, and $y = Ay_1 + By_2$.

To prove: $L(y) = 0$.

Proof. by calculation using Proposition 5:

$$L(y) = L(Ay_1 + By_2) = AL(y_1) + BL(y_2) = A \times 0 + B \times 0 = 0$$

Thus $L(y) = 0$, $\forall_A$ and $_B$. $\surd$

Remark. This result may also be stated/translated in pure verbal form, no math symbols or equations as:

Theorem 11 (on solutions to all homogeneous linear equations) *Known/given several solutions to a homogeneous linear equation. Then each/all linear combinations of these solutions is/are [many] more solutions to the same equation.*

Remark. Theorems are usually stated in verbal form because they are easier to remember. It is difficult to remember and easy to garble a formula/equation form of a theorem. It is useful to refer to a theorem by name; referring to a theorem by number is mainly done in textbooks.

Remark. This theorem provides a two-step algorithm for solving any and all homogeneous linear equations, namely:

Step 0. Observe/Check that the equation is indeed a homogeneous linear equation

Step 1. Find the simplest solutions.

How? Make an educated guess, followed by checking it.

Step 2. Take all their linear combinations.

But, the first step is a bit vague. The standard algorithm is more elaborate. We will model it in the next example.

Example 12 (of a good shock absorber) *Let us find many solutions to the differential equation:*

$$\ddot{x} + 11\dot{x} + 28x(t) = 0.$$

Step 0. *Set*:

$$L(x) = \ddot{x} + 11\dot{x} + 28x = 0.$$

Observe that $L(x) = 0$ *is a homogeneous linear equation*, since L is a linear operator.

Step 1. *Try* $x = x(t) = e^{rt}$, [where r is an unknown, to-be-found constant number.]

Step 2. *Plugging in*:

$$L(e^{rt}) = r^2e^{rt} + 11re^{rt}1 + 28e^{rt} = 0.$$

Factoring out e^{rt}, yields: $e^{rt}(r^2 + 11r + 28) = 0$. Hence

$$r^2 + 11r + 28 = 0,$$

which is called the *Characteristic Equation* for the given differential equation.

Step 3. *Solve for* r in the *Characteristic Equation*; this yields: $r = -4$ and -7.

We obtain two special solutions $x_1 = e^{-4t}$ and $x_2 = e^{-7t}$.

Step 4. *All the linear combinations of solutions to homogeneous linear equations are also solutions* (Theorem 11, on the form of solutions to homogeneous linear equations); thus (since we noted in Step 0 that $L(x) = \ddot{x}+11\dot{x}+28x = 0$ is a homogeneous linear equation.)

$$x(t) = Ae^{-4t} + Be^{-7t},\ \forall_{A \text{ and } B}$$

are many solutions to the differential equation: $\ddot{x} + 11\dot{x} + 28x = 0$.

Remark. We luck out that these are all the solutions; there are no other solutions. This is not proven here; take my word for it.

Remark. Among the many solutions $x(t) = Ae^{-4t} + Be^{-7t}$, $\forall_{A \text{ and } B}$, the two we found first, $x_1 = e^{-4t}$ and $x_2 = e^{-7t}$, are indeed much simpler than the bulk of the others. The power of Theorem 11 is that it provides us with many, more-difficult solutions after we find the easy ones.

Remark. We will use an analogous set of steps (same form but more complicated substance) later when we solve pairs of simultaneous, parametric homogeneous linear differential equations with constant coefficients. For example: $\dot{x} = 8x - y$ and $\dot{y} = -x + 2y$. (As a challenge, try to find a solution now using some ideas from high school algebra, together with material from this section.)

Remark. Observe that the coefficients in the "Characteristic Equation" are the same as the coefficients in the differential equation.

Definition. The *Characteristic Equation* for the *homogeneous linear differential equation with constant coefficients* $A\ddot{x} + B\dot{x} + Cx = 0$ is $Ar^2 + Br + C = 0$.

Remark. We could define the term "Characteristic Equation" to be anything at all, but in order for it to be *useful*, there must be a simple direct *connection* between the roots of the Characteristic Equation and some solutions of the corresponding differential equation.

Proposition 13 *The function* $y = e^{rt}$ *is a solution of the differential equation* $A\ddot{x} + B\dot{x} +$ $+Cx = 0$ *if and only if* r *is a root of the Characteristic Equation* $Ar^2 + Br + C = 0$.

Remark. You may establish this proposition as an Exercise **??**.

Remark. The above presentation is appropriate for instruction, but it is a bit wordy for tests. Here is test/homework-appropriate version.

Example 14 (of a test/homework-appropriate solution) *Finding solutions to the differential equation:*

$$\ddot{x} + 11\dot{x} + 28x(t) = 0.$$

Step 0. Set: $L(x) = \ddot{x} + 11\dot{x} + 28x = 0$, a homogeneous linear equation.

Step 1. Try: $x = e^{rt}$.

Step 2. $L(e^{rt}) = e^{rt}(r^2 + 11r + 28) = 0$.

Step 3. $(r + 4)(r + 7) = 0 \implies r = -4$ and -7.

$x_1 = e^{-4t}$ and $x_2 = e^{-7t}$.

Step 4. Since, linear combinations of solutions to homogeneous linear equations are also solutions:

$$x(t) = Ae^{-4t} + Be^{-7t}, \ \forall_{A \text{ and } B}.$$

Example 15 (An initial value problem) *Let us find the particular solution to the differential equation:*

$$\ddot{x} + 11\dot{x} + 28x = 0.$$

which satisfies given initial conditions: $x(0) = 0$ *and* $\dot{x}(0) = -6$.

Calculations: First one solves the linear differential equation; this was done in the last example.

$$x(t) = Ae^{-4t} + Be^{-7t}, \ \forall_{A \text{ and } B},$$

We need to pick out the solution which satisfies the given initial conditions. We plug in and calculate:

$$\text{At } t = 0: \quad 0 = x(0) = Ae^0 + Be^0 = A + B$$

$$\dot{x} = -4Ae^{-4t} - 7Be^{-7t}.$$

$$\text{At } t = 0: \quad -6 = \dot{x}(0) = A(-4)e^0 + B(-7)e^0 = -4A - 7B.$$

That is: $0 = A+B$ and $-6 = -4A-7B$. Solving these two equations for A and B yields: $A = -2$ and $B = 2$. Hence

$$\boxed{x(t) = -2e^{-4t} + 2e^{-7t}}$$

is the particular solution to the equation:

$$\ddot{x} + 11\dot{x} + 28x = 0$$

which satisfies the given initial conditions: $x(0) = 0$ and $\dot{x}(0) = -6$.

Check. $x(0) = -2e^0 + 2e^0 = 0.$ $\surd$

$\dot{x}(t) = 8e^{-4t} - 14e^{-7t}$, and $\dot{x}(0) = 8e^0 - 14e^0 = -6.$ $\surd$

EXERCISES

Exercise 24 *Using Example 12 or Example 14 as a model, find many solutions to this equation:* $\ddot{x} - 4\dot{x} - 5x = 0$*; do not skip steps.*

Exercise 25 (Dick and Jane) *Dick and Jane are investigating an engineering system which is governed by the differential equation* $\ddot{x} - 4\dot{x} - 5x = 0$*. (The general solution was found in the preceding exercise.) Separately, each of them measures the initial conditions and calculates the particular solution; finally each of them calculates* $\lim_{t\to\infty} x(t)$*.*

(a) Dick's measurements of the initial conditions are

$$x(0) = 3.14 \textit{ and } \dot{x}(0) = -3.146.$$

What particular solution does Dick calculate? According to Dick, what is $\lim_{t\to\infty} x(t)$*?*

(b) Jane's measurements are

$$x(0) = 3.14 \textit{ and } \dot{x}(0) = -3.134.$$

According to Jane, what is $\lim_{t\to\infty} x(t)$*?*

(c) Plot Dick's and Jane's solutions, together, on a computer (or your graphing calculator).

(d) What are your reactions to the answers to this exercise?

Exercise 26 *Use* **DSolve** *and* **NDSolve** *to solve this equation:* $\ddot{x} - 4\dot{x} - 5x = 0$*, together with the initial conditions:* $x(0) = 3.14$ *and* $\dot{x}(0) = -3.140$*. have the computer graph the two solutions together. Explain what happens.*

Exercise 27 *(a) Find the paricular solution to this equation:* $\ddot{x} - 4\dot{x} - 5x = 0$ *which satisfies the initial conditions:* $x(0) = 0$ *and* $\dot{x}(0) = 0$*.*

(b) Find the paricular solution to this equation: $\ddot{x} + 11\dot{x} + 28x = 0$ *which satisfies the initial conditions:* $x(0) = 0$ *and* $\dot{x}(0) = 0$*.*

Exercise 28 *Using the results from the previous exercise as data, guess more general situations where similar things happen. Check your guesses.*

Exercise 29 *State and prove the theorem on the form of solutions to homogeneous linear equations.*

Exercise 30 *Find the paricular solution to this equation:* $\ddot{x} - 4\dot{x} - 5x = 0$ *which satisfies the initial conditions:* $x(0) = 0$ *and* $x(\pi) = 0$*.*

Exercise 31 *Find the solutions to the shock absorber problem* $\ddot{x} + 11\dot{x} + 28x(t) = 0$*, when*

(a) $x(0) = 1$ *and* $\dot{x}(0) = 1$*,*

(b) $x(0) = 1$ *and* $\dot{x}(0) = -1$

(c) $x(0) = 1$ *and* $\dot{x}(0) = -5.5$

Graph these solutions.

Exercise 32 *Consider* $x(t) = Ae^{-rt} + B\ e^{-st}$*, with both* r *and* s *positive. When* A *is positive and* B *is negative show that the graph has exactly one local max or min, but not both. Same when* B *is positive and* A *is negative.*

Sketch all possible types of graphs for $x(t) = Ae^{-rt} + B\ e^{-st}$*, with both* r *and* s *positive.*

Exercise 33 *State and prove the theorem on the form of solutions to homogeneous linear equations.*

Complex Exponentials and Real Homogeneous Linear Equations

If we try to use the method of Example 12, on the equation $\ddot{x} + x = 0$, we would calculate

$$0 = L(e^{rt}) = r^2 e^{rt} + e^{rt} = (r^2 + 1)e^{rt},$$

and hence

$$0 = r^2 + 1 \;\Rightarrow\; r^2 = -1 \;\Rightarrow\; r = \pm i \;\Rightarrow\; x = e^{it} \text{ is a solution.}$$

Suddenly the *real* equation has a complex solution, that is a function with *complex* numbers. But, *complex* final answers to *real* equations are usually useless. In this course, all final answers must be *real* functions, never *complex*. Fortunately, its the complex exponential to the rescue.

Example 16 *(a) The linear operator* $L(f) = f'' + 3if' + 4f$ *sends* real *functions to* complex *functions. Here* $L(x^2) = 2 + 6ix + 4x^2$

(b) Linear operators of the form $L(f) = af'' + bf' + cf$*, where* a, b *and* c *are* real *numbers, send* real *functions to real functions.*

We are about to demonstrate how to “convert” undesirable complex solutions of real equations into desirable real solutions.

If/Given: A real linear transformation L (and a complex function y_0 such that)

$L(y_0) = 0$ and $y_0 = u_1 + iu_2$ (with u_1 and u_2 real functions.).

Then/To show: $0 = L(u_1)$ and $0 = L(u_2)$.

Proof. by calculation:

$$0 = L(y_0) = L(u_1 + iu_2) = L(u_1) + iL(u_2),$$

since L is a linear operator. We have:

$$0 + 0i = 0 = L(u_1) + iL(u_2)$$

Two complex numbers are equal when their real parts are equal and their imaginary parts are equal. Thus

$$0 = L(u_1) \text{ and } 0 = L(u_2)$$

√

Remark. This result may also be stated/translated in pure verbal form, no math symbols or equations as:

Theorem 17 (on complex solutions to real homogeneous linear equations) *Known/given a complex solution to a real homogeneous linear equation, Then both the real and the imaginary parts of this complex solution, are two more solutions to the (same) equation.*

Remark. This theorem tells us what to do when a complex solution raises its undesirable head, namely, take its real and imaginary parts, but only when trying to solve a real homogeneous linear equations,.

Example 18 (The basic spring-block equation) *Find many solutions to:*

$$\ddot{x} + \omega^2 x(t) = 0,$$

where ω *is a given constant real number.*

Calculations using the algorithm of Example 12, modified to take advantage of Theorem 17.

Step 0. *Set*:

$$L(x) = \ddot{x} + \omega^2 x(t) = 0.$$

Observe that $L(x) = 0$ *is a homogeneous linear equation*, since L is a linear operator.

Step 1. *Try* $x = e^{rt}$, [where r is an unknown, to-be-found constant number].

Step 2. *Plugging in* $x = e^{rt}$:

$$\begin{aligned} L(e^{rt}) &= r^2 e^{rt} + \omega^2 e^{rt} = (r^2 + \omega^2) e^{rt} \\ 0 = L(e^{rt}) &= (r^2 + \omega^2) e^{rt} \;\Rightarrow\; r^2 + \omega^2 = 0 \end{aligned}$$

which is the Characteristic Equation.

Step 3. *Solve for* r in the *Characteristic Equation*; this yields:

$$r = \pm \omega i$$

We obtain a complex solution $x_1(t) = e^{\omega i t} = \cos \omega t + i \sin \omega t$.

Step 3 Part C. *The real and the imaginary parts of a complex solution to a real homogeneous linear equation, are two (more) solutions.* (Theorem 17)

We obtain two special solutions

$$x_3(t) = \cos \omega t \quad \text{and} \quad x_4(t) = \sin \omega t$$

Step 4. *All the linear combinations of solutions to homogeneous linear equations are also solutions* (Theorem 11 on the form of solutions to homogeneous linear equations, which is applicable since we noted in Step 0 that $L(x) = 0$ is a homogeneous linear equation.) Thus:

$$\boxed{x(t) = A \cos \omega t + B \sin \omega t, \ \forall_{A} \text{ and } {}_{B}}$$

are all solutions to $L(x) = \ddot{x} + \omega^2 x = 0$.

Remark. This basic spring-block equation is important; memorize its solution.

Remark. We luck out that these are all the solutions; there are no other solutions. This is not proven here; take my word for it.

Remark. We ignored the other "fundermental" solution, $x_2(t) = e^{-\omega i t}$. Using it would *not* have added other solutions. Doing Exercise 34 will demonstrate this.

Remark. The above presentation is appropriate for instruction, but it is a bit wordy for tests. Here is test/homework-appropriate version.

Example 19 (of a bad shock absorber) *Find many solutions to*

$$4\ddot{x} + 8\dot{x} + 9x = 0.$$

Calculations using the test/homework-appropriate-version steps of Example 14. Phrases and sentences enclosed on braces "[ ]" may be considered as understood, even when they are not written

Step 0. Set: $L(x) = 4\ddot{x}+8\dot{x}+9x = 0$. Note that this is a real homogeneous linear equation.

Step 1. Try $x = e^{rt}$, [where r is an unknown, to-be-found constant number].

Step 2. $0 = L(e^{rt}) = (4r^2 + 8r + 9)e^{rt}$

Step 3. Hence $r = \frac{-8\pm\sqrt{64-144}}{8} = -1 \pm \frac{\sqrt{5}}{2}i.$

Then $x_1(t) = e^{(-1+\frac{\sqrt{5}}{2}i)t} = e^{-t}e^{i\frac{\sqrt{5}}{2}t} = e^{-t}\cos\frac{\sqrt{5}}{2}t + ie^{-t}\sin\frac{\sqrt{5}}{2}t.$

Step 3 Part C. Since both the real and the imaginary parts of a complex solution of a real homogeneous linear equation are two (more) solutions:

$$x_3 = e^{-t}\cos\frac{\sqrt{5}}{2}t \text{ and } x_4 = e^{-t}\sin\frac{\sqrt{5}}{2}t$$

Step 4. Since, linear combinations of solutions to homogeneous linear equations are also solutions:

$$\boxed{x(t) = Ae^{-t}\cos\tfrac{\sqrt{5}}{2}t + Be^{-t}\cos\tfrac{\sqrt{5}}{2}t,\ \forall_A \text{ and } {}_{B\in R}.}$$

Remark. It is important to memorize the definition of a linear operator and the statements of Theorems 11, 17 and 25. You should be able use them. Be able to prove Theorem 11.

Exercises

Remark. When solving linear differential equations, use Examples 18 and 19 as a model. Do not skip steps. Number the steps.

Exercise 34 **a.** *Redo Steps 3, 3C and 4 of Example 18 using the other "fundermental" solution,* $x_2(t) = e^{-\omega i t}$. *Does the answer seem different?*

b. *Using your solution from Part a, find the particular solution to:* $\ddot{x}+\omega^2 x(t) = 0$, *which satisfies the initial conditions:* $x(0) = 2$ *and* $\dot{x}(0) = -3$.

c. *Using the solution from Example 18, find the particular solution to:* $\ddot{x} + \omega^2 x(t) = 0$, *which satisfies the initial conditions:* $x(0) = 2$ *and* $\dot{x}(0) = -3$. *Does this answer differ from the one in Part b?*

Exercise 35 *Using Examples 18 and 19 as a model, find the general solution to this equation:*

$$\ddot{x} + 2\dot{x} + 5x = 0.$$

Number the steps.

Exercise 36 *(a) Find the general solution to this equation:*

$$4\ddot{x} + 8\dot{x} + 6.25x = 0.$$

Number the steps.

(b) Find the particular solution which satisfies the initial conditions $x(0) = 1$ *and* $\dot{x}(0) = 4$.

(c) Find the particular solution which satisfies the initial conditions $x(0) = 0$ *and* $\dot{x}(0) = 0$.

Exercise 37 *Find the particular solution to*

$$4\ddot{x} + \frac{1}{10}\dot{x} + 9x = 0$$

which satisfies the initial conditions $x(0) = 1$ *and* $\dot{x}(0) = -1$. *Number the steps.*

Exercise 38 *Graph the solution to the preceding exercise, together with the solution to* $4\ddot{x} + 9x = 0$ *with the same initial conditions over the interval* $[0, 5]$, *(2 solutions, 2 graphs, 1 set of axes). Does there appear to be a significant difference between the two solutions? If so, what is it?*

Exercise 39 *Find the general solution to*

$$4\ddot{x} + \frac{1}{10}\dot{x} + 9x = 0.$$

Graph together (1 set of axes), the two functions $\pm 5e^{-t/80}$ *and the 6 solutions to* $4\ddot{x} + \frac{1}{10}\dot{x} + 9x = 0$, *with the coefficients* $c_1 = \pm 3$ *and* $c_2 = \pm 4$, $c_1 = 5$ *and* $c_2 = 0$ *and* $c_1 = 0$ *and* $c_2 = 5$ *over the interval* $[0, 5]$. *Repeat this set of graphs over the interval* $[100, 105]$. *Repeat with one or more intervals of your choice. Describe what the graphs demonstrate. What is interesting?*

For these solutions, calculate $\lim_{t\to\infty} x(t) = 0$.

Exercise 40 *Again, graph the pair of solutions from Exercise 38, this time over the interval* $[100, 105]$. *Repeat with one or more intervals of your choice. What is interesting? How and where are these two solutions similar? How and where are these two solutions different?*

Do these graphs change the perception you had of the two solutions after you completed Exercise 38

Remark. The "friction" term $\frac{1}{10}\dot{x}$ in Exercises 38 and 40 looks small. The two equations are almost the same. In the short run, the solutions are almost the same. Over time, "friction" uses up the "energy" of the damped system, $4\ddot{x} + \frac{1}{10}\dot{x} + 9x = 0$; this results in very different solutions in the long run.

Exercise 41 *Consider the two equations:*

$$\ddot{x} + 2.01\dot{x} + x = 0 \quad \textit{and} \quad \ddot{x} + 1.99\dot{x} + x = 0.$$

The first represents a good shock absorber; the second a bad one. Suppose that they both satisfies the initial conditions $x(0) = 1$ *and* $\dot{x}(0) = -1$. *Graph both solutions together over the interval* $[0, 3]$, *(2 solutions, 2 graphs, 1 set of axes). Does there appear to be a significant difference between the two solutions? If so, what is it?*

Linearly Independent Sets

Definition Two functions form a *linearly independent set* if they are not multiples of each other. [4]

That is, two nonzero functions, $\{y_1 \text{ and } y_2\}$, form a *linearly independent set* if

$$y_2 \neq cy_1, \quad for\ any\ constant\ c.$$

Example 20 *Check that* $\{e^{4x} \text{ and } e^{7x}\}$ *form a linearly independent set.*

Calculations. Can $e^{4x} = ce^{7x}$, for some constant c? If so then $c = \frac{e^{4x}}{e^{7x}} = e^{-3x}$, which is not possible since e^{-3x} is not a constant function. $\surd$

Example 21 *Check that* $\{\sin x \text{ and } \cos x\}$ *form a linearly independent set.*

Calculations. Can $\sin x = c\cos x$, for some constant c? If so then $c = \frac{\sin x}{\cos x} = \tan x$, which is not possible because $\tan x$ is not a constant function. $\surd$

Definition When the general solution, to a homogeneous linear differential equation, is written as the set of all linear combinations of a linearly independent set (of solutions), then

[4] In general, a set of functions forms a *linearly independent set* if no one of these functions is a linear combination of the others.

the linearly independent set is called *a fundamental set of solutions* or a *basis for the solution space.*

Based on Example 12, we see that:

Example 22 *A fundamental set of solutions and a basis for the solution space for the differential equation,* $\ddot{x} + 11\dot{x} + 28x = 0$ *is* $\{e^{-4t}$ *and* $e^{-7t}\}$.

Theorem 23 *The general solution to each second order homogeneous linear differential equation has a basis or fundamental set consisting of two solutions. That is, given a second order homogeneous linear differential equation (on a domain in which the coefficients are continuous),* $L(y) = 0$*, there are two [non-zero] solutions* $\{y_1$ *and* $y_2\}$*, such that* $y_2 \neq cy_1$*, for any constant c, and the general solution to* $L(y) = 0$ *is:*

$$y = c_1 y_1 + c_2 y_2 \quad \forall_{numbers \quad c_1 \quad and \quad c_2}.$$

Remark. This theorem implies that the "many' solutions that we have been finding are actually the general solutions. There are no other solutions (hidden anywhere).

Remark. The last theorem and the two examples enable us to *avoid* the use of the "Wronskian".

Exercises

Exercise 42 *State and prove the theorem on solutions to homogeneous linear equations.*

Exercise 43 *State and prove the theorem on complex solutions to real homogeneous linear equations.*

Exercise 44 *State the theorem on the form of solutions to non-homogeneous linear equations.*

Exercise 45 *Check that the following are (equivalent) to homogeneous linear equations. Or, equivalently, check that a linear combinations of any two solutions is another solution to the same equation.*

(a) The Wave Equation in one space variable

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

(b) The Wave Equation in 3-space:

$$\nabla^2 u(x,y,z,t) \stackrel{def}{=} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$$

(c) The Heat Equation in 3-space

$$\nabla^2 u(x, y, z, t) = \frac{\partial u}{\partial t}$$

Non-homogeneous linear equations

Definition: Given a linear operator L, then equations with the form $L(y(x)) = g(x)$ are called *non-homogeneous linear equations*. The equation $L(u(x)) = 0$ is referred to as its *associated homogeneous equation*. Similarly, equations with the form $L(x(t)) = g(t)$ are called *non-homogeneous linear equations*. The equation $L(u(t)) = 0$ is its *associated homogeneous equation*.

Note that the left side is a linear operator, whose inputs are the *dependent* variable and the right side is a function of the *independent* variable *only*.

Remark. The function, g on the right side is called the "power" term by engineers since it usually represents the power sources. Mathematicians use the more general phrase, the "forcing" term.

Example 24 *The equation* $y''(x) - 11y' + 28y = \sin 2x$ *is a non-homogeneous linear equation and* $u''(x) - 11u'(x) + 28u(x) = 0$ *is its associated homogeneous equation.*

The equation $\ddot{x} - 11\dot{x} + 28x = \sin 2t$ *is a non-homogeneous linear equation and* $\ddot{u} - 11\dot{u} + 28u(t) = 0$ *is its associated homogeneous equation.*

The equation $y'(x) + \frac{2}{x}y = 4x$ is a non-homogeneous linear equation and $L(u(x)) = u'(x) + \frac{2}{x}u = 0$ is its associated homogeneous equation. We found that

$$y = x^2 + cx^{-2}, \ \forall_c$$

is the general solution to the non-homogeneous equation. There is a significance to each term of this sum, namely: The term, $y_1 = x^2$, is a single solution, by itself, to the non-homogeneous equation $y'(x) + \frac{2}{x}y = 4x$, (it occurs when $c = 0$). The other term, $u(x) = cx^{-2}$, $\forall_c$, is the general solution to the associated homogeneous equation.

The equation $\dot{x} + 2x(t) = 6$ is a non-homogeneous linear equation and $L(u(t)) = \dot{u} + u(t) = 0$ is its associated homogeneous equation. We found that

$$x(t) = 3 + ce^{-2t}, \ \forall_c$$

is the general solution to the non-homogeneous equation. The asymptotically-stable (steady-state) constant solution is $x_1(t) = 3$ (it occurs when $c = 0$). The other part is $u(t) = ce^{-2t}$, $\forall_c$, which is the general solution to the associated homogeneous equation.

It always happens that the general solution to a non-homogeneous linear equation is the sum of any solution (to the non-homogeneous linear equation) plus the general solution to its associated homogeneous equation.

Theorem 25 (Form of solutions to non-homogeneous linear equations) *Given a linear equation* $L(y) = g(x)$*, then*

$$\left\{ \begin{array}{c} \textit{General Solution} \\ \textit{to } L(y) = g(x) \end{array} \right\} = \left\{ \begin{array}{c} \textit{Any particular} \\ \textit{solution} \end{array} \right\} + \left\{ \begin{array}{c} \textit{General Solution} \\ \textit{to } L(u) = 0 \end{array} \right\}.$$

Mostly proven by your solutions to Exercises 20 and 22.

Remark. This theorem provides a (partial) algorithm for solving non-homogeneous linear equations, namely:

Step 1. Check that the equation is indeed a non-homogeneous linear equation

Step 2. Find the solution to the associated homogeneous equation.

Step 3. Find the "easiest" solution to the non-homogeneous equation.

How? Make an educated guess, followed by checking it.

Step 4. Add them.

Remark. We will use linear operators explicitly, as we find particular solutions to non-homogeneous linear differential equations.

Example 26 *Let us find the general solution to the differential equation:*

$$\ddot{x} + 11\dot{x} + 28x(t) = 56.$$

Calculations.

1. In Example 4, we checked that $L(x) = \ddot{x} + 11\dot{x} + 28x(t)$ is a linear transformation. Since 56 is not a function of the dependent variable x, $L(x) = \ddot{x} + 11\dot{x} + 28x(t) = 56$ is a linear equation. $\surd$

2. The associated homogeneous equation is $L(u) = \ddot{u} + 11\dot{u} + 28u(t) = 0$. In Example 12, we calculated its general solution as

$$u(t) = Ae^{-4t} + Be^{-7t}, \ \forall_A \text{ and}_B$$

3. Calculate any solution to $L(x) = 56$. Since the output (56) is a constant, we guess that there is a constant solution, $x_1(t) = 1$. For this guess, $\dot{x}_1 = 0 = \ddot{x}_1$. We calculate:

$$L(1) = 0 + 0 + 28 = 28.$$

Since $56 = 2 \times 28$, and since L is a linear operator, we try $x_2(t) = 2$:

$$L(2) = 2L(1) = 2 \times 28 = 56$$

Thus $x_2(t) = 2$ is a solution.

4. Add these solutions.

$$\left\{\begin{array}{c}\text{General Solution}\\ \text{to } L(x)=56\end{array}\right\} = \left\{\begin{array}{c}\text{Any particular}\\ \text{solution}\end{array}\right\} + \left\{\begin{array}{c}\text{General Solution}\\ \text{to } L(u)=0\end{array}\right\}.$$

Thus

$$x(t) = 2 + Ae^{-4t} + Be^{-7t} \; \forall_{A} \text{ and }_{B}$$

is the general solution to $\ddot{x} + 11\dot{x} + 28x(t) = 56$.

Correct Wording: "The solution to the associated homogeneous equation" or "The solution to $L(x) = 0$", when $L(x)$ has been defined. Incorrect Wording: "The homogeneous solution".

Next, a "power" term is added to a spring-block system, which results in a non-homogeneous equation.

Example 27 (of a spring-block system with "forcing") *Find the general solution to*

$$4\ddot{x} + 16x(t) = 40\sin 3t.$$

Calculations

1. We set $L(x(t)) = 4\ddot{x} + 16x$ and observed that it is a linear operator. Since $40\sin 3t$ is not a function of x, $L(x) = 4\ddot{x} + 16x(t) = 40\sin 3t$ is a linear equation. $\surd$

2. The associated homogeneous equation is $L(u) = 4\ddot{u} + 16u(t) = 0$. Equivalently (dividing by 4), $\ddot{u} + 4u(t) = 0$. We recognize this as the basic spring block equation; its general solution (which you have memorized) is

$$u(t) = A\cos 2t + B\sin 2t, \; \forall_{A} \text{ and }_{B}$$

(Note: Usually you have to solve the associated homogeneous equation here.)

3. Calculate any solution to $L(x) = 4\ddot{x} + 16x(t) = 40\sin 3t$. We know (from Exercises 2 and 3) that $L(\sin 3t)$ is a multiple of $\sin 3t$, [5] so we try: $x_1(t) = \sin 3t$. For this guess, $\ddot{x}_1 = -9\sin 3t$. Hence:

$$L(\sin 3t) = 4\ddot{x} + 16x(t) = -36\sin 3t + 0 + 16\sin 3t = -20\sin 3t.$$

We need $40\sin 3t$, not $-20\sin 3t$. No problem, since L is linear, we simply multiply by -2:

$$40\sin 3t = -2L(\sin 3t) = L(-2\sin 3t).$$

Hence a particular solution is $x_2(t) = -2\sin 3t$.

[5] When $\sin 3t$ is a solution to the associated homogeneous equation, a different guess is needed.

4. Add these solutions.

$$\left\{ \begin{array}{c} \text{General Solution} \\ \text{to } L(x) = 40\sin 3t \end{array} \right\} = \left\{ \begin{array}{c} \text{Any particular} \\ \text{solution} \end{array} \right\} + \left\{ \begin{array}{c} \text{General Solution} \\ \text{to } L(u) = 0 \end{array} \right\}.$$

$$x(t) = -2\sin 3t + A\cos 2t + B\sin 2t,\ \forall_A \text{ and } _B$$

is the general solution to $L(x) = 4\ddot{x} + 16x(t) = 40\sin 3t$.

Example 28 *Find a particular solution to the non-homogeneous differential equation:* $L(x) = \ddot{x} - 11\dot{x} + 28x = te^{2t}$.

Note that only the independent variable appears on the right side. Thus $L(x)$ is a linear combination of $x(t)$ and its derivatives. Hence, it is a linear operator (Example 4 in notes).

Guess/try $x_1 = te^{2t}$. Plug this guess into the differential equation:

$$L(te^{2t}) = 10\ \underbrace{te^{2t}}_{\text{needed}}\ +6\ \underbrace{e^{2t}}_{\substack{\text{not} \\ \text{needed}}}$$

$$L(e^{2t}) = 10e^{2t}$$

We seek a linear combination of these two equations, in which the e^{2t}-terms will cancel:

$$10L(te^{2t}) - 6L(e^{2t}) = 100te^{2t}$$

Using the fact that L is a linear operator:

$$L(10te^{2t} - 6e^{2t}) = 100te^{2t}$$

$$L[\frac{1}{100}(10te^{2y} - 6e^{2t})] = te^{2t}$$

Thus $\boxed{x = \frac{1}{10}te^{2t} - \frac{6}{100}e^{2t}}$ is a particular solution to $L(x) = \ddot{x} - 11\dot{x} + 28x = te^{2t}$.

Example 29 *Find a particular solution to the non-homogeneous differential equation:* $y'' - 3y' - 4x = 2\sin t$.

Calculations: Set $L(y) = y'' - 3y' - 4y$, and note that L is a linear operator. We remember that $L(\sin t)$ and $L(\cos t)$ are both linear combinations of $\sin t$ and $\cos t$. We calculate:

$$L(\sin t) = -5\sin t - 3\cos t$$

$$L(\cos t) = 3\sin t - 5\cos t$$

We seek a linear combination of these two equations, in which the $\cos t$-terms will cancel. Multiply the first equation by -5 and the second equation by 3, then add the equations. The sum is:

$$-5L(\sin t) + 3L(\cos t) = 34\sin t$$

We use the fact that L is a linear operator:

$$L(-5\sin t + 3\cos t) = 34\sin t$$

We need the coefficient of $\sin t$ to be 2, not 34. so we divide by 17:

$$\frac{1}{17}L(-5\sin t + 3\cos t) = 2\sin t$$

We use Rule (ii), of the definition of linear operator to obtain:

$$L(-\frac{5}{17}\sin t + \frac{3}{17}\cos t) = 2\sin t,$$

$\surd$ YEA

Thus: $\boxed{y_1(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t}$ is a particular solution to $L(y) = y'' - 3y' - 4y = 2\sin t$.

Exercises

Exercise 46 *Using the methods of this section, find the general solution to each of these equations.*

$$\dot{x} + 7x(t) = 3.$$

$$\dot{x} + 7x(t) = e^{3t}.$$

$$\dot{x} + 7x(t) = \sin 3t.$$

Exercise 47 *Find a particular solution to each of these equations.*

$$\ddot{x} + 3x(t) = \sin 2t.$$

$$\ddot{x} + 3x(t) = \sin 1.7t.$$

$$\ddot{x} + 3x(t) = \sin 1.732t.$$

Any comment on solutions.

Exercise 48 *Find the particular solution to this equation* $\ddot{x}+11\dot{x}+28x=56$ *which satisfies the initial conditions:*

$x(0)=0$ *and* $\dot{x}(0)=0$*.*

Exercise 49 *State and (partially) prove the theorem on the form of solutions to non-homogeneous linear equations.*

Exercise 50 *Given* $\dot{x}=-x+f(t)$*, where* $f(t)$ *is an unknown function. Let* $x_1(t)$ *and* $x_2(t)$ *be two particular solutions to this equation. Show that* $\lim_{t\to\infty} x_1(t)-x_2(t)=0$*.*

Exercise 51 *Use a computer software to twice solve and graph the solution to:*

$$\ddot{x}=6x(t)+30\sin 3t \quad \textit{and} \quad x(0)=2 \quad \textit{and} \quad \dot{x}(0)=12,$$

*first with an exact symbolic differential equation solver (***dsolve***),*

*second with a good numerical differential equation solver (***ode45***).*

Now, use the exact symbolic differential equation solver, to solve the same equation with the same initial "position", $x(0)=2$*; but with the following initial "velocities":*

$$\dot{x}(0)=12\pm 10^{-1}, 12\pm 10^{-5}, 12\pm 10^{-10}, 12\pm 10^{-14}, 12\pm 10^{-16}.$$

Graph these solutions together. Choose a large enough domain so that one can see all the solutions move away from the exact solution with initial "velocity": $\dot{x}(0)=12$*.*

Find the general solution to

$$\ddot{x}=6x(t)+30\sin 3t.$$

Please comment on the graphs of the solutions.

Summary of results for spring-block systems

Here is a summary of results on simple systems consisting of a block with mass m and a linear spring with Hooke's constant k, sometimes with a linear frictional force with constant b; the distance from the rest position is x. Details may be found in the differential equations text by Boyce and DiPrima, which will be referred to by "[BD]" below. as the

(1) Basic spring block system with no friction and no driving force: $m\,\ddot{x}+kx=0,\ m,k>0$

General solution to $\ddot{x}+\omega^2 x = 0$ is $x = A\ \cos\omega t + B\ sin\ \omega t,\ \forall_{A \text{ and } B}$; or equivalently:

$$x = C\ \cos(\omega t - \delta),\ \forall_{C \text{ and } \delta},$$

where $C = \sqrt{A^2+B^2})$ is the "amplitude" of the sinosoidal motion. The latter form is especially useful for graphing.

(2) Shock absorbers, a spring block system with a linear frictional force and no driving force:

$$m\ddot{x} + b\dot{x} + kx = 0,\ \forall_{m,b \text{ and } k>0}.$$

(i) Good shocks: The friction constant, "b" is large enough that the roots of the auxillary equation are real (and negative).

General solution: $x = Ae^{rt} + Be^{st},\ \forall_{A \text{ and } B \text{ and } r,s<0}.$

Graphs look like Figure 3.8.6 of [BD], when $AB > 0$; graphs look like exponential decay when $AB < 0$.

(ii) Bad shocks: The friction constant, " b " is small enough that the roots of the auxillary equation are complex – not real; the real part is negative.

General solution: $x = C\ e^{rt}\ \cos(\omega t - \delta),\ \forall_{c \text{ and } \delta}.$

Graph look like Figure 3.8.5 of [BD].

(3) Resonance occurs when there is a sinosoidal driving force whose frequency equals the "natural" frequency and there is no friction:

$$\ddot{x} + \omega^2 x = \sin\ \omega\ t.$$

Graph of solutions (with "zero" initial conditions: $x(0) = 0 = \dot{x}(0)$) look like Figure 3.9.2 of [BD]; the graph describes sinosoidal bouncing with the "amplitude" increasing at a constant rate.

(4) "Beats" occur when there is a sinosoidal driving force whose frequency does not equal the "natural" frequency and there is no friction:

$$\ddot{x} + \omega^2 x = \sin\ \omega_2 t,\ \text{when } \omega_2 \neq \omega.$$

Graph of solutions look like Figure 3.9.1 of [BD].

Systems of Linear Differential Equations

We will be calling pairs of equations like this:

$$\dot{x} = 2x + 3y$$

$$\dot{y} = 4x + 5y,$$

a "system of differential equations". They are also "simultaneous" equations and "parametric" equations; although those terms are rarely used after freshmen calculus. This is a good time to review parametric equations in your calculus book.

Parametric Equations

As background, we begin with a short description of "vector-valued" functions. If $x(t) = t$ and $y(t) = 2t$, then $v(t) = (x(t), y(t))$ means that $v(t) = (t, 2t)$ and v is a function of t, whose range is the set of coordinate vectors in the plane. This function $v(t)$ is called a *vector-valued function* of t. Sometimes, $x(t) = t$ and $y(t) = 2t$ are described as *parametric* equations with parameter t. Here, when $t = 10$, then $x = 10$ and $y = 20$, and $v = (10, 20)$. Here $y = 2x$, $\forall_t$; hence as t goes from 1 to 3, $v(t)$ travels along the straight line $y = 2x$ from (1,2) to (3,6).

Exercises: Graphing solution curves for Parametric Equations

Remark. Arrows should be placed on each curve to indicate in which direction the point (determined by the parametric functions) is moving as time increases.

Exercise 52 *Let* $x(t) = e^{2t}$ *and* $y(t) = e^t$, $\quad -\infty < t < \infty$. *Draw the graph in the* $xy-$*plane.*

Exercise 53 *Let* $r(t) = t = \theta(t)$, $\quad -\infty < t < \infty$, *for polar coordinates* (r, θ). *Draw the graph in the* $xy-$*plane.*

Exercise 54 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = 2x(t)$ *and* $\dot{y} = 4y(t)$. *Find (graph) the solution curve which goes through the point* $(x, y) = (3, 9)$.

Set-up: First find the general solutions for $x(t)$ and $y(t)$.

Exercise 55 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = 2x(t)$ *and* $\dot{y} = 3y(t)$. *Graph 5 different solution curves in the* $xy-$ *plane.*

Note that this is an autonomous pair of equations, since $\dot{x}$ *and* $\dot{y}$ *are defined in terms of the dependent variables only – there is no* t *in their formulas. What do you remember about solutions to autonomous equations (like* $\dot{x} = f(x)$*.)*

Exercise 56 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = 4x(t)$ *and* $\dot{y} = 2y(t)$*. Graph 5 different solution curves in the* $xy-$ *plane.*

Exercise 57 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = -2x(t)$ *and* $\dot{y} = 4y(t)$*. Graph 5 different solution curves in the* $xy-$ *plane.*

Exercise 58 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = 2x(t)$ *and* $\dot{y} = -4y(t)$*. Graph 10 different solution curves in the* $xy-$ *plane.*

Exercise 59 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = -7x(t)$ *and* $\dot{y} = -7y(t)$*. Graph 5 different solution curves in the* $xy-$ *plane.*

Exercise 60 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = -7x(t)$ *and* $\dot{y} = 0$*. Graph 5 different solution curves in the* $xy-$ *plane.*

Exercise 61 *Let* $x(t)$ *and* $y(t)$ *be (unknown) parametric functions of time. Given* $\dot{x} = 0$ *and* $\dot{y} = 7y(t)$*. Graph 5 different solution curves in the* $xy-$ *plane.*

Introduction to matrices

Matrix algebra is a very useful mathematical shorthand system for both representing and for doing algebraic calculations on systems of simultaneous linear equations (linear algebraic equations, linear differential equations, etc.). The matrix shorthand will make many calculations easier to do, easier to understand and easier to remember.

We begin by introducing "column vectors". *Column vectors* are simply coordinate vectors written vertically (top to bottom). The arithmetic (addition and scalar multiplication) for column vectors is the same as the arithmetic for "horizontal" vectors. For example:

$$3\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 10\begin{pmatrix} 10 \\ -1 \end{pmatrix} = \begin{pmatrix} -97 \\ 16 \end{pmatrix}$$

$$\begin{pmatrix} 5e^2 + & 3\sin 2 \\ 5e^3 + & 3\sin 3 \end{pmatrix} = 5\begin{pmatrix} e^2 \\ e^3 \end{pmatrix} + 3\begin{pmatrix} \sin 2 \\ \sin 3 \end{pmatrix}.$$

The difference between column vectors and horizontally written coordinate vectors is one of form only, not of substance. Sometimes, to squeeze a column vector into a single line of type, we use the "transpose" notation $(x, y)^T$ for $\begin{pmatrix} x \\ y \end{pmatrix}$. (The superscript T stands for transpose).

Notation. The zero vector $(0, 0)$ is the origin in the xy-plane; it is often represented by $\mathbf{0}$, or $\underline{0}$, or $\overline{0}$. When writing the zero vector, it is best to use $\underline{0}$, or $\overline{0}$ in order to distinguish it from the number 0.

When used with matrices, coordinate vectors are written vertically as column vectors.

For motivation, let us start by considering this pair of two linear equations:

$$\begin{aligned} a_1x + a_2y &= c \\ b_1x + b_2y &= d \end{aligned}$$

The left sides are sums of products, and therefore we may follow the "slogan": Write sums of products as dot products. Applying this slogan to the two equations yields:

$$\begin{aligned} c &= a_1x + a_2y = (a_1, a_2) \cdot (x, y) \\ d &= b_1x + b_2y = (b_1, b_2) \cdot (x, y) \end{aligned}$$

Now we rewrite these two equations in the form of column vectors:

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a_1x + a_2y \\ b_1x + b_2y \end{pmatrix} = \begin{pmatrix} (a_1, a_2) \cdot (x, y) \\ (b_1, b_2) \cdot (x, y) \end{pmatrix}.$$

Next we introduce the matrix shorthand for the right side:

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} (a_1, a_2) \cdot (x, y) \\ (b_1, b_2) \cdot (x, y) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix},$$

where the equation on the right is an example of "matrix-times-vector multiplication" alias a "matrix-vector product":

Thus the pair of two linear equations,

$$\begin{array}{l} a_1x + a_2y = c \\ b_1x + b_2y = d \end{array} \text{ is represented by } \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}.$$

Note: The entries of the matrix are the coefficients in the linear equations.

Definition. A 2×2-*matrix* M is a square array of numbers consisting of 2 rows and 2 columns. [6] We write:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

.

We may denote the rows of the matrix M by $\{r_1, r_2\}$, that is

$$r_1 = (a, b) \text{ and } r_2 = (c, d).$$

We may denote the matrix M in terms of its rows by: $M = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$. Then the *matrix-times-column-vector product* or simply the *matrix-vector-product* Mv is defined as

$$Mv = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \times v = \begin{pmatrix} r_1 \cdot v \\ r_2 \cdot v \end{pmatrix},$$

for all column vectors v in the plane. This is commonly referred to as "multiply the columns by the [corresponding] rows".

A common matrix is the *identity matrix*, I:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \times 2$$

We observe that multiplying by the identity matrix is like multiplying by one, since:

$$Iv = v, \ \forall_{vectors\ v}.$$

Another common matrix is:

$$rI = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$$

Example 30 *Given* $M = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$ *and* $v = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$, *then*

[6] A $n \times m$-*matrix* M is a rectangular array of numbers consisting of n rows and m columns. Much of what we say about 2×2-matrices will be valid for "larger" ones.

$$Mv = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} 2 \\ 10 \end{pmatrix} \begin{aligned} &= \begin{pmatrix} (1,2)\cdot(2,10) \\ (4,5)\cdot(2,10) \end{pmatrix} \\ &= \begin{pmatrix} 1\times 2 + 2\times 10 \\ 4\times 2 + 5\times 10 \end{pmatrix} \\ &= \begin{pmatrix} 22 \\ 58 \end{pmatrix}. \end{aligned}$$

The first couple of times that you do a matrix-vector multiplication, you should put in *all* the steps as in this example. By the middle of this chapter, you should be calculating simple matrix-vector products in your head.

Example 31 *Write the two equations*

$$\dot{x} = 2x + 3y$$

$$\dot{y} = 4x + 5y$$

in matrix vector form.

Solution: Note that the matrix is obtained by reading off the coefficients of x and y.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix}.$$

Check this equation by doing the matrix vector multiplication

Labelling the matrix and vectors: Set $v = \begin{pmatrix} x \\ y \end{pmatrix}$. Hence (by differentiating both sides with respect to t) $\dot{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$. Set matrix $M = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$.

Now the matrix-vector equation (and hence the pair of differential equations) may be written "formally" in matrix shorthand as:

$$\dot{v} = Mv$$

Linearity of matrix-vector multiplication

As always, when there is a "multiplication", there are rules for "permissible" operations. Following the rules results in *correct* calculations. Not following the rules or creatively making up rules (without checking) often results in *incorrect* calculations.

An alias for linear operator is *linear transformation*. The term "linear transformation" is usually used when the "inputs" from the domain are vectors, not necessarily functions.

Proposition 32 (Linearity Rules for Matrix-vector Multiplication) [7] *The "matrix function" $M(v) = M \times v$ is a linear transformation, that is for each 2×2 -matrix M:*

[7] A *proposition* is a minor theorem.

(i) $M(v+w) = M(v) + M(w), \forall_{vectors\ v\ and\ w \in R^2}$.

(ii) $M(kv) = k(M(v)), \forall_{vector\ v \in R^2 and\ each\ scalar\ k}$.

You may prove this proposition by doing the calculations of Exercise 62.

Proposition 33 *Let* M *be a* 2×2 *matrix and let* $v(t) = \binom{x(t)}{y(t)}$ *be a differentiable vector valued function of time. The transformation* $L(v) = \dot{v} - M \times v(x)$ *is a linear transformation.*

Proof using the results of the preceding proposition.

(i) $L(v+w) = (v+w)' - M(v+w) = v' - M(v) + w' - M(w) = L(v) + L(w), \forall$ $_{vectors\ v}$ and $_{w \in R^2}$.

(ii) $L(kv) = (kv)' - M(kv) = k(v' - M(v)), \forall_{vectors\ v \in R^2}$ and $_{eachscalar\ k}$.

$\surd$ YEA

Observation 34 *Let* M *be a* 2×2 *matrix and let* $v(t) = \binom{x(t)}{y(t)}$ *be a differentiable vector valued function of time. Then* $\dot{v} = Mv$ *is equivalent to a homogeneous linear equation.*

Proof. We rewrite the equation and define:

$$L(v) = \dot{v} - M \times v(t) = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The preceding proposition showed that this operator $L(v)$ is linear. $\surd$ YEA

The definition for linear combinations of functions is naturally extended to "linear combinations of vectors":

Definition: A *linear combination* of two vectors v and w is any vector with the form $Av + Bw$, $\forall_{numbers\ A\ and\ B}$.

The function $7e^{3t}\binom{2}{1} - 3e^{-5t}\binom{-2}{1}$ is an example of a linear combination of the two vector functions $e^{3t}\binom{2}{1}$ and $e^{-5t}\binom{-2}{1}$.

We rewrite Theorem 11; it is also valid for vector-valued functions of time.

Theorem 11 (on solutions to all homogeneous linear equations) Known/given several solutions to a homogeneous linear equation. Then all the linear combinations of these solutions are many more solutions to the (same) equation.

That is if $L(v)$ is a linear transformation/operator, and if u and w are two solutions to $L(v) = \mathbf{0}$, then $v = Au + Bw$, $\forall_{numbers\ A\ and\ B}$ are all solutions to $L(v) = \mathbf{0}$.

The algorithm that we used to solve homogeneous linear equations starts with finding very special solutions; then taking all linear combinations of those very special solutions. We will do the same thing here.

Combining this statement of Theorem 11 and Observation 34 produces:

Corollary 35 *Let* M *be a [square] matrix. Then* all *linear combinations of solutions to* $\dot{v} = Mv$ *are many more solutions to* $\dot{v} = Mv$.

That is if v_1 *and* v_2 *are two solutions to* $\dot{v} = Mv$ *, then* $v = c_1 v_1 + c_2 v_2$, $\forall_{numbers\ c_1\ and\ c_2}$ *are all solutions to* $\dot{v} = Mv$.

Exercise

Exercise 62 *Let*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad and \quad v = \begin{pmatrix} x \\ y \end{pmatrix} \quad and \quad w = \begin{pmatrix} e \\ f \end{pmatrix}.$$

Set $u = v + w$ *and* $v_1 = Mv$ *and* $w_1 = Mw$.

(i) *Calculate* u, v_1 *and* w_1.

(ii) *Set* $u_1 = Mu$. *Calculate* u_1 *and* $v_1 + w_1$.

(iii) *Observe that* $u_1 = v_1 + w_1$, *which is the same as* $M(v + w) = M(v) + M(w)$ *which is rule (i) for linear operators.*

(iv) *Let* k *be a number. Set* $v_2 = kv = \binom{kx}{ky}$. *Calculate* $v_3 = Mv_2$. *Observe that* $v_3 = kv_1$, *which is the same as* $M(kv) = kM(v)$ *which is rule (ii) for linear operators.* $\surd$
YEA

We want to solve $\dot{v} = Mv(t)$. We know that this is a homogeneous linear equation. Therefore, we need only find a few very special solutions, since all their linear combinations will be many solutions. How to find a single solution? – That is the question.

The matrix-vector equation $\dot{v} = Mv(t)$ is a vector-matrix analogue of the scalar equation $\dot{x} = rx(t)$. Bell! This is the exponential growth/decay equation. Its solution is $x = Ae^{rt}$. Analogously, we guess $\binom{x}{y} = v = e^{rt}\binom{A}{B}$, where r and A and B are unknown, to-be-found constants. Label the vector $\binom{A}{B} = w_0$.

Thus our guess for a very special solution to the matrix-vector equation $\dot{v} = Mv(t)$ is

$$v = e^{rt}w_0 = \begin{pmatrix} Ae^{rt} \\ Be^{rt} \end{pmatrix},$$

where r is an unknown, to-be-found constant number and w_0 is an unknown, to-be-found constant vector.

differentiating provides:

$$\dot{v} = \begin{pmatrix} Are^{rt} \\ Bre^{rt} \end{pmatrix} = re^{rt}w_0$$

Plugging into equation $\dot{v} = Mv(t)$ yields:

$$re^{rt}w_0 = Me^{rt}w_0$$

Moving the scalar function e^{rt} to the left yields:

$$re^{rt}w_0 = e^{rt}M_{w_0}$$

Thus:

$$\boxed{rw_0 = Mw_0.}$$

Warning. There is *no* division by vectors. *Do not cancel* the w_0.

We have shown:

Proposition 36 *If $v = e^{rt}w_0$ is a solution to $\dot{v} = Mv(t)$, then $Mw_0 = rw_0$.*

This proposition motivates the following definitions.

Definitions. Given a matrix M, whenever $M\mathbf{w}_0 = r\mathbf{w}_0$, for a special number r and a special vector $\mathbf{w}_0 \neq \mathbf{0}$, then r is called an *eigenvalue* [8] of M and $\mathbf{w}_0$ is an associated

[8] "Eigen" is a german word meaning characteristic.

eigenvector. The general solution to $M\mathbf{w} = r\mathbf{w}$, is the [9] *eigenspace* of M associated with r .

We will use "eigendata" of a matrix as jargon for the "eigenvalues for the matrix, and a corresponding eigenvector of each. We will refer to $v = e^{rt}w_0$ as an "eigenvalue-eigenvector solution" to $\dot{v} = Mv(t)$, when r is an eigenvalue of M, and $\mathbf{w}_0$ is an associated eigenvector.

Remark. The terms *characteristic value* and *characteristic vector* are aliases for *eigenvalue* and *eigenvector*, respectively.

Remark. How does one remember which is the eigenvalue and which is the eigenvector? Eigenvectors or characteristic vectors are always vectors. Eigenvalues or characteristic values are always "values", that is numbers. Eigenspaces are always "spaces", that is (special) sets of vectors.

Remark. The eigenspace, associated with an eigenvalue r, is the set of all eigenvectors associated with an eigenvalue r, together with the zero vector, alias the origin.

Remark. The eigenspaces of 2×2-matrices are straight lines through the origin; the only exceptions occur when the matrix is a multiple of the identity matrix.

The defining equation for eigenvalues and eigenvectors, $rw_0 = Mw_0$ implies that

$$\begin{aligned} &Mw_0 - rw_0 = \mathbf{0} \\ &Mw_0 - rIw_0 = \mathbf{0} \\ &(M - rI)w_0 = \mathbf{0} \end{aligned}$$

Set $A = M - rI$, and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the last equation may be written as:

$$\begin{matrix} A & v & = & \mathbf{0} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} x \\ y \end{pmatrix} & = & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{matrix}$$

and as:

$$\begin{aligned} ax + by = 0 \\ cx + dy = 0 \end{aligned}$$

Usually, these two equations represents two *different* straight lines through the origin; then the origin $(0, 0)$ is the only solution.

The exceptional cases occur when the two equations represent the *same* straight line through the origin. This occurs when when both lines have the *same* slope.

Example 37

$$\begin{aligned} x + 2y = 0 \\ 2x + 4y = 0 \end{aligned} \quad \textit{and in general} \quad \begin{aligned} ax + by = 0 \\ cx + dy = 0 \end{aligned},$$

when both lines have the same slope.

[9]**Grammar:** "an" versus "the". We write "r is *an* eigenvalue of M" since M usually has several eigenvalues. We write "$\mathbf{w}_0$ is *an* associated eigenvector" because there is an infinite number of eigenvectors, all associated with a single, specific eigenvalue. We write "*the* eigenspace of M associated with r" since there is only a single, unique eigenspace associated with each eigenvalue of M.

These two (possible) different equations have the same slope when:

$$-\frac{a}{b} = -\frac{c}{d} \quad \text{or} \quad ad - bc = 0$$

We rewrite the two equations of Example 37 in matrix form:

$$\begin{matrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} x \\ y \end{pmatrix} & = & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ A & v & = & \mathbf{0} \end{matrix}$$

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, there is a line of solutions to $Av = \mathbf{0}$ when $ad - bc = 0$.

Definition. The *determinant* of the 2×2-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Rewriting these results using the term, "determinant":

Proposition 38 *Given a square matrix A. The matrix-vector equation $Av = \mathbf{0}$ has many solutions when the determinent of A is zero, that is* $\det\ A = 0$.

The *characteristic polynomial* for a square matrix M, is the polynomial $\det(M - rI)$; The *characteristic equation* for a square matrix M, is the equation $0 = \det(M - rI)$.

Combining these results yields:

Proposition 39 *A number r is an eigenvalue of a matrix M, if and only if r is a root of the characteristic polynomial,* $\det(M - rI)$, *that is, r is a solution of the characteristic equation,* $0 = \det(M - rI)$.

Remark. This proposition provides an algorithm for calculating eigenvalues, namely, solve the characteristic equation.

Example 40 (for finding eigenvalues and eigenvectors.) *Given* $M = \begin{pmatrix} -1 & 8 \\ 2 & -1 \end{pmatrix}$. *Find all eigenvalues and a single eigenvector associated with each eigenvalue. Also, find many solution to* $\dot{\mathbf{v}} = M\mathbf{v}$.

Calculations. To find the eigenvalues: Solve the characteristic equation: $0 = \det(M - rI) = \begin{vmatrix} -1-r & 8 \\ 2 & -1-r \end{vmatrix}$

$$0 = (-1-r)^2 - 16 \Rightarrow (-1-r)^2 = 16 \Rightarrow -1 - r = \pm 4 \Rightarrow r = -1 \pm 4 = -5, 3$$

Remark. Having found the eigenvalues, the definition of eigenvectors provides an algorithm for calculating them.

To find eigenvectors:

Solve for $\mathbf{w}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$ in the defining equation: $M\mathbf{w}_1 = r\mathbf{w}_1$. Set $\mathbf{w}_1 = (x, y)$.

a) For $r = -5$: $\begin{pmatrix} -1 & 8 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -5 \begin{pmatrix} x \\ y \end{pmatrix}$

$$\left.\begin{array}{r} -x\ + 8y = -5x \\ 2x - y = -5y \end{array}\right\} \quad \left.\begin{array}{r} 4x + 8y = 0 \\ 2x + 4y = 0 \end{array}\right\} \quad \boxed{\text{x = -2y, the eigenspace.}}$$

Remark. All the points on the line $x = -2y$, except the zero vector, are eigenvectors. We will need only one.

Choose $\mathbf{w}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. (Note $\mathbf{w}_1 = \begin{pmatrix} 200 \\ -100 \end{pmatrix}$ is correct also.)

<u>Check:</u> $M\mathbf{w}_1 = \begin{pmatrix} -1 & 8 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \end{pmatrix} = -5 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. $\surd$

Observation 41 *The eigenspace $x = -2y$ contains the solution curves,* $\mathbf{v}_1(t) = Ae^{-5t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\forall_{A \in R}$, *for the equation* $\dot{\mathbf{v}} = M\mathbf{v}$..

b) For $r = 3$: $\begin{pmatrix} -1 & 8 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix}$

$$\left.\begin{array}{r} -x\ + 8y = 3x \\ 2x - y = 3y \end{array}\right\} \quad \left.\begin{array}{r} -4x + 8y = 0 \\ 2x + 4y = 0 \end{array}\right\} \quad \boxed{\text{x = 2y, the eigenspace.}}$$

Choose $\mathbf{w}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

<u>Check:</u> $M\mathbf{w}_2 = \begin{pmatrix} -1 & 8 \\ 2 & -1 \end{pmatrix} ; \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\surd$.

The eigenspace is the line $x = 2y$, which contains the solution curves, $\mathbf{v}_2(t) = Be^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\forall_{B \in R}$, of the equation $\dot{\mathbf{v}} = M\mathbf{v}$.

Since $\dot{\mathbf{v}} = M\mathbf{v}$ is a homogeneous linear equation, all linear combinations of solutions are more solutions. Thus many solutions to $\dot{\mathbf{v}} = M\mathbf{v}$, are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{v} = Ae^{-5t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + Be^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \forall_A \text{ and } {}_{B \in R}.$$

Since $v = \binom{x}{y}$, we may present the answers as

$$x(t) = 2Be^{3t} - 2Ae^{-5t} \text{ and } y(t) = Be^{3t} + Ae^{-5t}, \forall_{numbers\ A\ and\ B}.$$

Warning and reminder. The zero vector is *never* an eigenvector. If your calculations for an eigenvector lead only to the zero vector, then there was an error in your calculations. *Recalculate!*

Exercises

Exercise 63 *(a) Find the eigenvalues and eigenspaces for this matrix:*

$$M_{10} = \begin{pmatrix} -10 & 80 \\ 20 & -10 \end{pmatrix} = 10 \begin{pmatrix} -1 & 8 \\ 2 & -1 \end{pmatrix}$$

(b) What connection do you see between the eigendata for M_{10} and the eigendata for $\left(\begin{smallmatrix} -1 & 8 \\ 2 & -1 \end{smallmatrix}\right)$?

Exercise 64 *(a) Find the eigenvalues and eigenspaces for these matrices.*

$$T_1 = \begin{pmatrix} -12 & -24 \\ 0 & 6 \end{pmatrix} \quad \textit{and} \quad T_2 = \begin{pmatrix} -12 & 0 \\ -24 & 6 \end{pmatrix}$$

Check your answers.

(b) What is interesting about the results? What theorem would you conjecture?

Exercise 65 *(a) Find the eigenvalues and eigenspaces for these matrices.*

$$M_1 = \begin{pmatrix} 0 & 1 \\ 5 & 4 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 4 & 5 \\ 1 & 0 \end{pmatrix}$$

Check your answers.

(b) Does your answer to part (a) conflict with your conjecture for the preceding exercise? If yes, then reword that conjecture.

Exercise 66 *(a) Find the eigenvalues and eigenspaces for these matrices.*

$$\begin{pmatrix} 0 & 9 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise 67 *Let $x(t)$ and $y(t)$ be (unknown) functions of time. Given $\dot{x} = 2x(t)$ and $\dot{y} = 3y(t)$. Write these equations in matrix-vector form. Solve these equations using matrix methods.*

Exercise 68 *Given*

$$\dot{x} = 9y \quad \textit{and} \quad \dot{y} = x.$$

Solve these equations using matrix methods. Check that these solutions represent motion on a hyperbola $x^2 - 9y^2 = C$, for some constant C.

Set up: After finding $x(t)$ and $y(t)$, calculate $x^2 - 9y^2$, then simplify to a constant.

Exercise 69 *Find eigendata for the matrix* $M = \begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}$.

Exercise 70 (Change-of Coordinates) *In the Example 40, with* $M = \begin{pmatrix} -1 & 8 \\ 2 & -1 \end{pmatrix}$, *we calculated that many solutions to* $\dot{\mathbf{v}} = M\mathbf{v}$, *are:*

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{v} = Ae^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + Be^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \forall_A \text{ and } _{B\in R}.$$

(a) *What useful information can you get from this solution?*

(b) *What happens as* $t \to \infty$?

(c) *Note that this is an autonomous system of equations, since* $\dot{x}$ *and* $\dot{y}$ *are defined in terms of the dependent variables only – there is no* t *in their formulas. What do you remember about autonomous equations (* $\dot{x} = f(x)$.*)*

(d) *Graph a few solutions to* $\dot{\mathbf{v}} = M\mathbf{v}$ *in the* xy*-plane.*

(e) *Graph the solutions when* $A = 0$ *and* $B = 5$*; when* $A = 0$ *and* $B = -5$*; when* $A = 3$ *and* $B = 0$*; when* $A = -3$ *and* $B = 0$.

(f) *Change-of-variables*

Note that the solutions may be written as:

$$x = -2Ae^{-5t} + 2Be^{3t} \text{ and } y = Ae^{-5t} + Be^{3t}$$

Guess functions $x_1(t)$ *and* $y_1(t)$*, such that*

$$x = -2x_1 + 2y_1 \text{ and } y = x_1 + y_1.$$

This change-of-variables can be written in vector notation as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Remark. *The columns of this matrix are eigenvectors of matrix* M.

(g) *Plug these formulas for* x *and* y *into the original equation* $\dot{\mathbf{v}} = M\mathbf{v}$*; to convert it into new differential equations in terms of* x_1 *and* y_1.

(h) *Solve these differential equation for* x_1 *and* y_1.

Graph many solutions in the x_1y_1-plane.

(i) *How might the graphs in the x_1y_1-plane be brought over to the xy-plane? What is the x_1-axis in the xy-plane?*

Connection between a single second order linear differential equation and a system of two first order ones.

(A partial explanation of the Dick and Jane problem).

Let us consider the equation from the "Dick and Jane problem":

$$\ddot{x} - 4\dot{x} - 5x = 0$$

Set $\dot{x} = y$ (hence $\ddot{x} = \dot{y}$). This converts the equation into:

$$\dot{y} - 4y - 5x = 0\dot{y} = 5x + 4y$$

We now have this pair of equations:

$$\dot{x} = y,$$

which we rewrite in matrix-vector form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The "eigendata" and "fundermental" solutions are:

$$M \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } v_1(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$M \begin{pmatrix} 1 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 5 \end{pmatrix} \text{ and } v_2(t) = e^{5t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Real matrices with complex eigenvalues

When solving $\dot{\mathbf{v}} = M\mathbf{v}$, sometimes the characteristic polynomial has *complex* roots. When this occurs, largely continue calculating as you would with real numbers; making ajustments analogous to the ones made for second order linear differential equations.

Example 42 *Find an eigenvector for the matrix* $M = \begin{pmatrix} 2 & -4 \\ 4 & 2 \end{pmatrix}$*; an eigenvalue is* $2 \pm 4i$.

To find an eigenvector we solve the defining equation: $M\mathbf{w}_3 = r\mathbf{w}_3$. Set $\mathbf{w}_3 = (x, y)$.

$$\begin{pmatrix} 2 & -4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (2+4i)\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 2x - 4y = 2x + 4ix \\ 4x + 2y = 2y + 4iy \end{matrix} \Rightarrow \begin{matrix} -y = ix \\ x = iy \end{matrix}$$

An eigenvector is $\mathbf{w}_3 = (-1, i)$.

<u>Check:</u>

$$M\mathbf{w}_3 = \begin{pmatrix} 2 & -4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ i \end{pmatrix} = \begin{pmatrix} -2-4i \\ -4+2i \end{pmatrix} \text{ and } (2+4i)\mathbf{w}_3 = (2+4i)\begin{pmatrix} -1 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} -2-4i \\ 2i-4 \end{pmatrix}.$$

Remark.

We rewrite Theorem 17; it is also valid for vector-valued functions of time.

Theorem 17 (on complex solutions to real homogeneous linear equations) Known/given a complex solution to a real homogeneous linear equation. Then both the real and the imaginary parts of this complex solution, are two more solutions to the (same) equation.

Example 43 *Let us find many solutions to the system*

$$\begin{aligned} x'(t) &= 2x - 4y \\ y'(t) &= 4x + 2y. \end{aligned}$$

This is a matrix equation $v'(t) = Mv(t)$, where $M = \left(\begin{smallmatrix} 2 & -4 \\ 4 & 2 \end{smallmatrix}\right)$, is the matrix of Example 42. Hence $M\left(\begin{smallmatrix} -1 \\ i \end{smallmatrix}\right) = (2+4i)\left(\begin{smallmatrix} -1 \\ i \end{smallmatrix}\right)$.

Therefore the system has an eigenvalue-eigenvector solution:

$$v_1(t) = e^{(2+4i)t}\begin{pmatrix} -1 \\ i \end{pmatrix} = e^{2t}e^{4it}\begin{pmatrix} -1 \\ i \end{pmatrix}$$

Substituting $\cos 4t + i \sin 4t = e^{4it}$ yields:

$$v_1(t) = e^{2t}(\cos\ 4t + i\ \sin 4t)\begin{pmatrix} -1 \\ i \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} -e^{2t} & \cos & 4t \\ -e^{2t} & \sin & 4t \end{pmatrix}}_{Re\ v_1(t)} + i \underbrace{\begin{pmatrix} -e^{2t} & \sin & 4t \\ e^{2t} & \cos & 4t \end{pmatrix}}_{Im\ v_1(t)}$$

The matrix equation $v'(t) = M\ v(t)$ is equivalent to the real linear homogeneous equation

$$L(v) = v'(t) - Mv(t) = \mathbf{0}.$$

Therefore Theorem 17 is applicable and two real solutions to the system are just the real and imaginary parts of the complex solutions; they are:

$$v_3(t) = Re\ v_1(t) \quad = \begin{pmatrix} -e^{2t} & \cos\ 4t \\ -e^{2t} & \sin\ 4t \end{pmatrix}$$

$$v_4(t) = Im\ v_1(t) \quad = \begin{pmatrix} -e^{2t} & \sin\ 4t \\ e^{2t} & \cos\ 4t \end{pmatrix}.$$

Thus, using Theorem 11 many solutions are $v = Av_3 + Bv_4$, $\forall_A$ and $_B$

$$v = Ae^{2t}\begin{pmatrix} \cos\ 4t \\ \sin\ 4t \end{pmatrix} + Be^{2t}\begin{pmatrix} -\sin\ 4t \\ \cos\ 4t \end{pmatrix},\ \forall_A \text{ and } _B$$

Having found the solution via $\cos 4t + i \sin 4t = e^{4it}$, let us check the pair of fundermental solutions.

Check

(i) For $x_3(t) = e^{2t} \cos\ 4t$ and $y_3(t) = e^{2t} \sin\ 4t$

$$\begin{aligned} x_3' &= 2e^{2t} \cos\ 4t - 4e^{2t} \sin\ 4t = 2x - 4y \\ y_3' &= 2e^{2t} \sin\ 4t + 4e^{2t} \cos\ 4t = 2y + 4x \end{aligned}$$

$\surd$

(ii) For $x_4(t) = -e^{2t} \sin\ 4t$ and $y_4(t) = e^{2t} \cos\ 4t$

$$\begin{aligned} x_4'(t) &= -2e^{2t} \sin\ 4t - 4e^{2t} \cos\ 4t = 2x = 4y \\ y_4'(t) &= 2e^{2t} \cos\ 4t - 4e^{2t} \sin\ 4t = 2y + 4x \end{aligned}$$

$\surd$

Example 44 *(Finding a complex eigenvector). Given*

$$\begin{matrix} M & w_0 & = & r & w_0 \\ \begin{pmatrix} 4 & -2 \\ 4 & 0 \end{pmatrix} & \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} & = & (2+2i) & \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \end{matrix}$$

Top coordinate:

$$4x_0 - 2y_0 = (2+2i)x_0 = 2x_0 + 2ix_0$$

Solve for y_0:

$$\begin{aligned} 2y_0 &= 2x_0 - 2ix_0 = 2(1-i)x_0 \\ \underbrace{y_0}_{1-i} &= (1-i)\underbrace{x_0}_{1} \end{aligned}$$

$w_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$

Check

$$\begin{aligned} \begin{pmatrix} 4 & -2 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i-i \end{pmatrix} &= \begin{pmatrix} 4-2+2i \\ 4 \end{pmatrix} &= \begin{pmatrix} 2+2i \\ 4 \end{pmatrix} \\ (2+2i)\begin{pmatrix} 1 \\ 1-i \end{pmatrix} &= \begin{pmatrix} 2+2i \\ 2-2i^2 \end{pmatrix} &= \begin{pmatrix} 2+2i \\ 4 \end{pmatrix} \end{aligned}$$

$\checkmark$

Remark. We found the eigenvector by bligthly solving for x and y in the equation for the top coordinate alone; while ignoring the equation for the bottom coordinate. That this rashness is all right was confirmed by the check. This shortcut *only* works for (complex) eigenvectors of 2×2-matrices; it does *not* work for 3×3-matrices or larger matrices.

Remark. If we had calculated the equation for the bottom coordinate instead, we would have found that: $4x_0 = (2+2i)y_0$. Then we would have chosen $w_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$ as an eigenvector.

Remark. In spite of looking very different w_0 and w_1 are actually multiples since $w_1 = (1+i)w_0$.

Exercise 71 *Solve these equations:* $\dot{x} = 6x(t) + 7y(t)$ *and* $\dot{y} = -7x(t) + 6y(t)$, $\forall_t$.

Exercise 72 *Given*

$$\dot{x} = -9y \text{ and } \dot{y} = x.$$

Solve these equations using matrix methods. Show that these solutions represent motion on an ellipse $x^2 + 9y^2 = C$, *for some constant* C.

Set up: After finding $x(t)$ *and* $y(t)$, *calculate* $x^2 + 9y^2$, *then simplify to a constant.*

Exercise 73 *Solve the equation* $\dot{v} = Mv$, *twice for the matrix* $M = \begin{pmatrix} 4 & -2 \\ 4 & 0 \end{pmatrix}$ *of Example 44.*

(a) *First, using the eigenvector* $w_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$.

(b) *Second, using the eigenvector* $w_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$.

(c) *Comment on the difference in the form of the answers.*

Exercise 74 *For each of the two forms of the solution found in the preceding exercise, find the particular one that also satisfies these initial conditions:* $v(0) = \binom{1}{2}$.

Did the difference in the form of the answers remain?

An alternate method for the double root case for 2×2-matrices only.

Example 45 *For* $M = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$, *we calculate that* $M \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. *The only eigenvalue is 2, and* $(1, -1)$ *is an eigenvector.*

An eigenvalue/eigenvector solution to $\dot{\mathbf{v}} = M\mathbf{w}(t)$ *is:* $\mathbf{v}_1(t) = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Another fundermental solution is needed.

Two other solutions have the form

$$\mathbf{v}_2(t) = te^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ A \end{pmatrix} \quad \text{and} \quad \mathbf{v}_3(t) = te^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t} \begin{pmatrix} B \\ 0 \end{pmatrix}.$$

Both $\mathbf{v}_2$ *and* $\mathbf{v}_3$ *type solutions work for almost all double root* 2×2*-matrix cases. The exceptional cases look like:* $M = \begin{pmatrix} r & 0 \\ a & r \end{pmatrix}$. *If the* $\mathbf{v}_2$ *type solution does not work, then the* $\mathbf{v}_3$ *type solution will. Note that both* $\mathbf{v}_2$ *and* $\mathbf{v}_3$ *contain* $t \times \mathbf{v}_1$.

Plug $\mathbf{v}_2(t)$ *into* $L(\mathbf{v}) = \dot{\mathbf{v}} - \mathbf{Mv(t)} = \mathbf{0}$ *and solve for* A.

Here $A = -1$, *and hence:* $\mathbf{v}_2(t) = te^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

Since $\dot{\mathbf{v}} = M\mathbf{v}$ *is a homogeneous linear equation, all linear combiations of solutions, are more solutions. The general solution is*

$$\mathbf{v}(t) = Ke^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C \left[te^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right], \forall_{K \text{ and } C}.$$