ABSTRACT

Title of Dissertation: Complex Hyperbolic Triangle Groups

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We present several approaches to proving discreteness of triangle groups in $PU(2,1)$ and prove several results. The various methods used are algebraic, geometric, and real-world algorithmic (meant to be implemented on a present-day computer).

Complex Hyperbolic Triangle Groups

by

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Introduction

The subject of the discreteness of triangle groups in $PU(2,1)$ is a beautiful and far-reaching one. There has been some exploration in this area following research in the early 1990's by Bill Goldman and John Parker. Their conjecture summarizing the discreteness of ideal triangle groups was tackled unsuccessfully but with some very interesting methods by Hanna Sandler, and then successfully by Richard Schwartz in the late 1990's using an entirely different approach. Since then we have, along with Richard Schwartz, done some research into discreteness of the nonideal cases as well as fundamental domain construction and exploration of the structure of the limit sets and domains of discontinuity. This latter exploration has led to some fascinating results by Schwartz, while the former has led to this thesis.

In this thesis we push the envelope a little by proving some results pertaining to the cases where the complex geodesics intersect within complex hyperbolic space, and when they do not intersect at all. These two categories make up the two main parts of the thesis, with the latter being more thoroughly touched upon than the former. We prove some sufficient conditions for discreteness, some for indiscreteness, and make several conjectures.

I would like to thank Bill Goldman and Richard Schwartz for introducing me to this fascinating area and giving me inspiration when it was sorely lacking. I would also like to thank John Parker and Larry Triplett for several inspiring conversations relating to this research.

1.1 Summary of Results

The results of this thesis may be broken down into three major areas (bundled into chapters 2 and 3) and one minor area (chapter 4).

First we examine some ultra-ideal triangle groups, where none of the generating geodesics meet. We find a two-dimensional family of these groups which are discrete. This discreteness is proved in two ways, one being through the analysis of a subset of the boundary of $H^2_{\mathbb{C}}$ which is taken off itself nontrivially by all elements of the group (Theorem 2.2.0.14), and the other being through the construction of explicit fundamental domains for the group actions (section 2.3).

Second, by doing extensive computer analysis on the behavior of certain objects in the boundary of $H^2_{\mathbb{C}}$, we hypothesize that our discreteness may be extended further (Section 2.4), and we construct an algorithm which will prove the discreteness for specific parameter values outside the range proved previously (section 2.5).

Thirdly we examine some triangle groups where the generating geodesics intersect (Chapter 3). We parametrize a family of these, namely the (n, n, ∞)

cases, and we examine various attributes of these groups. In Sections 3.2 and 3.3 we look more closely at the $(4, 4, \infty)$ groups, prove some discreteness results and make several conjectures. In Section 3.4 we then go on to look at the (n, n, ∞) cases for large enough n that we can see the connection with the Goldman-Parker Conjecture.

Lastly (chapter 4) we give a short, sweeping discreteness criterion for another large family of ultra-ideal groups.

1.2 Preliminaries

Consider complex hyperbolic n-space $H^n_{\mathbb{C}}$, which is defined as the complex projectivization of the negative vectors in $\mathbb{C}^{n,1}$ with the Hermitian inner product defined by $\langle z, w \rangle = z_0\bar{w}_0 + ... + z_n\bar{w}_n - z_{n+1}\bar{w}_{n+1}$. We call a vector $v \in \mathbb{C}^{2,1}$ *positive*, *null*, or *negative* if its Hermitian norm is positive, zero, or negative respectively. Since a negative vector z must have $z_{n+1} \neq 0$, $H^n_{\mathbb{C}}$ may be identified with the unit ball in $\mathbb{C}^n$ by normalizing the last co-ordinate and then ignoring it.

We will be concerned with the case where $n = 2$. Call the ball model in this case B. Projectivizing the null vectors, which must also have nonzero last co-ordinate, we get $\partial H^2_{\mathbb{C}}$, which may be identified with S^3.

Let $PU(n, 1)$ be the group of automorphisms of $\mathbb{C}^{n,1}$ which preserve the Hermitian norm. The elements of $PU(n, 1)$ fall roughly into three categories as follows. Elliptic elements have at least one fixed point in $H^2_{\mathbb{C}}$ and may have others on the boundary. Parabolic elements have a single fixed point which lies in $\partial H^2_{\mathbb{C}}$, roughly correspond to a rotation around a boundary point. Hyperbolic elements have two fixed points which lie in $\partial H^2_{\mathbb{C}}$, and corresponds to rotation around a point which lies outside $H^2_{\mathbb{C}} \cup \partial H^2_{\mathbb{C}}$.

For completeness' sake, we mention that an elliptic element is regular elliptic if its eigenvalues are all distinct. Parabolic elements are divided into two categories. If an element may be written as element of $U(2,1)$ with 1 as its only eigenvalue, it is said to be unipotent. If it is not unipotent, it is ellipto-parabolic (also called screw-parabolic). An ellipto-parabolic element preserves a unique complex geodesic, on which it acts as a parabolic element of $PU(1,1)$.

In [Go], Goldman gives a discriminant function which acts on traces of matrices in $PU(2,1)$ and categorizes the elements accordingly. More precisely, let

$$\bar{f}(z) = |z|^4 - 8Re(z^3) + 18|z|^2 - 27$$

Then for $A \in PU(2,1)$, denote the cube root of unity by ω_3, then:

1. A is regular elliptic iff $\bar{f}(\mathrm{Tr}(A)) < 0$.
2. A is hyperbolic iff $\bar{f}(\mathrm{Tr}(A)) > 0$.
3. A is ellipto-parabolic iff A is not elliptic and $\mathrm{Tr}(A) \in \bar{f}^{-1}(0) - 3\{1, \omega_3, \omega_3^2\}$.
4. A is a complex reflection iff A is elliptic and $\mathrm{Tr}(A) \in \bar{f}^{-1}(0) - 3\{1, \omega_3, \omega_3^2\}$.
5. A represents a unipotent automorphism iff $\mathrm{Tr}(A) \in 3\{1, \omega_3, \omega_3^2\}$.

Definition 1.2.0.1. *We define the* ***Hermitian Cross Product***

$$\boxtimes : \mathbb{C}^{2,1} \times \mathbb{C}^{2,1} \to \mathbb{C}^{2,1}$$

by

$$v \boxtimes w = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\bar{v} \times \bar{w}) = \begin{bmatrix} \bar{v}_3\bar{w}_2 - \bar{v}_2\bar{w}_3 \\ \bar{v}_1\bar{w}_3 - \bar{v}_3\bar{w}_1 \\ \bar{v}_1\bar{w}_2 - \bar{v}_2\bar{w}_1 \end{bmatrix}.$$

The Hermitian cross product plays the same role in the Hermitian vector space that the standard cross product plays in Euclidean space. That is, the cross product of two vectors yields a vector perpendicular to the two.

Definition 1.2.0.2. *Given two points x and y in $H^2_{\mathbb{C}} \cup \partial H^2_{\mathbb{C}}$, we define the **complex geodesic** $C \subset H^2_{\mathbb{C}} \cup \partial H^2_{\mathbb{C}}$ containing these points by lifting x and y to $\tilde{x}$ and $\tilde{y}$ respectively, and then taking $\tilde{C}$ to be the complex span of $\tilde{x}$ and $\tilde{y}$. Let C be the projectivization of $\tilde{C}$, which is a projective subspace of complex dimension 1. C is unique by construction.*

It is worth noting that by consideration of dimensions, two complex geodesics are either identical, disjoint, or meet in a single point. It is impossible for two complex geodesics to meet in a real line.

Definition 1.2.0.3. *The **polar vector** for C is the unique vector perpendicular to both $\tilde{x}$ and $\tilde{y}$ and given by $\tilde{x} \boxtimes \tilde{y}$. Polar vectors are always positive and therefore any positive vector corresponds to a complex geodesic.*

Definition 1.2.0.4. *Two complex geodesics that do not intersect inside $H^2_{\mathbb{C}}$ either intersect in the boundary, in which case they are called **parallel** or **asymptotic**, or are disjoint, in which case they are called **ultraparallel**.*

Given a complex vector space V, there is an underlying real vector space where the vectors are identical, but scalars are taken only from $\mathbb{R}$. Denote this real vector space by $V_{\mathbb{R}}$.

Definition 1.2.0.5. *Let C_1 and C_2 be two complex geodesics with polar vectors c_1 and c_2 respectively. Denote by $span_{\mathbb{R}}(c_i)$ the span of c_i under multiplication by real scalars. Thus $span_{\mathbb{R}}(c_i)$ is a real linear subspace of $\mathbb{C}^{2,1}_{\mathbb{R}}$. Define the* ***angle***

$$\measuredangle(C_1, C_2) = \measuredangle(c_1, c_2) = \min_{c_1, c_2} \{\angle(c_1, c_2) : c_i \in span_{\mathbb{R}}(c_i)\},$$

where $\angle$ denotes the angle between the two vectors measured normally. Since $span_{\mathbb{R}}(c_i)$ is invariant under multiplication by -1, the angle necessarily satisfies $0 \le \measuredangle(C_1, C_2) \le \pi/2$.

Lemma 1.2.0.6. *Suppose c_1 and c_2 are polar vectors for two complex geodesics C_1 and C_2 respectively, and suppose c_1 and c_2 have been normalized. There are three possibilities.*

1. *$|< c_1, c_2 >| < 1$, in which case $|< c_1, c_2 >| = \cos(\theta)$, where θ is the angle of intersection between C_1 and C_2, which meet inside $H^2_{\mathbb{C}}$. In this case $c_1 \boxtimes c_2$ is a negative vector, corresponding to the point of intersection.*

2. *$|< c_1, c_2 >| = 1$ in which case C_1 and C_2 are parallel. In this case, $c_1 \boxtimes c_2$ is a null vector, corresponding to the intersection point on the boundary.*

3. *$|< c_1, c_2 >| > 1$, in which case $|< c_1, c_2 >| = \cosh(\rho/2)$, where ρ is the distance between C_1 and C_2. In this case, $c_1 \boxtimes c_2$ is a positive vector, and is the polar vector corresponding to the unique complex geodesic orthogonal to and intersecting C_1 and C_2.*

Definition 1.2.0.7. *Given a complex geodesic C, there is a unique involution $i \in PU(2,1)$ which fixes C, called* ***inversion*** *in C. This involution is given in $\mathbb{C}^{2,1}$ by the standard inversion in a vector subspace. Specifically, if p denotes the polar vector for C, we have $V \mapsto \Pi_{p^\perp}(V) - \Pi_p(V)$.*

Definition 1.2.0.8. *Given a complex geodesic C and a unit complex number μ, there is an elliptic element which rotates "around" the geodesic C by μ radians. This will be called a μ-**reflection** in C.*

For example, when C has polar vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

the diagonal matrix $(\mu, 1, 1)$ is the μ-reflection. All others are $PU(2,1)$-conjugate to this one.

An inversion in a complex geodesic is then simply a μ-reflection with $\mu = -1$. Clearly, if μ is an n-th root of unity, then the μ-reflection has order n.

Consider that a complex geodesic, in the ball model, is homeomorphic to a disc. Thus its intersection with the boundary is homeomorphic to a circle. We call circles that arise in this manner *chains*. From two distinct points on a chain we can retrieve the complex geodesic, so there is a bijection between chains and complex geodesics. We can therefore, without loss of generality, talk about reflections in chains rather than in complex geodesics.

Complex geodesics are one of two types of totally geodesic submanifolds of complex hyperbolic space, and the metric on $H^2_{\mathbb{C}}$ restricts on a complex geodesic to the Poincaré model of the unit disk. The other type of totally geodesic submanifolds are totally real. More concretely, let U be a subspace of $\mathbb{C}^{2,1}_{\mathbb{R}}$. Then U is said to be totally real if $\mathbb{J}(U)$ is orthogonal to U, where orthogonality is determined under the nondegenerate real-valued symmetric bilinear form $\mathrm{Re}\langle *, * \rangle$ and $\mathbb{J} : \mathbb{C}^{2,1}_{\mathbb{R}} \to \mathbb{C}^{2,1}_{\mathbb{R}}$ such that $\mathbb{J} \circ \mathbb{J} = -\mathrm{Id}$. (Such a $\mathbb{J}$ is called a complex structure.) The projectivization of U is then a totally real totally geodesic submanifold of $H^2_{\mathbb{C}}$ to which the metric restricts yielding the Klein model.

The simplest totally real totally geodesic submanifold is the fixed point set of the automorphism $(z, w) \mapsto (\bar{z}, \bar{w})$ in the ball model of $H^2_{\mathbb{C}}$. All others are images of this one under $PU(2, 1)$.

The boundaries of these submanifolds are called $\mathbb{R}$-circles. We will not go far into depth into the structure of these, we will only mention the few types we shall need. For a more complete description see [Go].

Definition 1.2.0.9. *Given three complex geodesics γ_0, γ_1 and γ_2, consider the representation $\Gamma : G = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \to PU(2, 1)$, the former having generators g_0, g_1 and g_2, which takes g_i to inversion in γ_i. We call the image of Γ a* ***triangle group****. Abusing notation, we also say Γ is a triangle group, and we say that g_0, g_1 and g_2 generate the triangle group Γ.*

Naturally various types of triangle groups depend on how the geodesics γ_i intersect one another. We isolate three particular cases for discussion. If the γ_i meet pairwise in $\partial H^2_{\mathbb{C}}$, we call Γ an ideal triangle group. If the γ_i meet pairwise inside $H^2_{\mathbb{C}}$ (or perhaps some but not all meet in $\partial H^2_{\mathbb{C}}$), we call Γ a sub-ideal triangle group. Lastly, if the γ_i do not meet inside $H^2_{\mathbb{C}}$ (or again some in $\partial H^2_{\mathbb{C}}$) we call Γ an ultra-ideal triangle group.

In all three cases, the question may be asked of when a given triangle group is discrete. By discrete we mean that the group is discrete as a subgroup of $GL_3(\mathbb{C})$, the group of 3×3 matrices over $\mathbb{C}$.

The boundary $\partial H^2_{\mathbb{C}}$ is homeomorphic to S^3, and one of the representations we choose for this is $\mathbb{C} \times \mathbb{R} \cup \{\infty\}$, with points either ∞ or (z, r) with $z \in \mathbb{C}$ and $r \in \mathbb{R}$, intuitively the first co-ordinate corresponds to the xy-plane and the second to the z-axis as we are familiar with it. Let $\mathcal{H}$ denote this representation, that is, $\mathbb{C} \times \mathbb{R} \cup \{\infty\}$. We have the homeomorphism $\mathcal{P}$ taking S^3 to $\mathcal{H}$ given by the standard stereographic projection:

$$
\begin{aligned}
&(z_1, z_2) \mapsto \left(\frac{z_1}{1+z_2}, -\mathrm{Im}\left(\frac{1-z_2}{1+z_2} \right) \right) \\
&(0, -1) \mapsto \infty
\end{aligned}
$$

There are many stereographic projections based from any point in the boundary. The technical definition is unnecessary here, but suffice to say we are using but one example based at $(0, -1)$.

The most important thing we need to know about the $\mathcal{H}$-representation is the way chains appear. Specifically, the unit circle in $\mathbb{C} \times \{0\}$ and vertical lines (with the infinite point) are all chains. It is straightforward to show that the only chains through ∞ are vertical. Other chains are various ellipses (perhaps circles) which project to circles via $\mathbb{C} \times \mathbb{R} \to \mathbb{C}$. See [Go] for more details.

Definition 1.2.0.10. *For $z \in \mathbb{C}$, the* **z-*chain*** *is the chain having polar vector*

$$\begin{bmatrix} 1 \\ -\bar{z} \\ \bar{z} \end{bmatrix}.$$

The z-chain is the vertical chain in $\mathcal{H}$ through the point $(z, 0)$.

Definition 1.2.0.11. *For $z, r \in \mathbb{R}$ the* **(z, r)*-chain*** *is the chain having polar vector*

$$\begin{bmatrix} 0 \\ 1 + r^2 + iz \\ 1 - r^2 - iz \end{bmatrix}.$$

The (z, r)-chain is the circle of radius r centered at the origin in $\mathbb{C} \times \{z\} \subset \mathcal{H}$.

Definition 1.2.0.12. *The* ***Clifford Torus*** *T is*

$$T = \left\{ (z, w) \in B : |z| = |w| = \sqrt{\frac{1}{2}} \right\}.$$

Define the map $\Phi : B \to \mathbb{C}$ by $(z, w) \mapsto w$. Φ is the orthogonal projection onto the complex geodesic having the 0-chain as its boundary. The image of this map is the unit disk Δ in $\mathbb{C}$, and in particular the 0-chain is taken to the unit circle and T is taken to the circle of radius $\sqrt{1/2}$. Since Δ is a complex geodesic, it carries the Poincaré model of real hyperbolic space.

Chapter 2

Isosceles Ultra-Ideal Triangle Groups

2.1 Preliminaries

First we introduce some notation.

Definition 2.1.0.13. *Define an* $[\mathbf{m}, \mathbf{l}, \mathbf{k}]$ ***triangle group****, for* $m, l, k \geq 0$ *to be the triangle group whose generating complex geodesics are pairwise distance* m, l *and* k *apart. We extend this definition to say that if one of these is 0, then the corresponding geodesics are asymptotic.*

This notation complements the standard notation of a (p, q, r) triangle group for which the geodesics intersect or are asymptotic.

Inversion in the (z, r)-chain is given by the matrix

$$i_{z,r} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \dfrac{1+z^2+r^4}{2r^2} & \dfrac{-1-2iz+z^2+r^4}{2r^2} \\ 0 & -\dfrac{-1+2iz+z^2+r^4}{2r^2} & \dfrac{1+z^2+r^4}{2r^2} \end{bmatrix}.$$

This map commutes with Φ ($B \to \mathbb{C}$ taking $(z, w) \mapsto w$) and takes Δ to itself. Restricted to Δ, $i_{z,r}$ is a fractional linear transformation with a single fixed point. That is, it is an elliptic element of $PU(1,1)$ of order two corresponding to the lower right two-by-two block in the matrix above.

The product $i_{z,r} i_{-z,r}$ restricted to Δ is a hyperbolic element of $PU(1,1)$, therefore fixes two points on the boundary $\partial\Delta$. These points correspond in the $\mathcal{H}$ model to the two fixed points on the vertical chain (invariant under the product) through 0, which derive from the attracting and repelling eigenvectors of the product of the inversions.

Let C be the chain with polar vector

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

In $\mathcal{H}$, C winds itself around the Clifford Torus. Inversion in C, denoted i_c, takes the Clifford Torus to itself and interchanges the two components of $\mathcal{H} - T$.

The collection of ultra-ideal triangles we consider have the property that one of the remaining two chains is inside the Clifford Torus and the other is outside. Unfortunately, there is no particularly nice way to arrange these last two chains in $\mathcal{H}$. Picking one of the last two complex geodesic and inverting it in C, we produce a triple of complex geodesics which do lie in a more convenient fashion. Clearly the group generated by this new triple is identical to that produced by the original. We take this approach in labelling our inversions. When we need to discuss distances between pairs of sides, though, we will switch back.

Fix $z, r \in \mathbb{Z}$. Let $i_+ = i_{z,r}$ and $i_- = i_{-z,r}$. Let Γ be the group generated by i_c, i_+ and i_-. Let Λ be the group generated by i_+ and i_-. Let C_+ and C_- be the chains (and by abuse of notation the geodesics) left invariant by i_+ and i_- respectively. Let $C'_- = i_c(C_-)$, so the triple C, C_+, C'_+ really form the triangle we are considering, while the triple C, C_+, C_- is easier to work with. It follows from symmetry that the triangle thus formed is isosceles.

2.2 First Discreteness Theorem

We prove the following theorem:

Theorem 2.2.0.14. Γ *is discrete given that*

1. *z and r satisfy*

$$z^2 + 2r^4z^2 + r^8z^2 + 2z^4 + 2r^4z^4 + z^6 - 8r^8 \geq 0,$$

2. *The (z, r)-chain is not contained within the Clifford Torus.*

Call the inequality in (1) *Condition D*.

The cleanest way to express (2) is to note that one foliation of the Clifford Torus by (w, s)-chains may be given by the collection of such chains satisfying $w = \sqrt{6s^2 - 1 - s^4}$, so that a (z, r)-chain is not inside the Clifford Torus iff $z^2 \geq 6r^2 - 1 - r^4$. Call this condition the *Torus Condition*.

Figure 2.1 illustrates the zeros set of both Condition D and the Torus Condition. The picture is one quadrant of a vertical slice through the z-axis of the $\mathcal{H}$-representation. The area of discreteness is the area above both curves.

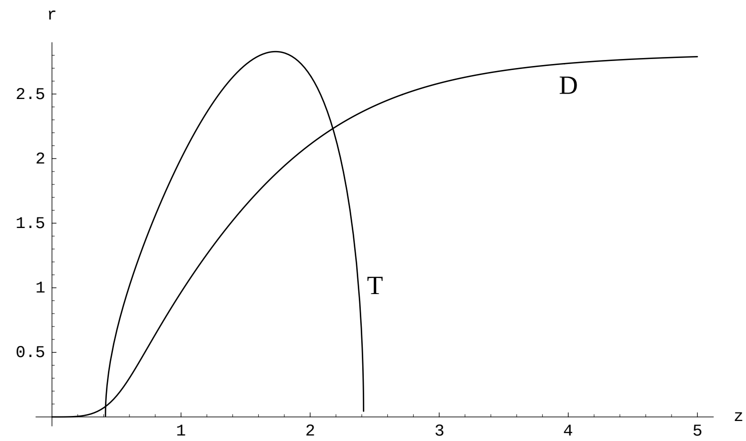

Figure 2.1: Plot of Condition D

2.2.1 Discreteness Criterion

We use the same discreteness criterion used by Richard Schwartz in [Sc].

Definition 2.2.1.1. *Suppose* Γ *acts on a set* S*, and* $U_1, U_2, V \subset S$ *with* $V \subsetneq U_1$*, we say that* $(U_1, U_2)_V$ *is* ***compressing*** *for* Γ *if the following two conditions are met:*

1. $i_c(U_1) = U_2$.
2. $i(U_2) \subsetneq V$ *for all non-identity* $i \in \Lambda$.

Definition 2.2.1.2. *We say* $(U_1, U_2)_V$ *is* ***semi-compressing*** *if only the second of these conditions is met.*

We say Γ is compressing if there exists a set S on which Γ acts, and three subsets U_1, U_2, and $V \subsetneq U_1$, for which $(U_1, U_2)_V$ is compressing. Note that Λ is important in this definition.

The following is elementary:

Lemma 2.2.1.3. *Let $\rho : A \to B$ be a surjective map of sets on which Γ acts. Suppose there exist U_1, U_2, and $V \subsetneq U_1$ all subsets of A such that Γ is semi-compressing with respect to the sets $\rho(U_1)$, $\rho(U_2)$, and $\rho(V)$ in B. Then if i_c interchanges U_1 and U_2, then $(U_1, U_2)_V$ is compressing for Γ.*

Lemma 2.2.1.4. *If Γ is compressing, then Γ is a discrete subgroup of $PU(2,1)$.*

Proof. Consider an element $g \in \Gamma$. There are four types of g and each has an action on either U_1 or U_2 which is isolated from the identity. Observe that g may be written as $i_1 i_c i_2 ... i_c i_n$ where all the i_j are in Λ. Then notice the four types and their effects in the following. For purposes of clarification for the four lines below assume the leading and trailing i_j's, if they exist, are non-identity.

1. $i_1 i_c i_2 ... i_c i_n(U_2) \subsetneq V$
2. $i_1 i_c i_2 ... i_c(U_1) \subsetneq V$
3. $i_c i_2 ... i_c i_n(U_2) \subsetneq i_c(V)$
4. $i_c i_2 ... i_c(U_1) \subsetneq i_c(V)$

In all four cases, g is nontrivial and is isolated from the identity. Thus Γ is discrete.

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2.2.2 Discreteness Proof

This implies discreteness of all the ideal triangle groups up to a certain limit, and many ultra-ideal groups.

Let U_1 be the outer component of $\mathcal{H} - T$ (the component containing the 0-chain) and let U_2 be the inner component. The map c switches U_1 and U_2. We will find V such that Γ is semi-compressing with respect to $(\Phi(U_1), \Phi(U_2))_{\Phi(V)}$, proving Γ is discrete.

Let $U = \phi(U_2)$, the inner component of $\Delta - \Phi(T)$.

It suffices to show that Λ takes U completely off itself for the parameters in question, and that the images of U under all elements of Λ are contained in a proper subset of $\Phi(U_1)$.

Let $h = i_+ i_-$, and consider h as a hyperbolic element of $PU(1,1)$ acting on Δ with the Poincaré model. Explicitly computing the eigenvectors of h, we find the fixed points of h to be

$$\left(\frac{1 - r^4 - z^2}{1 + r^4 + z^2}\right) \pm \left(\frac{2\sqrt{r^4 + z^2}}{1 + r^4 + z^2}\right) i.$$

The first of these corresponds to the attracting eigenvector and as such shall be denoted v_a. The latter of these corresponds to the repelling eigenvector and shall be denoted v_r. Denote by γ the real geodesic connecting the two.

Definition 2.2.2.1. *Suppose $\alpha \in PU(2,1)$ is hyperbolic with invariant geodesic λ not passing through the origin. Define the* ***clockwise end*** *of λ to be the end on the clockwise side of the smaller component (in the Euclidean metric) of $\Delta - \lambda$.*

Definition 2.2.2.2. *We say that two hyperbolic elements in $PU(2,1)$ are* ***agreeable*** *if they have the same attracting and repelling fixed points.*

Definition 2.2.2.3. *Suppose α and β are agreeable geodesics. We say α in $PU(1,1)$ is* ***stronger*** *than β if α moves any given point further than β.*

Lemma 2.2.2.4. *Suppose α_1 and α_2 are agreeable geodesics, and λ_i is the attracting eigenvalue for α_i. Then α_1 is stronger than α_2 iff $\lambda_1 > \lambda_2$.*

Proof. Under $PU(1,1)$-equivalence it suffices to prove the lemma for the element

$$\begin{bmatrix} \frac{\lambda^2+1}{2\lambda} & i\frac{\lambda^2-1}{2\lambda} \\ -i\frac{\lambda^2-1}{2\lambda} & \frac{\lambda^2+1}{2\lambda} \end{bmatrix}$$

where $\lambda > 1$ (resp. $1/\lambda$) is the eigenvalue corresponding to the attracting fixed point i (resp. repelling fixed point $-i$). A simple computation (applying the element to any point in $\text{int}(\Delta)$) shows that increasing λ yields a stronger element.

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Definition 2.2.2.5. *Suppose ω is a geodesic in real hyperbolic space, and let $d \in \mathbb{R}_+$. Let W denote the set of all points whose distance to ω is less than or equal to d. W is bounded by two curves which are said to have distance d from ω. Let any curve which arises in this fashion be called* ***parallel*** *to ω.*

Definition 2.2.2.6. *Suppose α is a geodesic in real hyperbolic space. A curve parallel to α is called a* ***hypercycle***.

Lemma 2.2.2.7. *In the Poincaré model, the hypercycles are all Euclidean circles through the endpoints of the geodesic, except for one, which is the Euclidean straight line through the two points.*

Proof. The easiest way to see this is to map the unit disk to the upper half-plane model of real hyperbolic space by the map

$$z \mapsto \frac{1 - iz}{z - i}$$

with inverse

$$z \mapsto \frac{zi + 1}{z + i}.$$

In this model, geodesics are either vertical lines or semicircles perpendicular to the real axis. Let p be a point on the geodesic consisting of the complex axis and let x be the point of distance $\ln(d)$ measured along the perpendicular geodesic to the right of p. That is, x lies on the circle of radius $\mathrm{Im}(p)$ centered at the origin. Since distance is measured along semicircles centered at the origin via the formula

$$\mathrm{dist}(x, p) = \ln \frac{|x - \bar{p}| + |x - p|}{|x - \bar{p}| - |x - p|},$$

the point x satisfies

$$d = \frac{|x - \bar{p}| + |x - p|}{|x - \bar{p}| - |x - p|}.$$

Solving this, we find that

$$\mathrm{Im}(x) = \left(\frac{2d}{d^2+1}\right)\mathrm{Im}(p).$$

Since $|x|^2 = \mathrm{Im}(p)^2$, we compute

$$\mathrm{Re}(x) = \sqrt{1 - \left(\frac{2d}{d^2+1}\right)^2}\,\mathrm{Im}(p)$$

so that x lies on the line whose slope is the ratio of these. Specifically, the slope of the line is independent of p. Thus the hypercycle consisting of all such x for all such p consists precisely of the line with this slope. Note that this line passes through both the origin and infinity.

Since both of the maps above take circles and lines to circles and lines, the hypercycle is taken back to either a circle or a line in the disk model as claimed.

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Lemma 2.2.2.8. *Suppose α is the invariant geodesic for a hyperbolic element g. The orbit of a point p under g lies on a hypercycle parallel to α.*

Proof. We measure distance from p to α along the unique geodesic perpendicular to α. Since g preserves angles and distances, p is taken to a point whose distance to α is identical. The set of such points is a hypercycle by definition.

$\bigtriangledown$

In fact, if we take the infinitesimal generator of a hyperbolic element, the orbit of a point under that generator is precisely a hypercycle.

Suppose α is a hyperbolic element with attracting and repelling fixed points α_a and α_r respectively. For any β agreeable with α, $\beta(U)$ is contained within the two unique hypercycles joining α_a to α_r and tangent to U. It follows that if β is stronger than α and α takes U off itself, β will also.

The attracting eigenvalue for the element h is

$$\frac{r^4 + 2z(z + \sqrt{r^4 + z^2})}{r^4} = 1 + \frac{2z^2}{r^4} + 2z\sqrt{\frac{1}{r^{12}} + \frac{z^2}{r^{16}}}.$$

Increasing z and/or decreasing r will increase the attracting eigenvalue, and hence yield a stronger hyperbolic element. It follows that here is a "first case" of when U is off itself, and then it is off itself for all larger z and smaller r.

A straightforward calculation shows that there is a unique hypercycle $\tilde{\gamma}$ parallel to γ which is perpendicular to $\Phi(T)$.

Let t_a be the point in $\tilde{\gamma} \cap \Phi(T)$ on the same side of $\Phi(T)$ as v_a, and let t_r be the other point.

Since elements of $PU(1,1)$ preserve angles and preserve the property of being a circle, and since the effect of h is to "drag" $\Phi(T)$ toward v_a, insisting that h take $\Phi(T)$ to a tangent circle (the first case of U being taken off itself) is equivalent to the condition that $h(t_r) = t_a$.

The following is an elementary geometric computation.

Lemma 2.2.2.9. *Suppose g is a hyperbolic element with conjugate fixed points*

$$\alpha \pm i\sqrt{1 - \alpha^2}$$

and invariant geodesic γ. The unique invariant hypercycle parallel to γ and perpendicular to the circle of radius $\sqrt{1/2}$ intersects that circle at the points

$$\frac{2}{3}\alpha \pm i\sqrt{\frac{1}{2} - \frac{4}{9}\alpha^2}.$$

The vectors (in $\mathbb{C}^{2,1}$) corresponding to t_a and t_r are therefore given (up to equivalence) by

$$\begin{bmatrix} 0 \\ a \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ \bar{a} \\ b \end{bmatrix}$$

where

$$a = 4(1 - (r^4 + z^2)^2) + i\sqrt{18(1 + r^4 + z^2)^4 - 16(1 - (r^4 + z^2)^2)^2}$$

and

$$b = 6(1 + r^4 + z^2)^2.$$

The matrix (in $PU(2,1)$) for the element h is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \bar{\beta} & \bar{\alpha} \end{bmatrix}$$

where

$$\alpha = \frac{r^4(1 + iz) + iz(-i + z)^2}{r^4}$$

and

$$\beta = \frac{iz(1 + r^4 + z^2)}{r^4}.$$

It suffices to show that $\Phi h(t_r) = \Phi(t_a)$ for the data we have here. In the notation above,

$$\Phi h(t_r) = \frac{\alpha a + \beta b}{\bar{\beta} a + \bar{\alpha} b},$$

and

$$\Phi(t_a) = \frac{\bar{a}}{b}.$$

In order for these two to be equal, we solve the equation

$$(\alpha a + \beta b) b = (\bar{\beta} a + \bar{\alpha} b)\bar{a},$$

or equivalently, find the solutions of

$$\alpha a b + \beta b^2 - \bar{\beta} a \bar{a} - \bar{\alpha} b \bar{a} = 0.$$

Cancelling all denominators and simplifying the left-hand side of this equation gives us leaves us solving $(1 + r^4 + z^2)^2(1 + 34r^4 + r^8 + 34z^2 + 2r^4z^2 + z^4)(z^2 + 2r^4z^2 + r^8z^2 + 2z^4 + 2r^4z^4 + z^6 - 8r^8) = 0$. Since the first two terms are strictly positive, the condition is that the third term is zero, as we claimed.

Figure 2.2 shows the projection via Φ to Δ of Condition D and the Clifford Torus to Δ. The region corresponding to Condition D is the region above and below the top and bottom curves.

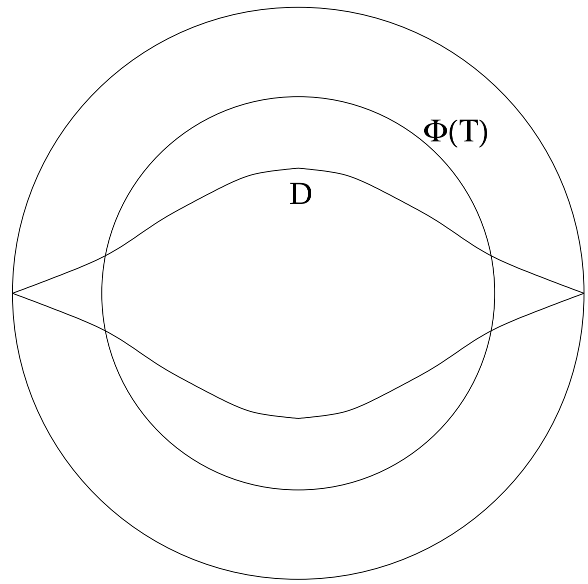

Figure 2.2: Projection of Condition D

Figure 2.3 shows an example of Condition D. Included are v_a, v_r, t_a, t_r, the geodesic γ connecting v_a and v_r, and the hypercycle $\bar{\gamma}$ parallel to γ and perpendicular to $\Phi(T)$. We also see both $\Phi(T)$ and its image under the map h.

By symmetry, $i_- i_+$ satisfies the same criteria. By the parabolicity of h, once U has left itself, all powers of h merely drag U closer towards v_a. Similarly, all powers of $i_- i_+$ drag U closer towards v_r.

All that's left (up to symmetry) is to show that for all $n \geq 0$, $i_- h^n(U)$ is disjoint from U.

Assume for convenience that v_a is contained in the first quadrant. The second quadrant argument is symmetric. γ, the invariant geodesic for h, is also left invariant by both i_+ and i_-. Let R be the union of U with the region in the first and second quadrants between the two (unique) hypercycles parallel to γ

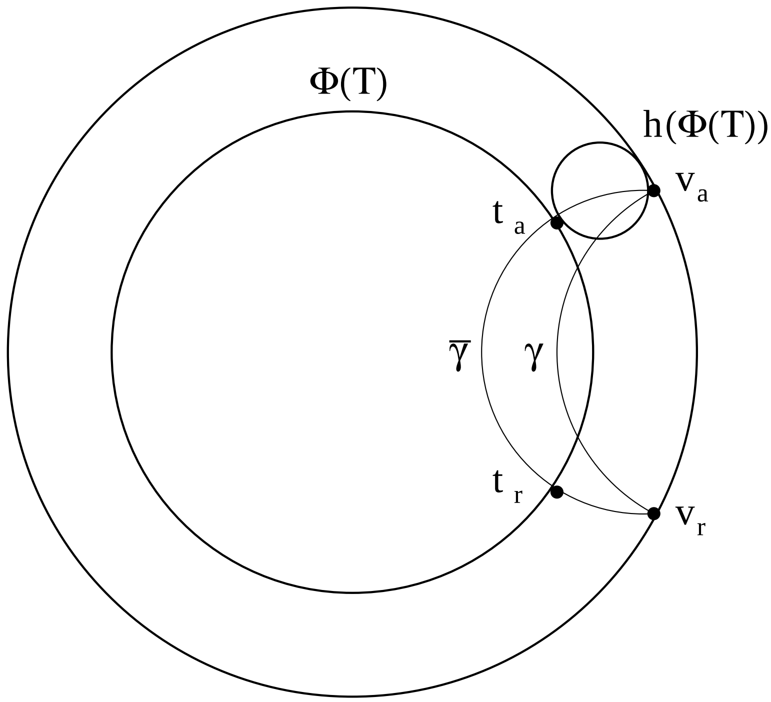

Figure 2.3: Example of Condition D

and tangent to $\Phi(T)$. Since these hypercycles are parallel to γ, they are left invariant by h and therefore since U is contained in R, all the images $h^n(R)$ are also contained in R. It suffices therefore to show that i_- takes R off itself.

It suffices to show that there exists a geodesic γ' through p_- which does not intersect R, since then i_- leaves this geodesic invariant and switches both sides, so $i_-(R) \cap R = \emptyset$.

Suppose $\gamma \cap U = \emptyset$, then γ is on the outside of the region R and so we may let $\gamma' = \gamma$.

On the other hand, if $\gamma \cap U \neq \emptyset$, then let α be the geodesic in the first and second quadrants tangent to $\Phi(T)$ and symmetric with respect to the x-axis. Note that inversion in points on α will take R off itself because α, being to the right of

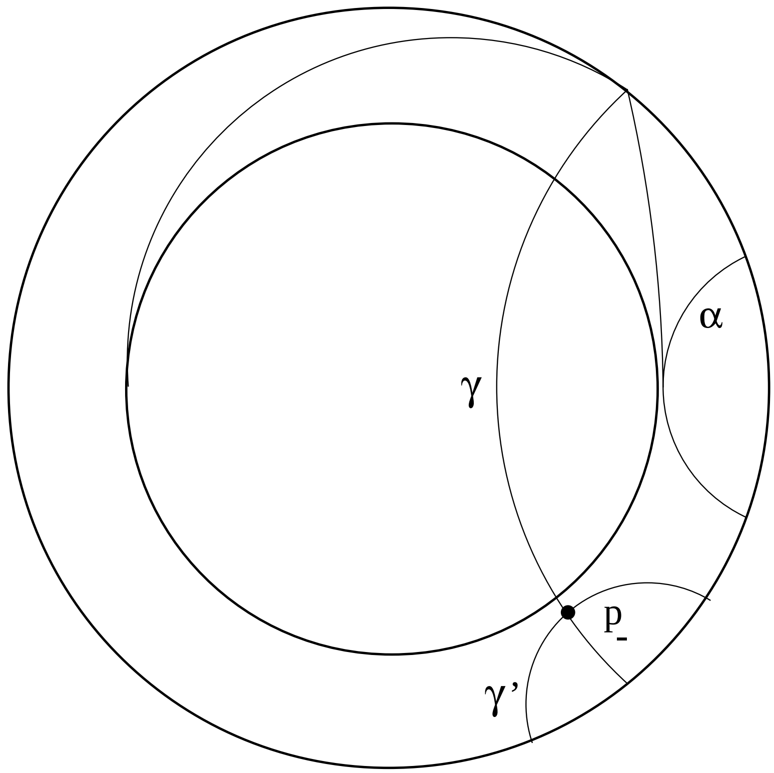

Figure 2.4: $\gamma \cup U \neq \emptyset$

γ, is therefore to the right of the hypercycles parallel to γ up until the last case when the hypercycle, α and $\Phi(T)$ are all tangent to one another on the right side of $\Phi(T)$. Thus α has all of R on its left-hand side. Let γ' be the unique geodesic passing through p_- with perpendicular bisector passing through the origin. Since the angle p_- makes with the x-axis is greater than zero, γ' is further clockwise than α and since $|p_-| \geq \sqrt{1/2}$, the endpoints of γ' are closer than or equal to α's, measured along the unit circle with the Euclidean metric. Putting these together gives us that the most counterclockwise endpoint of γ' is further clockwise than that for α. Therefore γ' does not intersect R so $i_-(R) \cap R = \emptyset$. Figure 2.4 shows a picture of this.

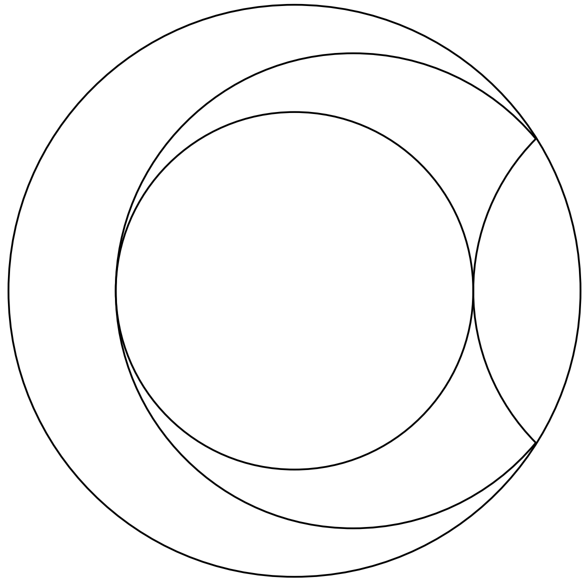

Figure 2.5: $R \cup S$, the Projection of the Fundamental Domain

A similar argument works for $i_+(i_-i_+)^n$, where we denote by S the region analogous to R.

Thus Λ takes U completely off itself. Since we surely missed parts of the first and fourth quadrants (or second and third), we are guaranteed that the images of U under all elements of Λ are contained in a proper subset of $\Phi(U_1)$, namely $R \cup S$. Thus we see that the sets $(\Phi(U_1), U)_{R \cup S}$ are semi-compressing for Γ.

In figure 2.5 we see the region $R \cup S$ between the bounding hypercycles tangent to $\Phi(T)$ and parallel to γ.

Now we may define V to be $\Phi^{-1}(R \cup S)$, which gives us $(U_1, U_2)_V$ compressing for Γ. Thus Γ is discrete.

$\bigtriangledown$

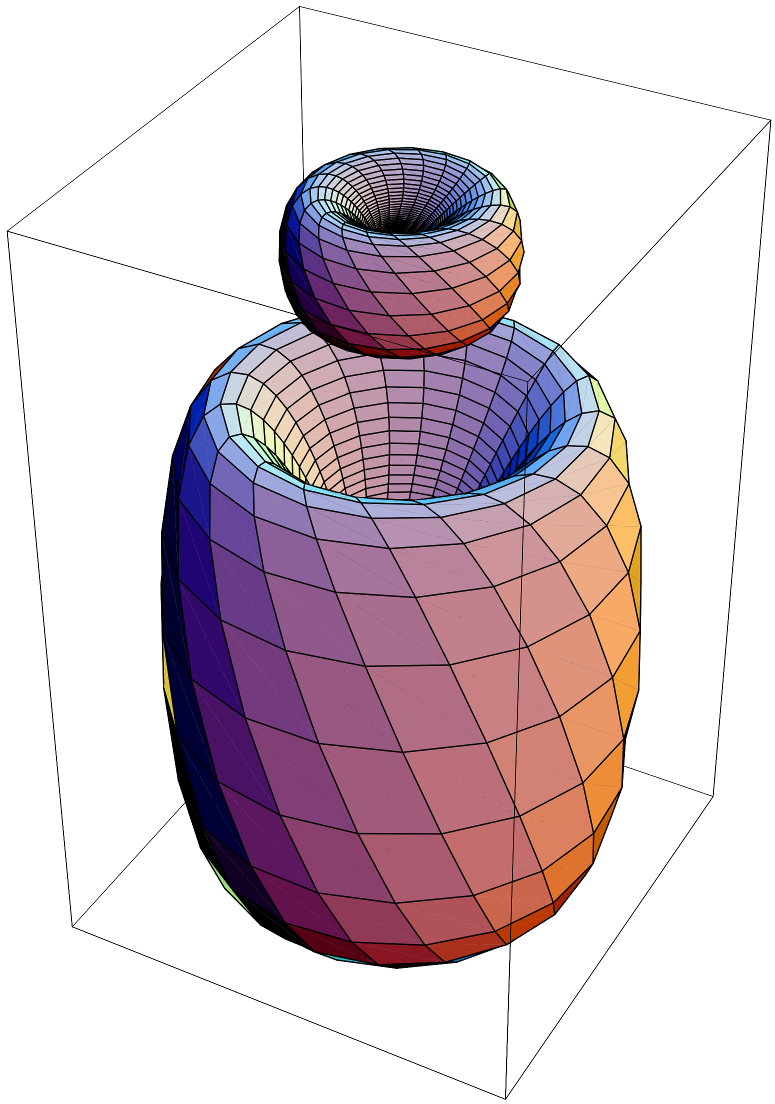

Figure 2.6: Clifford Torus and Image for Discrete Ultra-Ideal Group

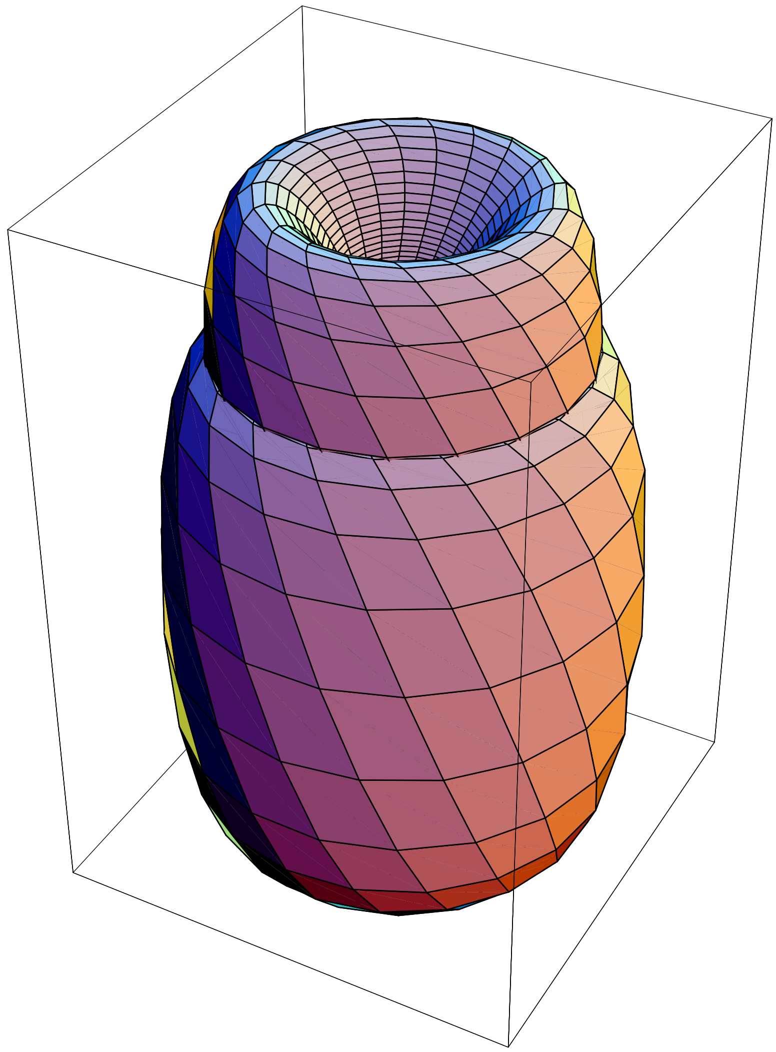

Figure 2.7: Clifford Torus and Image for Indiscrete Ultra-Ideal Group

Figure 2.6 shows the Clifford torus and its image under h for a discrete case, specifically for $z = 3$ and $r = 2$. The original torus is the darker, larger of the two. Figure 2.7 shows the torus and its image under h for an indiscrete case, for $z = 1.5$ and $r = 2$.

2.2.3 Alternate Expressions for Condition D

There are two other convenient ways of expressing Condition D. The first is simply to project it down into Δ and express it in terms of the point p_+. Doing this shows that if p_+ is given by $y = h + vi$, then the borderline for Condition D is given by

$$1 - 4h^2 + 6h^4 - 4h^6 + h^8 - 6v^2 + 8h^2v^2 - 14h^4v^2 + 4h^6v^2 + 2v^4 - 16h^2v^4 + 6h^4v^4 - 6v^6 + 4h^2v^6 + v^8 = 0$$

and so Condition D is satisfied if y lies above this curve.

Alternately we may convert the above to polar coordinates (θ, s). Doing so yields the borderline condition

$$1 - 5s^2 + 4s^4 - 5s^6 + s^8 + s^8 + (s + s^3)^2 \cos 2\theta = 0,$$

and so Condition D is satisfied when (θ, s) lies outside this curve.

2.2.4 What We Have Proved

For any pair of ultraparallel complex geodesics, we may compute the distance between them. We would like to know what triples of distances we have found to be discrete. At this point, calculation-wise, we need to switch back to the triple C, C_+ and C'_+ which form a triangle.

Denote by v_c, v_+ and v'_+ the normalized polar vectors for C, C_+ and C'_+ respectively. We have

$$v_c = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_+ = \begin{bmatrix} 0 \\ \dfrac{1+r^2+zi}{2r} \\ \dfrac{1-r^2-zi}{2r} \end{bmatrix} \text{ and } v'_+ = \begin{bmatrix} \dfrac{-1-r^2+zi}{2r} \\ 0 \\ \dfrac{-1+r^2-zi}{2r} \end{bmatrix}.$$

Computing distances, we see

$$\left|< v_+, v'_+ >\right| = \frac{(1-r^2)^2+z^2}{4r^2} \geq 1,$$

so that

$$\text{dist}(C_+, C'_+) = 2 \text{ arccosh} \left(\frac{(1-r^2)^2+z^2}{4r^2} \right).$$

We also have

$$|< v_+, v_c >| = \left|< v'_+, v_c >\right| = \sqrt{\frac{(1+r^2)^2+z^2}{8r^2}} \geq 1,$$

so that

$$\text{dist}(C_+, C) = \text{dist}(C'_+, C) = 2 \text{ arccosh} \sqrt{\frac{(1+r^2)^2+z^2}{8r^2}}.$$

Note that the two inequalities result from the Torus Condition.

Now then, let $d = \text{dist}(C_+, C)$. We will show that $\text{dist}(C_+, C'_+) = 2d$, thus proving that the triangle groups constructed are of distances $(d, d, 2d)$.

Using the two hyperbolic trigonometric identities

$$\cosh(2\theta) = \cosh^2\theta + \sinh^2\theta$$

and

$$\sinh\ \text{arccosh}\theta = \sqrt{\theta^2 - 1},$$

we calculate

$$\begin{aligned}
d &= \text{dist}(C_+, C)\\
d &= 2\ \text{arccosh}\sqrt{\frac{(1+r^2)^2+z^2}{8r^2}}\\
\cosh(d) &= \cosh(2\ \text{arccosh}\sqrt{\frac{(1+r^2)^2+z^2}{8r^2}})\\
\cosh(d) &= \frac{(1+r^2)^2+z^2}{8r^2} + \left(\sinh\ \text{arccosh}\sqrt{\frac{(1+r^2)^2+z^2}{8r^2}}\right)^2\\
\cosh(d) &= \frac{(1+r^2)^2+z^2}{8r^2} + \sqrt{\frac{(1+r^2)^2+z^2}{8r^2} - 1}\\
\cosh(d) &= \frac{(1-r^2)^2+z^2+4r^2}{8r^2} + \frac{(1-r^2)^2+z^2-4r^2}{8r^2}\\
\cosh(d) &= \frac{(1-r^2)^2+z^2}{4r^2}\\
d &= \ \text{arccosh}\left(\frac{(1-r^2)^2+z^2}{4r^2}\right)\\
2d &= \text{dist}(C_+, C'_+)
\end{aligned}$$

It follows from this calculation and the fact that d may be made as large or (positive) small as possible by appropriate choice of nonnegative z and r that for any d we sample at least one $[d, d, 2d]$ ultra-ideal triangle group; In fact, we sample infinitely many.

2.2.5 Ideal Triangle Groups and the Goldman-Parker Results

For the remainder of this subsection, assume the two chains C_+ and C_- lie on the Clifford torus. That is, Γ is an ideal triangle group. Simultaneously solving the Curve and Torus Condition gives us the solutions

$$(z, r) = \left(\frac{1}{11}\sqrt{302 - 36\sqrt{70}}, \sqrt{\frac{3}{11}\left(9 - \sqrt{70}\right)}\right)$$

and

$$(z, r) = \left(\frac{1}{11}\sqrt{302 + 36\sqrt{70}}, \sqrt{\frac{3}{11}\left(9 + \sqrt{70}\right)}\right)$$

For now let us examine the first pair. In searching for a correspondence to [Gp] we again must note that when C_+ and C_- do lie on the Clifford torus, the three chains C_+, C_-, and C do not form an ideal triangle, though inversions in them do yield an ideal triangle group. One triple which yields an ideal triangle is the triple C_+, $i_c(C_-)$, and C.

The polar vector for C_+ is

$$\begin{bmatrix} 0 \\ 38 - 3\sqrt{70} + i\sqrt{302 - 36\sqrt{70}} \\ -16 + 3\sqrt{70} - i\sqrt{302 - 36\sqrt{70}} \end{bmatrix},$$

for $i_C(C_-)$ is

$$\begin{bmatrix} -38+3\sqrt{70}+i\sqrt{302-36\sqrt{70}} \\ 0 \\ 16-3\sqrt{70}-i\sqrt{302-36\sqrt{70}} \end{bmatrix},$$

and for C is

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

The pairwise points of intersections of the corresponding complex geodesics are

$$\begin{bmatrix} 70-9\sqrt{70}+i\sqrt{302-36\sqrt{70}} \\ 70-9\sqrt{70}-i\sqrt{302-36\sqrt{70}} \\ 2(-54+6\sqrt{70}) \end{bmatrix},$$

$$\begin{bmatrix} -16+3\sqrt{70}+i\sqrt{302-36\sqrt{70}} \\ -16+3\sqrt{70}+i\sqrt{302-36\sqrt{70}} \\ 38-3\sqrt{70}+i\sqrt{302-36\sqrt{70}} \end{bmatrix},$$

and

$$\begin{bmatrix} -16+3\sqrt{70}-i\sqrt{302-36\sqrt{70}} \\ -16+3\sqrt{70}-i\sqrt{302-36\sqrt{70}} \\ 38-3\sqrt{70}-i\sqrt{302-36\sqrt{70}} \end{bmatrix}.$$

The Cartan angular invariant of these three points is $\arctan\sqrt{35}$, which is the positive part of the upper bound for discreteness found in [Gp].

Following similar computations for the second pair yields the Cartan angular invariant $-\arctan\sqrt{35}$, which is the negative part of the upper bound found for discreteness in [Gp].

2.3 An Alternate Proof by Construction of Related Fundamental Domains

We emulate the method for constructing a fundamental domain for an index two subgroup in [Gp] with some modifications necessary to cover the greater range of groups we are dealing with. This will also result in an alternate proof of discreteness to the one we stated earlier, which may seem quicker at first, but relies upon the following lemmas and definitions.

The following may all be found in [Gp].

Definition 2.3.0.1. *Let $x, y \in H^n_{\mathbb{C}}$ and define the **half-space***

$$\mathcal{H}(x,y) = \{z \in H^n_{\mathbb{C}} : \rho(x,z) < \rho(y,z)\},$$

which has as its boundary the ***equidistant hypersurface***

$$\mathcal{E}(x,y) = \{z \in H^n_{\mathbb{C}} : \rho(x,z) = \rho(y,z)\}.$$

The construction of Dirichlet fundamental domains for discrete groups extends in a straightforward fashion to $H^n_{\mathbb{C}}$, so we may define, for G acting discretely on $H^n_{\mathbb{C}}$ and for $x \in H^n_{\mathbb{C}}$, the Dirichlet fundamental domain for G based at x, denoted $D_G(x)$, by

$$\begin{aligned} D_G(x) &= \{z \in H^n_{\mathbb{C}} : \rho(x,z) < \rho(gx,z), \forall g \in G - Id\} \\ &= \bigcap_{g \in G - Id} \mathcal{H}(x, gx) \end{aligned}$$

The following lemma is a version of the Klein Combination Theorem. We have taken this from [Gp].

Lemma 2.3.0.2. *Let G_1, G_2 be discrete subgroups of $PU(n,1)$ with connected fundamental domains D_1 and D_2. Let E_1 and E_2 be the interior of the complement of D_1 and D_2 respectively. Suppose that $E_1 \cap E_2 = \emptyset$ and $D_1 \cap D_2 \neq \emptyset$. Then $G =< G_1, G_2 >$ is discrete and $D = D_1 \cap D_2$ is a fundamental domain for G. In particular, if D_1 and D_2 are Dirichlet polyhedra based at $x \in H^n_{\mathbb{C}}$ for G_1 and G_2 then D is the Dirichlet polyhedron based at x for G.*

Lemma 2.3.0.3. *Suppose S is a $\mathbb{C}$-linear subspace of $H^n_{\mathbb{C}}$ and let $\Pi_S : H^n_{\mathbb{C}} \to S$ be orthogonal projection onto S. Suppose that $x \in \mathbb{C}$ and G is a discrete group of automorphisms of $H^n_{\mathbb{C}}$ leaving S invariant. Then*

$$D_G(x) = \Pi_S^{-1}(D_G(x) \cap S).$$

Lemma 2.3.0.4. *Let $S = H^1_{\mathbb{C}}$ and let $p_1, p_2 \in S$ be distinct points. Let σ_i denote inversion in p_i and $\Sigma = \langle \sigma_1, \sigma_2 \rangle$. Let $x \in \mathcal{E}(p_1, p_2) \subset S$ and let $\gamma_i = \mathcal{E}(x, \sigma_i x)$ denote the geodesic bisecting the segment from x to $\sigma_i x$. Then*

$$D_\Sigma(x) = \mathcal{H}(x, \sigma_1 x) \cap \mathcal{H}(x, \sigma_2 x) \cap \mathcal{H}(x, \sigma_1\sigma_2 x) \cap \mathcal{H}(x, \sigma_2\sigma_1 x)$$

and has either two or four sides which are segments of the geodesics γ_1, γ_2, $\gamma_{12} = \mathcal{E}(x, \sigma_1\sigma_2 x)$, and $\gamma_{21} = \mathcal{E}(x, \sigma_2\sigma_1 x)$. The two-sided case occurs when the half-spaces spaces $\mathcal{H}(x, \sigma_1 x)$ and $\mathcal{H}(x, \sigma_2 x)$ are contained in the half-spaces $\mathcal{H}(x, \sigma_1\sigma_2 x)$ and $\mathcal{H}(x, \sigma_2\sigma_1 x)$ respectively.

Consider now the three complex geodesics C, C_+ and C_-. C_+ and C_- both lie in same component of $\mathcal{H} - T$. Let $D_+ = i_c(C_+)$ and $D_- = i_c(C_-)$, so that D_+ and D_- also lie in the same component of $\mathcal{H} - C$. Denote by $C_\pm$ (resp. $D_\pm$) the complex geodesic perpendicular to C_+ and C_- (resp. D_+ and D_-). Such geodesics exist and are unique because C_+ and C_- (resp. D_+ and D_-) are disjoint (the chains are unlinked). Conveniently, due to our construction, the polar vectors for $C_\pm$ and $D_\pm$ are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

respectively, so that they intersect at the origin of the ball.

Let us focus our attention on $C_{\pm}$, by symmetry the situation on $D_{\pm}$ is analogous. Since C_+ and C_- are perpendicular to $C_{\pm}$, inversion in both of these preserves $C_{\pm}$. Inversion restricts as a map on $C_{\pm}$ to inversion in a point, that is, an elliptic element of order two. We looked at the same restriction of inversion in §2.2.2, only this time we will take a different approach from here.

Consider the group $\langle i_-, i_+\rangle$ acting on $C_{\pm}$ where each inversion restricts to inversion in a point. By lemma 2.3.0.4 we can construct a Dirichlet domain based at the origin of the ball and by Lemma 2.3.0.3 we can extend it to a Dirichlet domain for the action of $\langle i_-, i_+\rangle$ on $H^2_{\mathbb{C}}$. It suffices by Lemma 2.3.0.2 to show that this extended Dirichlet domain does not intersect the corresponding one for $D_{\pm}$. It suffices therefore to show that the Dirichlet domain for the action on $C_{\pm}$ contains the circle of Euclidean radius $\sqrt{1/2}$. By symmetry the Dirichlet domain for the $D_{\pm}$ situation will also satisfy this criterion, and we will have the necessary hypothesis for discreteness of $\Gamma^* = \langle i_-, i_+, i_c i_- i_c, i_c i_+ i_c\rangle$ by Lemma 2.3.0.2. Since Γ^* is a subgroup of index two in Γ, this proves Γ is also discrete.

Consider then $C_{\pm}$ as Δ like we did before. The intersections of $C_{\pm}$ and C_+ (resp. C_-) occur at the points

$$p_+ = \frac{-1 + r^2 - iz}{-1 - r^2 + iz} = \frac{1 - r^4 - z^2}{(r^2+1)^2 + z^2} + \frac{2z}{(r^2+1)^2 + z^2}\ i$$

and its conjugate,

$$p_- = \frac{-1 + r^2 + iz}{-1 - r^2 - iz} = \frac{1 - r^4 - z^2}{(r^2+1)^2 + z^2} - \frac{2z}{(r^2+1)^2 + z^2}\ i.$$

Observe that since we are inspecting the Dirichlet domain at the origin, for any geodesic γ in Lemma 2.3.0.4, the point on γ closest to the origin is the one of smallest Euclidean norm. Again referring to Lemma 2.3.0.4, note that γ_- and γ_+ have closest point precisely p_+ and p_- respectively. A straightforward calculation shows that the Euclidean norms of p_+ and p_- are greater than or equal to $\sqrt{1/2}$.

Thus we only need look at γ_{-+} and γ_{+-}. By symmetry we can look only at the first of these. Computationally we find

$$i_-i_+(0) = \frac{-z(1+r^4+z^2)}{(z-i)(r^4+z(z-i))}$$

which has Euclidean norm $\sqrt{a}$, where

$$a = \frac{z^2(1+r^4+z^2)^2}{(z^2+1)(r^8+z^2+2r^4z^2+z^4)}.$$

Consider that γ_{-+} is the geodesic bisecting the segment from 0 to $i_-i_+(0)$. The point on this geodesic which is closest to 0 is precisely this point of bisection which is located at half the Poincaré distance from 0 to $i_-i_+(0)$.

The following is a straightforward geometrical computation.

Lemma 2.3.0.5. *Suppose $p' \in \Delta$ is located precisely halfway between 0 and a point $p \in \Delta$ in the Poincaré metric. Then*

$$\|p'\| = \frac{\sqrt{1+\|p\|}-\sqrt{1-\|p\|}}{\sqrt{1+\|p\|}+\sqrt{1-\|p\|}},$$

where $\|z\|$ denotes the Euclidean norm of z.

For p' therefore to have Euclidean norm greater than or equal to $\sqrt{1/2}$, a simple calculation shows that we must have

$$\|p\|^2 \geq \frac{8}{9}.$$

Assuming p has norm $\sqrt{a}$, we must have

$$a \geq \frac{8}{9}.$$

Solving this inequality gives us the solution set

$$z^2 + 2r^4z^2 + r^8z^2 + 2z^4 + 2r^4z^4 + z^6 - 8r^2 \geq 0,$$

which is precisely the same bound found in section 2.2.2.

2.3.1 Pictures of Fundamental Domains

The method used for construction in this section not only proves the discreteness of the group Γ^*, but also suggests how to construct fundamental domains for these groups. Since $H^2_{\mathbb{C}}$ is a four dimensional space, explicitly drawings of fundamental domains would be difficult, if not impossible, to construct or put on paper. The next best thing is to intersect the fundamental domains with the boundary $\partial H^2_{\mathbb{C}}$ and examine these. In light of this approach we offer the following definitions, taken from [Go].

Definition 2.3.1.1. *Given two points $x, y \in H^2_{\mathbb{C}}$, the **spinal sphere** $\mathcal{S}_{x,y}$ with vertices x and y is the set*

$$\partial H^2_{\mathbb{C}} \cap \mathcal{E}(x, y).$$

Definition 2.3.1.2. *Given a spinal sphere $\mathcal{S}_{x,y}$, the (unique) complex geodesic containing x and y is called the* ***complex spine*** *of $\mathcal{S}$ and is denoted $\mathcal{S}^{\mathbb{C}}$.*

Definition 2.3.1.3. *Given a spinal sphere $\mathcal{S}$, the spine of $\mathcal{S}$, denoted $\mathcal{S}^{\mathbf{s}}$, is defined to be the set of points in the complex spine which are equidistant from x and y. That is,*

$$\mathcal{S}^s_{x,y} = \mathcal{E}(x, y) \cap \mathcal{S}^{\mathbb{C}}.$$

In other words, this is the perpendicular bisector in $\mathcal{S}^{\mathbb{C}}$ of the real geodesic joining x and y.

Definition 2.3.1.4. *The points x and y above are called* ***covertices*** *of $\mathcal{S}_{x,y}$, while the endpoints of the spine $\mathcal{S}^s$ are called the* ***vertices*** *of $\mathcal{S}_{x,y}$.*

We should mention that the vertices of a spinal sphere are unique. That is, there is a bijection between pairs of distinct points in $\partial H^2_{\mathbb{C}}$ (or real geodesics in $H^2_{\mathbb{C}}$) and spinal spheres. The same is not true of covertices, any two points will suffice as covertices for a given spinal sphere provided that the hypersurface $\mathcal{E}(x, y)$ used to define the spinal sphere is equidistant from the two points.

The following may be found in [Gp].

Theorem 2.3.1.5. *(Mostow)*

Let $\mathcal{E}(x, y)$*,* $\mathcal{S}^{\mathbb{C}}$*, and* $\mathcal{S}^s$ *be as above. Let* $\Pi : H^2_{\mathbb{C}} \to \mathcal{S}^{\mathbb{C}}$ *be orthogonal projection. Then*

$$\mathcal{E}(x, y) = \Pi^{-1}(\mathcal{S}^s).$$

Definition 2.3.1.6. *This is called the* ***Mostow Slice Decomposition*** *of* $\mathcal{E}$*.*

Using the notation from this section, the group $\langle i_-, i_+\rangle$ has fundamental domain based at the origin bounded by portions of the equidistant hypersurfaces $\Pi^{-1}(\gamma_-)$, $\Pi^{-1}(\gamma_+)$, $\Pi^{-1}(\gamma_{-+})$, and $\Pi^{-1}(\gamma_{+-})$. The intersection of these hypersurfaces with $\partial H^2_{\mathbb{C}}$ is then composed of pieces of the spinal spheres with γ_-, γ_+, γ_{-+}, and γ_{+-} as spines. The vertices of the four spinal spheres are then the endpoints of the four spines, while the pairs $(0, i_-(0))$, $(0, i_+(0))$, $(0, i_-i_+(0))$ and $(0, i_+i_-(0))$ are covertices for each of the spheres.

Let us drop back down to $C_\pm$ again for a moment. Consider $p_+ = C_\pm \cap C_+$ and $p_- = C_\pm \cap C_-$ calculated earlier, where both were taken to be points in $C_\pm$ with the unit disc model. Since these two are conjugate to one another with p_+ having positive imaginary part, the following facts may be easily verified by construction.

We assume p_+ is in the first quadrant, the second-quadrant argument is symmetric. The geodesic γ_{+-} was defined as being the geodesic equidistant between 0 and its image under i_+i_-. The point $i_+i_-(0)$ has larger Euclidean norm than the point $i_+(0)$, and also has larger imaginary part. It follows that the geodesic

γ_+ is an arc of a circle with larger Euclidean radius than the geodesic γ_{+-} and lies (rotationally) closer to the (Euclidean) top of the unit disk $C_\pm$. Thus it is impossible for the latter to extend further clockwise than the former. By symmetry the same is true of the geodesic γ_- with respect to γ_{-+}.

Since p_+ and p_- are conjugate and i_+ is an isometry,

$$\begin{aligned}\rho(0, i_+(0)) &= \rho(0, i_-(0)) \\ &= \rho(i_+(0), i_+ i_-(0))\end{aligned}$$

so that

$$i_+(0) \in \gamma_{+-} = \mathcal{E}(0, i_+ i_-(0)).$$

It follows that since $i_+(0)$ is outside of $\mathcal{H}(0, i_+(0))$ by definition and by the preceding paragraph, so is the most clockwise end of γ_{+-}.

To summarize, γ_{+-}'s clockwise end is further counterclockwise than γ_+'s, but is indeed outside of $\mathcal{H}(0, i_+(0))$.

Since i_+ takes γ_+ to itself and reverses its orientation, it follows that if i_+ intersects i_{+-} at a distance d from p_+, then another geodesic, namely γ_-, must intersect it at a distance $-d$, otherwise i_+ would take the segment of γ_+ contributing to the fundamental domain off itself, which would mean it would not act properly on the faces. The net effect here is that either γ_+ intersects both γ_{+-} and γ_-, or it intersects neither.

A similar argument holds for p_- in the lower half of the circle.

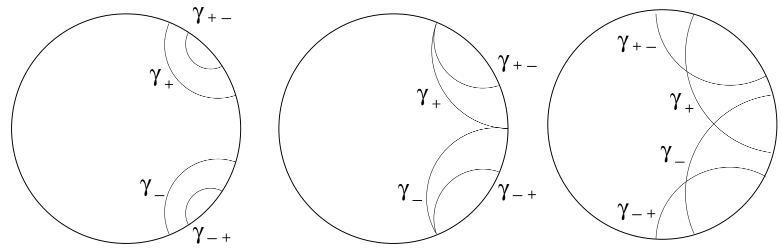

Figure 2.8: Type 2 and Type 4 Fundamental Domains in the Unit Disc

We therefore have two possibilities for the number of faces in the fundamental domain. Either γ_{+-} is completely outside of $\mathcal{H}(0, i_+(0))$, in which case we don't need to pay attention to $\mathcal{H}(0, i_+i_-(0))$ when constructing the fundamental domain, or γ_{+-} intersects γ_+. In the former case, we have two disjoint faces, and in the latter case, four faces that meet in a line. That is, γ_{+-} intersects γ_+, which intersects γ_-, which intersects γ_{-+}.

Figure 2.8 shows the two and four-faced cases and the intermediate case (also two-faced) in the middle.

This combinatorical argument lifts up to its analogy for the equidistant hypersurfaces, the $\mathcal{E}$'s. We have two cases, two $\mathcal{E}$'s with no intersection, or four $\mathcal{E}$'s with "linear" intersection. Call these combinatorical types Type 2 and Type 4 respectively.

A similar argument yields the fundamental domain for $\langle i_c i_- i_c, i_c i_+ i_c \rangle$ which consists of i_c applied to the hypersurfaces above, and whose intersection with $\partial H^2_{\mathbb{C}}$ is i_c applied to the spinal spheres above. Intersecting these two yields the fundamental domain $\mathcal{F}^*$ for Γ^*, the free product of the two. Note that depending upon the position of the spinal spheres and due to symmetry, $\mathcal{F}^*$ is either four or eight-faced.

The fundamental domain $\mathcal{F}^*_\partial$ for the action of the group Γ^* on $\partial H^2_{\mathbb{C}}$, is $\partial H^2_{\mathbb{C}} \cap \bar{\mathcal{F}}^*$, where $\bar{\mathcal{F}}^*$ is taken to be the extension of $\mathcal{F}^*$ to the boundary $\partial H^2_{\mathbb{C}}$. In $\partial H^2_{\mathbb{C}}$, we construct the fundamental domain by deleting the bounded region of $\partial H^2_{\mathbb{C}} - \{$the four or eight spinal spheres$\}$ from $\partial H^2_{\mathbb{C}}$. That is, $\mathcal{F}^*_\partial$ is homeomorphic to S^3 with four or eight D^3's deleted, where the D^3's may overlap.

Figure 2.9 shows several pictures of fundamental domains. The top two are the fundamental domains for the group generated by i_- and i_+ and by Γ^* in the Type 2 case, while the bottom two are the Type 4 cases. For the top two, we clearly see the spheres do not intersect, while for the latter the spheres intersect in two large groups in the manner previously mentioned.

2.3.2 The Quotient Space

For simplicity, we shall denote $\mathcal{H}(0, g(0))$ for $g \in \Gamma$ by $\mathcal{H}(g)$ and similarly for $\mathcal{E}(0, g(0))$. Since the fundamental domain we constructed before is the intersection of two fundamental domains, one based on $C_\pm$ and one on $D_\pm$, in the discussion that follows, any object (such a face, spinal sphere, etc.) which relates to $C_\pm$ will be called $C_\pm$'s object, and similarly for $D_\pm$.

To construct the quotient space $\mathcal{Q}$ for the action of a discrete Γ on $H^2_{\mathbb{C}}$, consider the subgroup Γ^* used before to compute the quotient space $\mathcal{Q}^*$ for this, and then take the quotient of i_c. We shall explicitly calculate the face-pairing transformations on $\mathcal{F}^*$ (and hence on $\mathcal{F}^*_\partial$) in order to calculate $\mathcal{Q}^*$ and $\mathcal{Q}^*_\partial$ (the quotient space on the boundary) respectively.

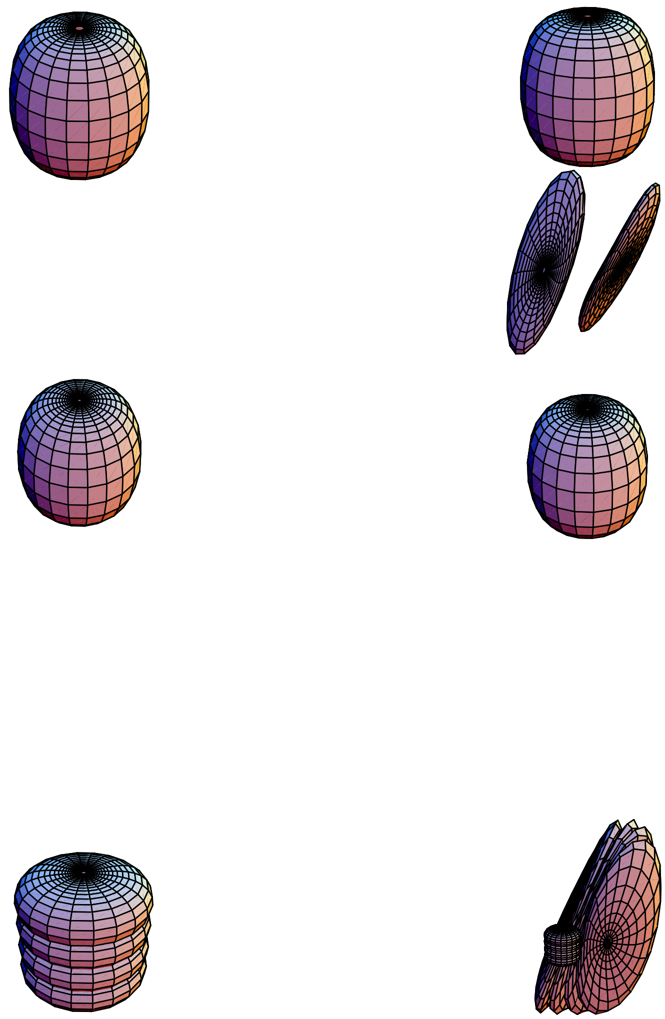

Figure 2.9: Type 2 and Type 4 Fundamental Domain in $\partial H^2_{\mathbb{C}}$

To construct the quotient space $H^2_{\mathbb{C}} \diagup \Gamma^*$ we must examine the face identifications of Γ^* on $\mathcal{F}^*$. The faces of $\mathcal{F}^*$ do not intersect for some parameter values, but in some they do. Recall that the $\mathcal{F}^*$ is either four or eight-sided, so when we mention the "sides" or "faces" of $\mathcal{F}^*$, we mean whichever ones exist for the parameter values being considered.

Denote by S the set $\{i_-, i_+, i_-i_+, i_+i_-, i_ci_-i_c, i_ci_+i_c, i_ci_-i_+i_c, i_ci_+i_-i_c\}$. Each $g \in S$ fixes a real geodesic mentioned in the construction of the fundamental domains, so these (in addition to notation before) will be denoted $\gamma(g)$ so that, for example, $\gamma(i_+i_-)$ is identical to γ_{+-} etc. Note that the first four of these are $C_\pm$'s while the second are $D_\pm$'s.

Lemma 2.3.2.1. *The faces of $\mathcal{F}^*$ are mapped via the identifications*

$$g^{-1} : \mathcal{E}(g) \mapsto \mathcal{E}(g^{-1})$$

for $g \in S$.

Proof. Let $x \in \mathcal{E}(g)$ so that $\rho(x, 0) = \rho(x, g(0))$. Then

$$\begin{aligned} \rho(g^{-1}(x), 0) &= \rho(x, g(0)) \\ &= \rho(x, 0) \\ &= \rho(g^{-1}(x), g^{-1}(0)) \end{aligned}$$

so that $g^{-1}(x) \in \mathcal{E}(g^{-1})$.

$\triangledown$

It therefore follows that the quotient space is formed by taking $H^2_{\mathbb{C}}$, slicing off four or eight possibly intersecting "caps" (corresponding to the interiors of the complements of the $\mathcal{H}$'s) from the edge of the ball and identifying the exposed edges (corresponding to the $\mathcal{E}$'s) in four self-identifications or four self-identifications and two pairs. The former case arises when we have:

$$i_- : \mathcal{E}(i_-) \leftrightarrow \mathcal{E}(i_-)$$

$$i_+ : \mathcal{E}(i_+) \leftrightarrow \mathcal{E}(i_+)$$

$$i_c i_- i_c : \mathcal{E}(i_c i_- i_c) \leftrightarrow \mathcal{E}(i_c i_- i_c)$$

$$i_c i_+ i_c : \mathcal{E}(i_c i_+ i_c) \leftrightarrow \mathcal{E}(i_c i_+ i_c)$$

and the latter when we have, in addition to the above,

$$i_+ i_- : \mathcal{E}(i_- i_+) \mapsto \mathcal{E}(i_+ i_-)$$

$$i_c i_+ i_- i_c : \mathcal{E}(i_c i_- i_+ i_c) \mapsto \mathcal{E}(i_c i_+ i_- i_c)$$

The intersection of this with $\partial H^2_{\mathbb{C}}$ is the same as $\mathcal{Q}^*_\partial$, $\partial H^2_{\mathbb{C}}$ (a copy of S^3) with four or eight 3-discs (each bounded by a spinal sphere) removed in the correct consistent manner which leaves a connected space remaining, and then identifying these spinal spheres in the corresponding manner.

Consider next the identifications on the faces. Recall that $\mathcal{E}(g) = \Pi^{-1}(\gamma(g))$ has $\gamma(g)$ as its spine. Notice that from by the Mostow slice decomposition,

$$\Pi^{-1}(\gamma(g)) = \bigcup_{x \in \gamma(g)} \Pi^{-1}(x),$$

where each $\Pi^{-1}(x)$ is a complex geodesic and is called a *slice at* x of the face. The general picture here is that the face is one-parameter stack of discs put together. Notice also that for $g = i_-$ (resp. i_+) we have the slice at $C_\pm \cap C_-$ (resp. $C_\pm \cap C_+$) is precisely C_- (resp. C_+) and for $g = i_c i_- i_c$ (resp. $i_c i_+ i_c$) we have the slice at $D_\pm \cap D_-$ (resp. $D_\pm \cap D_+$) is precisely D_- (resp. D_+). These can be thought of as the equators of the face. The two discs at the ends of the spine are degenerate, in that they are simply points.

Because the elements in S preserve either $C_\pm$ or $D_\pm$ (depending upon which they relate to) and send faces to faces, is follows that the spines are sent to spines in the correlative manner that the faces are.

Two spines contained within the same complex geodesic (known as cospinal) intersect if and only if their corresponding spinal spheres do (This is not necessarily true for any two spines.) Thus intersection of faces is characterized completely by intersections of spines within (without loss of generality due to symmetry) $C_\pm$. If two spines intersect, then only the parts of the spines which border the equidistant hypersurface restricted to $C_\pm$ will be used in the construction of the face of the fundamental domain. The meeting point of the spines (of which there can only be one) will correspond to the unique slices where the $\mathcal{E}$'s intersect.

Consider the self-identification arising from i_-; the other three work similarly. Recall that γ_+'s orientation is reversed by i_-. Correspondingly, $\mathcal{E}(i_+)$ will undergo an orientation reversing transformation in the sense that a slice at distance d from the point p_+ will be identified with a slice at distance $-d$. Another way to say this is that $\mathcal{E}(i_+)$ would be identified with its reflection in the slice at p_+.

Consider now the other intersections. We will look at $i_- i_+$ since the other three will again arise similarly. Since p_+ and p_- are conjugate to one another, the (real) geodesic fixed by i_-i_+ is symmetric with respect to the x-axis. By construction, $\gamma(i_+i_-)$ and $\gamma(i_-i_+)$ are symmetric with respect to the x-axis, and since i_-i_+ is parabolic, its effect is to reflect $\gamma(i_+i_-)$ in the x-axis, taking it to $\gamma(i_-i_+)$ with reversed orientation.

Recall the two combinatoric types we saw in construction of the fundamental domain and which we use here. We will discuss $< i_-i_+ >$ specifically because the same result holds for $< i_ci_-i_c, i_ci_+i_c >$. The face identification arising from i_- and i_+ are all that are considered in Type 2, and involve removing caps from the open ball that $H^2_{\mathbb{C}}$ is, and identifying each boundary component $\mathcal{E}$ with itself via a reflection through its equator. The effect of this is easier to visualize in two or three real dimensions, and is that the newly exposed face simply closes up on itself and seals the open ball back up. Thus in the Type 2 case we are cutting out two disks and then sealing up the space. Thus in this case, the quotient space $\mathcal{Q}^*$ is homeomorphic to the original space, that is, the open unit ball in $\mathbb{C}^2$.

In the Type 4 case, the faces intersect in the order

$$\mathcal{H}(\gamma_+) \cap \mathcal{H}(\gamma_{+-}) \neq \emptyset,$$
$$\mathcal{H}(\gamma_{+-}) \cap \mathcal{H}(\gamma_{-+}) \neq \emptyset,$$
$$\mathcal{H}(\gamma_{-+}) \cap \mathcal{H}(\gamma_-) \neq \emptyset,$$

and the union of these four faces is homeomorphic to a large cap. The identifications on the individual faces combine to an identification which collapses the face and (like in Type II) closes the hole originally created by removing this cap.

Thus in this case also, $\mathcal{Q}^*$ is homeomorphic to an open unit ball in $\mathbb{C}^2$.

Passing now to computing $\mathcal{Q}$, we quotient $\mathcal{Q}^*$ by the element i_c. It is clear that we are simply identifying two halves of an open ball in $\mathbb{C}^2$, which results in an open hemiball.

In looking at the boundary of the quotient space $\mathcal{Q}_\partial$, we are removing either two open balls bounded by spinal spheres and which do not intersect, or removing four open balls which intersect in a "line" and are bounded by spinal spheres. In the former case, the identifications map the top half of each spinal sphere to its bottom half, closing up the hole in $\partial H^2_{\mathbb{C}}$. In the latter case, the boundary of the four balls that are removed is composed of parts of four spinal spheres, which close up all together in an analogous fashion to the equidistant hypersurfaces.

In either case, we end up with S^3, which, when we quotient out by the final inversion i_c, gives us $\mathcal{Q}_\partial$ being homeomorphic to half of S^3, that is, a hemisphere.

It is certainly worth mentioning that the boundary of the quotient space is quite different from the quotient space for the action of the group on the boundary. For the latter, we must first note that the action of the group on the boundary is *not* properly discontinuous, since we have accumulation points both at attracting points of hyperbolic elements and at fixed points of parabolic elements, among perhaps other places. We must remove all these limit points and look at the remaining space, called the domain of discontinuity. The action of the group on this space is properly discontinuous. We have not explored this here.

2.4 New Frontiers: Second Discreteness Theorem

Here is a general statement of our next theorem. After introducing some preliminaries, this is more precisely stated as Theorem 2.4.2.1

Theorem 2.4.0.2. *There is a one-parameter family of $[l, l, 2l]$ triangle groups lying beyond the range proven in Theorem 2.2.0.14 for which several values yield discrete groups. We conjecture that they all do.*

2.4.1 Some Preliminaries

In [Sc], Schwartz proves that in the ideal case, the upper bound for discreteness may be pushed further than the angular invariant $\arctan\sqrt{35}$ first proved in [Gp]. In fact, Schwartz shows that the ideal triangle group is discrete if and only if the product of the generators is not elliptic. This product is parabolic at the angular invariant $\arctan\sqrt{125/3}$, and elliptic afterwards. One may ask the following question about the ultra-ideal triangle groups. If we perturb the borderline ideal case by pulling the two chains off the Clifford Torus, precisely as we have in the previous sections, is there a way to preserve discretenessΓ In other words, can the discreteness of the ultra-ideal triangle groups be pushed beyond Condition DΓ

By pulling the chains in and down along a certain curve, we preserve the parabolicity of the product of the generators (by generators we mean the three that actually yield the triangle). To compute this curve, note that the trace of

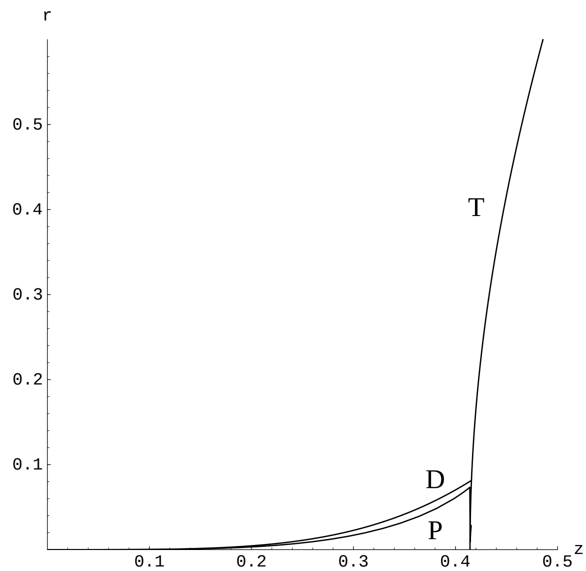

Figure 2.10: The Parabolic Ridge

the product $i_+i_-i_0 = i_+i_0(i_0i_-i_0)$ is

$$\tau = \frac{i(i+z)(r^4+z^2+zi)}{r^4},$$

so we simply plug this trace into the discriminant function f. We have avoided writing the actual equation for the curve here because it is quite complicated. Suffice to say it passes through the origin and lies completely below the curve for Condition D. Call this curve the Parabolic Ridge. See figure 2.10 for a picture of the Parabolic Ridge, Condition D, and the edge of the Clifford Torus Condition.

We will find a fundamental domain for the group Γ^* generated by i_-, i_+, $i_0i_-i_0$ and $i_0i_+i_0 >$, thereby proving discreteness of the ultra-ideal triangle groups along the Parabolic Ridge.

Between the Parabolic Ridge and Condition D lies a no-man's land of probable discreteness. We think that a modification of Schwartz's Dented Torus approach, or the more recent work of Parker might give methods with which to settle this.

Suppose C is a chain and $c \in C$. For $x \notin C$ there is a unique $\mathbb{R}$-circle joining x and c and passing through $C - \{c\}$. To see this, apply an appropriate transformation so that C is the vertical chain through the origin and c is ∞. Then the relevant $\mathbb{R}$-circle is the radial $\mathbb{R}$-circle through x and the vertical axis. For any given pair (C, c) and point $x \notin C$, define

$$\Omega(C, c; x)$$

to be the segment of the aforementioned $\mathbb{R}$-circle which joins x to c but does not touch $C - \{c\}$.

Definition 2.4.1.1. *For any set X such that $X \cap C = \emptyset$, define*

$$\Omega(C, c; X) = \bigcup_{x \in X} \Omega(C, c; x).$$

In both cases we say that $\Omega(C, c; X)$ is the ***cone of*** $\mathbf{X}$ *to c, mentioning C only when necessary.*

If g is an ellipto-parabolic element preserving the pair (C, c), we often write

$$\Omega(g; X)$$

instead.

Define $\Pi_{\mathbb{C}} : H \to \mathbb{C}$ to be vertical projection. Letting V be the vertical chain through the origin, define $\Upsilon : H - V \to S_1 \times \mathbb{R}$ to be $(z, y) \mapsto (\arg(z), y)$, so Υ takes $H - V$ to a cylinder.

2.4.2 The Four Spheres

Given that $\xi = i_+ i_0 i_-$ is ellipto-parabolic, we let (P, q) be the chain-point pair such that $\xi(q) = q$ and $\xi(P) = P$. ξ acts on the complex geodesic associated to P by a parabolic automorphism of $PU(1,1)$. Let p denote the polar vector associated to P.

Let $C'_+ = i_0(C_+)$ and $C'_- = i_0(C_-)$. Define

$$\Sigma_*^{\downarrow} = \Omega(\xi; C_*)$$

and

$$\Sigma_*^{\uparrow} = C_*(\Sigma_*^{\downarrow})$$

for $* = +, -$. Lastly, define

$$\bar{\Sigma}_\alpha^\beta = i_0(\Sigma_\alpha^\beta)$$

for $\alpha = +, -$ and for $\beta = \uparrow, \downarrow$.

Lastly, define

$$\Sigma_* = \Sigma_*^{\uparrow} \cup \Sigma_*^{\downarrow}$$

and

$$\bar{\Sigma}_* = \bar{\Sigma}_*^{\uparrow} \cup \bar{\Sigma}_*^{\downarrow}$$

for $* = +, -$.

Call the set $(\Sigma_+, \Sigma_-, \bar{\Sigma}_+, \bar{\Sigma}_-)$ the four spheres. We have yet to prove that they are in fact spheres, but the notation gives us some direction.

Call q and $i_*(q)$ the ends of Σ_*, for $* = +, -$. Let the ends of the $\bar{\Sigma}$'s be the images of the ends of the Σ's under i_0. Note the following:

1. Σ_+ and Σ_- meet at q.
2. One end of $\bar{\Sigma}_-$ is $i_0 i_-(q) = i_+(q)$, so $\bar{\Sigma}_-$ meets Σ_+ at $i_+(q)$.
3. One end of $\bar{\Sigma}_+$ is $i_0 i_+(q) = i_-(q)$, so $\bar{\Sigma}_+$ meets Σ_- at $i_-(q)$.
4. $\bar{\Sigma}_+$ and $\bar{\Sigma}_-$ meet at $i_0(q)$.

Let C_* the equator of Σ_* for $* = +, -$. Define $\bar{C}_*$ similarly. Note that inversions in the equators generate the group Γ^*.

This may seem confusing at first, but we would like the reader to keep in mind figure 2.11.

The picture implies that the points q, $i_0(q)$, $i_+(q)$ and $i_-(q)$ are the only places where the four spheres meet, and in fact, this is precisely what needs to be proved. We present our results and conjecture now, and then go on to prove the necessities.

Theorem 2.4.2.1. *1. The four Σ's are in fact spheres, each of which is left invariant by inversion in its equators, which also interchanges the interior and exterior of the sphere.*

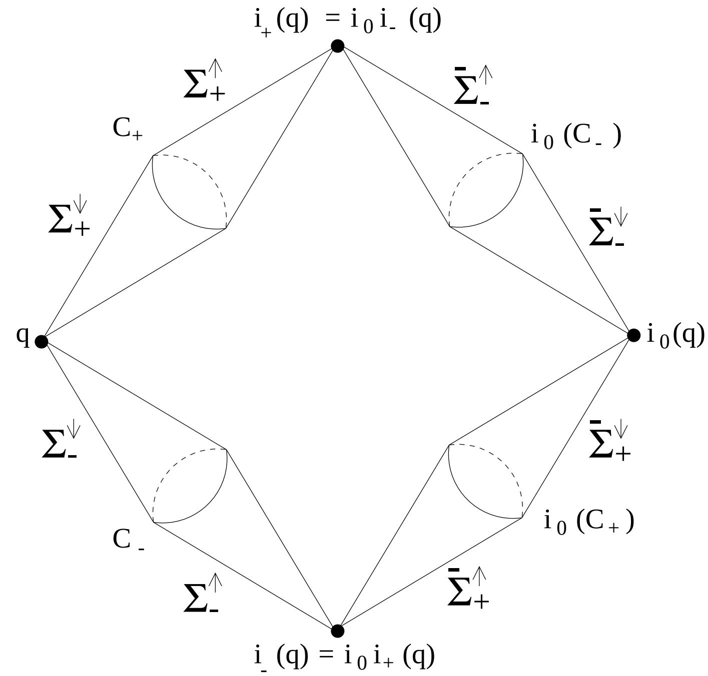

Figure 2.11: The Four Spheres

2. *For the values $r = 0.3$, $r = 0.35$, $r = 0.41$, $r = 0.4$ and $r = .39$ with appropriate t (along the Parabolic Ridge) the spheres do not intersect except at the points mentioned. It follows that the group is discrete for these values.*

Conjecture 2.4.2.2. *For all parameter values, the spheres do not intersect except at the points mentioned. The group is thus discrete along the entire Parabolic Ridge.*

We prove Theorem 2.4.2.1 with a mix of standard and computational computing methods. For the latter, we present an algorithm which will, for two computationally significantly disjoint hybrid cones, prove this disjointness in a finite amount of time. We then run this algorithm on all necessary hybrid cones for the parameters mentioned. The cases where the cones meet at a point are dealt with using standard mathematical proofs.

The following may be found in [Sc2].

Lemma 2.4.2.3. *The Σ's are in fact analytically embedded spheres.*

The fact that inversion in the equator does as is mentioned in Theorem 2.4.2.1 follows from construction of the spheres. We have thus proves part 1 of the theorem.

Let R be rotation of π radians around the $\mathbb{R}$-circle $C = \mathbb{R} \times \{0\} \subset H$. R takes C_0 to itself, P to itself, and switches C_+ and C_-. It follows that R switches $i_0(C_+)$ and $i_0(C_-)$. Note also that since R switches C_+ and C_-, we have $i_- = R \circ i_+ \circ R$, from whence it follows that R interchanges $i_+(q)$ and $i_-(q)$ and interchanges $i_+(P)$ and $i_-(P)$. Thus since R preserves the property of being a chain or an $\mathbb{R}$-circle and is an order 2 action on everything used to construct the four spheres, it acts as an element of order 2 on these spheres, interchanging $+$ and $-$.

Consider then that the group $F = \langle i_0, R \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on the four spheres. In order to show the requirements of Conjecture 2.4.2.2, we can mod out by the action of F first to simplify matters. More specifically, any disjointness necessity not involving Σ_+ may be written via F as one involving Σ_+. It is therefore only necessary that we show three things:

Lemma 2.4.2.4. *(The Initial Disjointness Lemma)*

(I) $\Sigma_+ \cap \Sigma_- = \{q\}$.

(II) $\Sigma_+ \cap \bar{\Sigma}_+ = \emptyset$.

(III) $\Sigma_+ \cap \bar{\Sigma}_- = \{i_0(q)\}$.

It is worth noting that for any pair of spheres meeting at a point, there is an antiholomorphic automorphism (R or $i_0 \circ R = R \circ i_0$) which interchanges the two, fixing that point.

Instead of looking at the Σ as a whole, we break them into pieces. In other words, we look at the disjointness criterion involving the parts $\uparrow$ and $\downarrow$. Thus we break each of I, II and III above into four pieces. Fortunately since i_0 switches $\Sigma_+^\downarrow$ and $\bar{\Sigma}_+^\downarrow$, any statement involving the latter but not the former may be ignored because there is an equivalent (under i_0) statement involving the former. Similarly, since R switches $\Sigma_+^\downarrow$ and $\bar{\Sigma}_-^\uparrow$, any statement involving the latter but not the former may be ignored. Lastly, since $i_0 \circ R = R \circ i_0$ switches $\Sigma_+^\downarrow$ and $\bar{\Sigma}_-^\downarrow$, any statement involving the latter but not the former may be ignored.

We thus have the following breakdown of the Initial Disjointness Lemma into pieces:

Lemma 2.4.2.5. *(The Main Disjointness Lemma)*

(I) $\Sigma_+ \cap \Sigma_- = \{q\}$.

(a) $\Sigma_+^\downarrow \cap \Sigma_-^\downarrow = \{q\}$.

(b) $\Sigma_+^\downarrow \cap \Sigma_-^\uparrow = \emptyset$.

(c) $\Sigma_+^\uparrow \cap \Sigma_-^\uparrow = \emptyset$.

(II) $\Sigma_+ \cap \bar{\Sigma}_+ = \emptyset$.

(a) $\Sigma_+^\downarrow \cap \bar{\Sigma}_+^\downarrow = \emptyset$.

(b) $\Sigma_+^\downarrow \cap \bar{\Sigma}_+^\uparrow = \emptyset$.

(c) $\Sigma_+^\uparrow \cap \bar{\Sigma}_+^\uparrow = \emptyset$.

(III) $\Sigma_+ \cap \bar{\Sigma}_- = \{i_0(q)\}$.

(a) $\Sigma_+^\uparrow \cap \bar{\Sigma}_-^\uparrow = \{i_+(q)\}$.

(b) $\Sigma_+^{\downarrow} \cap \bar{\Sigma}_-^{\downarrow} = \emptyset$.

(c) $\Sigma_+^{\downarrow} \cap \bar{\Sigma}_-^{\uparrow} = \emptyset$.

A note about a pair of cones: If two chains are coned to the same point then by the cone construction itself, if they meet other than at the cone point, then they meet for an entire $\mathbb{R}$-arc.

2.4.3 A More Detailed Look at the Structure of Chains

Chains have a fascinatingly symmetric structure which we will review here without proof. For a more technical discussion, see [Go]. Every nonvertical chain is uniquely determined by a center point in $\mathcal{H}$ and a real radius. The chain structure is preserved under rotation about the line $\{0\} \times \mathbb{R} \subset H$ and under vertical translation, so it suffices to consider chains whose center lies on the positive part of the axis $\mathbb{R} \times \{0\} \subset \mathbb{C} \times \mathbb{R}$. A chain whose center is at the origin is a circle, and moving the chain a distance along the axis from the origin yields a chain which is an ellipse projecting (via $\mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \{0\}$) to a circle, and which has an eccentricity proportional to its distance from the origin. Viewing $\mathcal{H}$ from above, the chain is angled up under a clockwise orientation.

For any given chain C, we may draw a horizontal radial line segment l which passes through the center point and passes through the chain in precisely two points. Under $\Pi_{\mathbb{C}}$, l is mapped to the radial line segment which is a diameter of $\Pi_{\mathbb{C}}(C)$. Applying Υ to a chain not intersecting V yields a curve which is then rotationally symmetric about $\Upsilon(l)$, which is a point. We take the image of Υ to be $[0, 2\pi] \times \mathbb{R}$ with the right and left-hand edges identified.

Lemma 2.4.3.1. *If C links V, then $\Upsilon(C)$ is a curve which has an upper segment to the left of $\Upsilon(l)$ and a lower segment to the right of $\Upsilon(l)$. The upper segment is concave down while the lower segment, by rotational symmetry, is concave up. Any nontrivial translation of $\Upsilon(C)$ yields a curve which then intersects $\Upsilon(C)$ in at most two points.*

Proof. From [Go] we have a parametrization of a finite chain in $\mathcal{H}$ with center (ζ_0, h_0) and radius r_0 as the collection of all points $(\zeta, h) \in H$ satisfying

$$|\zeta - \zeta_0| = r_0 \qquad h = h_0 - 2\mathrm{Im}(\bar{\zeta}_0\zeta).$$

Alternately we may convert to polar co-ordinates centered at $\zeta_0 = x_0 + y_0 i$ with $\zeta = (x_0 + r_0 \cos\theta) + (y_0 + r_0 \sin\theta)i$. Plugging this reparametrization into the equation for h above yields

$$h = h_0 - \mathrm{Im}((x_0 - y_0 i)((x_0 + r_0 \cos\theta) - (y_0 + r_0 \sin\theta)i)),$$

which simplifies to

$$h = h_0 - 2r_0(x_0 \sin\theta - y_0 \cos\theta).$$

which is then the function describing h as a polar function of θ, centered at ζ_0, under the map Υ. We then compute

$$\frac{d^2h}{d\theta^2} = -2r_0(y_0 \cos\theta - x_0 \sin\theta),$$

which equals 0 for defined h only when $y_0 \cos\theta = x_0 \sin\theta$, which occurs only twice on $[-\pi/2, \pi/2)$. The concavity changes only twice, therefore, and an easy check shows that in one segment it is concave up and in the other, concave down.

Now then, since C links V, shifting polar co-ordinates to be centered at any point inside C other than ζ_0 does not change the pattern of derivative changes in h when taken in polar co-ordinates to that new center. Specifically, polar co-ordinates centered at the origin yields the same two changes in concavity. Υ maps l to one of these points, and results in the rotational symmetry which is then evident in the formula for h above.

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The following lemma may be proved by straightforward calculation. We give the flavor here.

Lemma 2.4.3.2. *Suppose we have a chain with center* $(k, 0)$ *on the positive real axis and radius* l *with* $l < k$. *Then inversion in this chain followed by inversion in the chain of radius* 1 *(or really of any value) centered at* $(0, 0)$ *takes* C_+ *and* C_- *to chains with centers of conjugate first co-ordinate, negative second co-ordinate, and identical radius. Call such chains nicely related.*

Proof. Applying such inversions in the chain with center $(0, y)$ and radius r yields a chain with center having first co-ordinate

$$\frac{k(k^2 - l^2 - r^2 - yi)}{k^4 + l^4 - k^2(2l^2 + r^2 + yi)},$$

second co-ordinate

$$\frac{l^2(-2k^2 + l^2)y}{k^8 + l^8 - 2k^6(2l^2 + r^2) - 2k^2 l^4(2l^2 + r^2) + k^4(6l^4 + 4l^2 r^2 + r^4 + y^2)},$$

and radius

$$\sqrt{\frac{l^4 r^2}{k^8 + l^8 - 2k^6(2l^2 + r^2) - 2k^2 l^4(2l^2 + r^2) + k^4(6l^4 + 4l^2 r^2 + r^4 + y^2)}}.$$

$\triangledown$

Lemma 2.4.3.3. *$\Sigma_+^{\downarrow}$ and $\Sigma_+^{\downarrow}$, cannot meet along one or two $\mathbb{R}$-arcs.*

Proof. We show that one cannot happen, and then that two cannot.

Without loss of generality, by symmetry via R, if we have an $\mathbb{R}$-arc joining $p_+ \in C_+$ and q which passes through $p_- \in C_-$, we also have a $\mathbb{R}$-arc joining $R(p_+) \in C_-$ with $R(p_-) \in C_+$. Thus unless R interchanges p_+ and p_-, we have two $\mathbb{R}$-arcs where they meet. If R does interchange p_+ and p_-, then this implies that each lies on the $\mathbb{R}$-arc connecting the other to q. This is clearly impossible.

Suppose they meet along two $\mathbb{R}$-arcs. By the above argument, if we let $M \in PU(2,1)$ be as in 2.4.1, we have a point on $M(C_+)$ which is radially away from a point on $M(C_-)$ and vice versa. This implies, however, that $M(C_+)$ and $M(C_-)$ are in fact linked, which is not the case. It is easier to see this from the point of view of Υ, since under Υ we get two curves which intersect each other twice, with one intersection being the projection of $M(C_-)$ on top of $M(C_+)$, and one being the opposite.

$\triangledown$

Theorem 2.4.3.4. *(Ia) is true.*

Proof. Let $M \in PU(2,1)$ be as in 2.4.1. That is, $M(P) = V$ and $M(q) = \infty$. It suffices to show that $\Upsilon(M(C1)) \cap \Upsilon(M(C2)) = \emptyset$.

A straightforward calculation shows that P links both C_+ and C_- and has center on the positive real axis $\mathbb{R} \times \{0\} \subset H$. We also have q lying on this same axis. In fact, q is computed to correspond to the point in $\mathcal{H}$ $(k, 0)$ with

$$k = \sqrt{\frac{6r^4 - \sqrt{36r^8 - 4(1+z^2)(r^8 + z^2 + 2r^4z^2 + z^4)}}{2(1+z^2)}}.$$

We may then choose M to be inversion in a chain linking C_+ and C_- with center on the positive real axis (taking P to V and q to 0) followed by inversion in the chain of radius 1 centered at $(0, 0)$ (preserving V and taking 0 to ∞.) This we can do since a straightforward calculation shows that if we define $\bar{M}$ to be inversion in the chain with center $(k, 0)$ with $k \in \mathbb{R}$ and radius l, then the vertical chain through 0 is taken to the chain with center $(k - l^2/(2k), 0)$ and radius $l^2/(2k)$. We can thus pick l and k so that q is taken to the origin and P to vertical. Then define M to be the composition of the diagonal matrix $(-1, 1, -1)$ with $\bar{M}^{-1}$, that is, $M = (-1, 1, -1) \circ \bar{M}^{-1}$. The former is precisely the inversion in the chain of radius 1 centered at $(0, 0)$ and the latter finishes the job.

By the above lemma, $M(C_+)$ and $M(C_-)$ are nicely related. By the rotationally symmetric structure of chains, $M(C_+)$ and $M(C_-)$ are then related to each other by a vertical translation and a rotation. Under Υ, $\Upsilon(M(C_+))$ and $\Upsilon(M(C_-))$ are then related by a translation. It follows then that if $\Upsilon(M(C_+))$ and $\Upsilon(M(C_-))$ meet, they do so in at most two points. We saw above that $\Sigma_-^{\downarrow}$ and $\Sigma_+^{\downarrow}$ cannot meet in one or two points. Thus they do not intersect at all, and so the lemma is proved.

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Theorem 2.4.3.5. *(IIIa) is true.*

Proof. First note that $R \circ i_0$ maps $i_+(q)$ to itself, $i_-(q)$ to itself, switches q and $i_0(q)$, and interchanges the pairs $(C_+, i_0(C_-))$ and $(C_-, i_0(C_+))$. More specifically, $R \circ i_0$ sets up a symmetry which switches $\Sigma_+^\uparrow$ and $\bar{\Sigma}_-^\uparrow$. We can then use the first paragraph of Lemma 2.4.3.3, mutatis mutandi, to show that $\Sigma_+^\uparrow$ and $\bar{\Sigma}_-^\uparrow$ do not meet in either one or two $\mathbb{R}$-arc. Notice it is necessary here to note that C_+ and $i_0(C_-)$ do not meet. This is true because by construction C_0, C_+ and $i_0(C_-)$ form an ultra-ideal triangle.

Observe that $\Sigma_+^\uparrow = i_+(\Sigma_+^\downarrow)$ and $\bar{\Sigma}_-^\uparrow = i_+(i_+ i_0 i_-(\Sigma_-^\downarrow))$. It follows that the intersection $\Sigma_+^\uparrow \cap \bar{\Sigma}_-^\uparrow = i_+(\Sigma_+^\downarrow \cap i_+ i_0 i_-(\Sigma_-^\downarrow))$. Recall that under M, $\Sigma_+^\downarrow$ and $\Sigma_-^\downarrow$ are created by adding radial $\mathbb{R}$-arcs to $M(C_+)$ and $M(C_-)$ respectively, with the latter being a rotation and vertical translation of the former. Since $i_+ i_0 i_-$ is an ellipto-parabolic element fixing (P, q), $M \circ i_+ i_0 i_- \circ M^{-1}$ is the ellipto-parabolic element fixing (V, ∞). Such elements are screw-translations, that is, they have a rotation around V and a vertical shift. Summarizing, $i_+ i_0 i_-(\Sigma_-^\downarrow)$ is a rotation and/or shift of $\Sigma_+^\downarrow$. Thus $\Upsilon(C_+)$ and $\Upsilon(i_+ i_0 i_-(C_-))$ meet in at most two points. Since they do not meet in one or two, they don't meet. Thus $\Sigma_+^\uparrow \cap \bar{\Sigma}_-^\uparrow = i_+(\Sigma_+^\downarrow \cap i_+ i_0 i_-(\Sigma_-^\downarrow)) = i_+(q)$ as claimed.

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We now have seven disjointness criteria left to prove. We leave this to the next section with a different approach.

2.5 Foliated Patches and a Finitely Computable Discreteness Criterion

The methods here were developed by Schwartz, while the application is our own.

2.5.1 Motivation

Suppose we have two sets $A, B \subset \partial H^2_{\mathbb{C}}$ and we wish to show that $A \cap B = \emptyset$. It suffices of course to show that there exist A', B' containing A and B respectively and with $A' \cap B' = \emptyset$. If A and B are numerically difficult to handle, we might be better off picking A' and B' with $A \subseteq A'$ and $B \subseteq B'$ which are more manageable. The A and B we will be working with are the hybrid cones in the Main Disjointness Lemma, and this section will be concerned with a computable construction of satisfactory A' and B'.

2.5.2 Computational Issues

The method will be presented first in a purely geometric/algorithmic form with no regard for any actual application. We will then move on to the issues of coaxing this algorithm to function on a present-day computer. Lastly we will discuss some problems that can and do arise in certain situations.

2.5.3 The Algorithm

The boundary of Complex Hyperbolic Space represented as $\mathcal{H}$ may be given polar co-ordinates provided we ignore the point at infinity. This parametrization is very natural, with points (θ, r, h), and the obvious map

$$(\theta, r, h) \to (r\cos\theta + ir\sin\theta, h) \in \mathbb{C} \times \mathbb{R}.$$

Given two radii r_1 and r_2 with $0 < r_1 \leq r_2$, two angles θ_1 and θ_2 with $\theta_1 \leq \theta_2$, and two heights h_1 and h_2 with $h_1 \leq h_2$, we can look at the set P of points (θ, r, h) with $\theta_1 \leq \theta \leq \theta_2$, $r_1 \leq r \leq r_2$ and $h_1 \leq h \leq h_2$. P is a "block" in polar co-ordinates with three very natural foliations: P is foliated by arcs of chains which are circles centered on the vertical axis, also by arcs of vertical chains, and lastly by arcs of $\mathbb{R}$-circles which are horizontal lines passing through the vertical axis.

Definition 2.5.3.1. *Given any set of points as above, we abuse notation and define a* ***foliated patch*** P *to be either the set of points itself or the image of* $\mathcal{P}^{-1}(P)$ *under an element of* $PU(2,1)$*. If no map has been applied, we say the patch is* ***straightened****. Recall that* $\mathcal{P}$ *is stereographic projection, so* $\mathcal{P}^{-1}(P)$ *is a foliated patch in a "nice" position in* $\partial H^2_{\mathbb{C}}$*.*

Given a chain C which links V_0 the vertical chain through 0, we can define a foliated patch bounded by C given by all the data above except for the heights. The reason for this is straightforward, since for any two angles we can merely look at the maximum and minimum height of the chain between those angles. Taking those two heights, we get our patch.

The vertical projection map is injective on the hybrid cone $\Omega(V_0, \infty; C)$, so a foliated patch bounding C actually contains the piece of $\Omega(V_0, \infty; C)$ which is bounded by the same angles and radii. It follows that for any hybrid cone Ω in standard position, we may cover $\Omega - \{\infty\}$ by patches merely by picking a polar grid and taking a patch for each grid member, and then throwing out the patches which don't contain a piece of Ω.

Definition 2.5.3.2. *A* ***patching*** *is defined to be a collection of patches.*

Note that a patching of a cone covers the entire cone except the cone point.

Definition 2.5.3.3. *For any patch P in $\mathcal{H}$ in standard position, define the* ***co-patch*** *P^c to be the patch with the same heights and radii as P, but with the two angles equal to each other and to $\pi + (\theta_1 + \theta_2)/2$, where θ_1 and θ_2 are the angles for P.*

Note that P^c is two-dimensional; it has no angular width. P^c will play an integral role in doing computations on P.

Definition 2.5.3.4. *A straightened patch has eight vertices and a center, the latter with a canonical definition. Given a straightened patch P and an element $M \in PU(2,1)$, we can define the* ***center*** *of $M(P)$ to be $M(c)$ where c is the center of P, and we can define the* ***radius*** *of $M(P)$ to be maximum Euclidean distance (in the ball model) between $M(c)$ and $M(v)$, where v is taken over the eight vertices. Thus each patch, straightened or not, has a center and a radius. For any patch P, denote the former by $c(P)$ and the latter by $r(P)$.*

Definition 2.5.3.5. *Say a patch is* ***guaranteed*** *if is contained in the ball of radius* $r(P)$ *centered at* $c(P)$*. Say a patching is guaranteed if each patch in the collection is guaranteed. For* $M \in PU(2,1)$*, say a patch* P *is* M*-guaranteed if* $M(P)$ *is guaranteed.*

Definition 2.5.3.6. *Say two guaranteed patches (resp. two guaranteed patchings) are* ***ball disjoint*** *if the balls (resp. sets of balls) containing them are disjoint.*

Not all patches are guaranteed. If a patch is quite curved, bits of it could stick outside of the ball. One of the main parts of the computational part of the algorithm will be to find a guaranteed patching of the hybrid cones in question.

In [Sc3], Schwartz goes through the details associated to proving that a patch is guaranteed. We repeat the basic statements here without proof in order to give motivation for our algorithms, pseudocode and code. The reader interested in specifics can read [Sc3]. We have made minor changes to account for our own specific application.

S^3 is a Lie group. The group law is given by

$$(z_1, z_2) * (w_1, w_2) = (z_1 w_1 - z_2 \bar{w}_2, z_1 w_2 + z_2 \bar{w}_1).$$

Note that the element $(1, 0)$ is the identity here. Using this law, define right multiplication by $R_X(Y) = Y * X$. Let

$$\bar{R}_X(Y) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \circ R_X^{-1}(Y).$$

Note that $\bar{R}_p(p) = (0, -1)$ for any $p \in S^3$.

For a chain C, suppose the endpoints of an arc A of C are p_1 and p_2, and $q \in C$ is off this arc. Define a **diameter guarantee** for this arc, $\Delta(A)$ to be $\sqrt{2}/2$ if $\text{dist}(q, p_i) > \text{dist}(p_1, p_2)$ for either i and ∞ otherwise. $\Delta(A)$ satisfies $\lambda(A) \leq \Delta(A)$, where λ denotes the diameter of A; that is, the maximum distance between pairs of points on A.

For the above arc A, we define a **curvature guarantee**, $K(A)$ to be $2/s$, where s is the maximum distance between pairs in the triple $\{p_1, p_2, q\}$. $K(A)$ satisfies $\kappa(A) \leq K(A)$, where $\kappa(A)$ is the standard space curvature of A.

For an $\mathbb{R}$-circle R, suppose the endpoints of an arc A of R are p_1 and p_2, and $q \in R$ is off this arc. Define the diameter guarantee for A as

$$\Delta(A) = \frac{\epsilon_1(\epsilon_1 + \epsilon_2)}{\epsilon_3},$$

where

1. ϵ_3 is the Euclidean distance between p_1 and p_2.

2. ϵ_1 is half the length of the line segment $\Pi_{\mathbb{C}} \circ \mathcal{P} \circ \bar{R}_q(A)$. Notice that this latter expression is a line segment because C, the continuation of the arc A, passes through q, and under this map q passes through ∞ in $\mathcal{H}$. All such $\mathbb{R}$-circles are affine lines in $\mathcal{H}$, which then project to real lines in $\mathbb{C}$.

3. ϵ_2 is the smallest ϵ such that $\bar{R}_q(p_i) \in U_\epsilon$, where $U_\epsilon = \{(z, w) \in S^3 : |w + 1| \leq \epsilon\}$.

We define a curvature guarantee for A to be $K(A) = \sqrt{(6/\delta)^2 - 8}$, where δ

is the Euclidean diameter (in the ball model) of A. In practice, we find a lower bound for δ (by testing a few pairs of points) which yields an upper bound for $K(A)$. Since we check curvature by checking $K(A)$ instead, using the biggest $K(A)$ suffices.

Definition 2.5.3.7. *Say an arc A (for us, either an arc in a chain or an $\mathbb{R}$-circle) is **straight enough** for a ball B with radius r if the endpoints are contained in B and both*

$$\kappa r < 1 \qquad \text{and} \qquad 2\lambda < \frac{1}{\kappa r} - 1.$$

In [Sc3] Schwartz shows that if an arc is straight enough for a ball, it is contained in the ball.

The basic method for showing a patch is guaranteed is as follows: First, show that the copatch is guaranteed by showing first that its top and bottom bounding $\mathbb{R}$-arcs are straight enough, and then that each foliating vertical $\mathbb{C}$-arc is. This shows that the copatch is contained within a little ball. Next, show that each of the left and right faces of the patch are guaranteed for the patch's ball by the same method just given. Lastly we verify that all of the circular $\mathbb{C}$-arcs foliating the patch between the left and right sides are straight enough, simultaneously, based upon our knowledge of three points on each such arc, those three points being the intersections of the chain containing the arc with the three flat patches previously guaranteed.

More specifically, here is the algorithm. Let P be a patch with copatch P^c. Let r and r' denote the radii and c and c' denote the centers of the balls B and

B' corresponding to P and P^c. Denote by $d(,)$ the Euclidean distance in the ball model.

1. Use K and Δ on each (top and bottom) radial $\mathbb{R}$-arc of P^c to verify that each arc is straight enough for B'.

2. Suppose A is a vertical $\mathbb{C}$-arc in P^c with endpoints p_1 and p_2. Let q, the other points on A's chain, be $(0,-1)$. Since the endpoints of A are in the arcs checked in step 1, they are contained in B'. Thus $d(p_i, c') \le r'$ and so $s \ge d(q, p_1) \ge d(q, c') - d(p_1, c') \ge d(q, c') - r'$, where the second-to-last inequality is the triangle inequality. Since $\kappa \le K = 2/s \le 2/(d(q, c') - r')$, if $2r'/(d(q, c') - r') < 1$ then $\kappa r' < 1$.

3. For the same arbitrary A, the endpoints are within $2r'$ of each other (both are contained in B'), and each is at least $d(q, c') - r'$ from q. Put $\Delta = \sqrt{2}/2$ if $d(q, c') - r' > 2r'$, that is, if $3r' < d(q, c')$, and ∞ otherwise. Since $\lambda \le \Delta$, check if $2\lambda < 1/(\kappa r') - 1$ by checking if this is true for 2Δ. In other words, *first* check if $3r' < d(q, c')$ and if not, we fail. Otherwise, $\Delta = \sqrt{2}/2$ so we then check if $\sqrt{2} < 1/(\kappa r') - 1$, that is, if $\kappa r' < 1/(1 + \sqrt{2})$.

   At this point, steps 1-3 pass if the copatch is guaranteed. Note that our arbitrary choice of vertical $\mathbb{C}$-arc in steps 2 and 3 was absorbed via estimates of the endpoints, so the checks hold for any choice of such an arc.

4. Do the same three steps for the left and right faces of P for the ball B.

5. Let A be any circular $\mathbb{C}$-arc joining the left and right faces. The chain C containing A intersects P^c once. Let this intersection point be q, and let p_1 and p_2 be the endpoints of A. q is contained in B' since P^c is guaranteed.

Both p_i are within r of c since each is in a guaranteed face of P. Thus q is at least $d(c, c') - r - r'$ from either p_i. Like in step 2, $s \geq d(c, c') - r - r'$, so to show $\kappa r < 1$, it suffices to show that $2r/(d(c, c') - r - r') < 1$, since $\kappa r \leq (2/s)r = 2r/(d(c, c') - r - r')$.

6. This is like step 3. Since both p_i are within $2r$ of each other, and each is at least $d(c, c') - r - r'$ from q, put $\Delta = \sqrt{2}/2$ if $2r < d(c, c') - r - r'$ and ∞ otherwise. In other words, first check if $2r < d(c, c') - r - r'$ and if not, fail. Otherwise, check if $\kappa r < 1/(1 + \sqrt{2})$.

   These last two steps guarantee that the arbitrary A, hence any A, is straight enough for B. Thus P is guaranteed.

If the algorithm finishes with no problems, the patch is guaranteed.

One thing to note in the application of this algorithm. When checking the copatch P^c, we check if $\kappa r' < 1$ and then later that $\kappa r' < 1/(1 + \sqrt{2})$. The first check is redundant, so in practical application we check only the latter, which implies the former. The same thing is true three other times during the algorithm.

Definition 2.5.3.8. *Given a straightened patch P bounded by a chain C linking V_0, define a* ***refining of* P** *to be the collection of between one and four patches given by breaking P up into four equal pieces (via cuts at halfway between the angles and halfway between the radii) and then recalculating the heights of the individual pieces. Throw away any patch which does not intersect $\Omega(V_0, \infty; C)$. Note that at least one patch will remain.*

Definition 2.5.3.9. *A* ***refining*** *of a patching is the patching yielded by refining all its nonguaranteed patches.*

Frustratingly enough, patchings of cones are still infinite objects (that is, an infinite number of patches), and hence by construction inconvenient to compute with. Thus we define the following:

Definition 2.5.3.10. *An* **N-*patching*** *of a cone in standard form is a patching of the cone such that the patches cover the cone out to a radius of greater than N. For a cone not in standard form, we say an N-patching of the cone is the image of an N-patching.*

Since an N-patching is built from a finite subset of a cone in standard from, it leaves out a piece of the cone around the cone point, so that two N-patchings being disjoint does not prove that the cones are. We will find a separate patching which covers the cone point and which is computationally compatible with the rest of the patches.

Let U be the unit spinal sphere, the spinal sphere with vertices $(0, \pm 1)$ in $\mathcal{H}$. U is not a Euclidean sphere inside $\mathcal{H}$, instead it is the set of points $\{(z, y) \in \mathcal{H} : |z|^4 + y^2 = 1\}$. Let I be inversion in the (-1)-chain. $I(U)$ is contained in the single patch with angles 2.5 and 3.75 radians, radii 1 and 3, and heights -5 and 5. Call this patch P_∞. $I(P_\infty)$ is then a patch containing U. Let I_R be inversion in the $(0, R)$-chain. In the ball model, $I_R \circ I(P_\infty)$ is a patch containing $(0, -1)$, corresponding to ∞ in $\mathcal{H}$. Because I_R takes spinal spheres to spinal spheres, $I_R(U)$ is a spinal sphere, but note that in $\mathcal{H}$, I_R has *switched* the interior and exterior, so that $I_R \circ I(P_\infty)$ has boundary inside the spinal sphere $I_R(U)$ but the interior of the patch itself extends outward.

$I_R(U)$ is a spinal sphere which extends no further than the cylinder of radius

R^2 in $\mathcal{H}$. It follows that for a cone Ω in standard form and $M \in PU(2,1)$, we can guarantee $M(\Omega)$ with a patching comprised of M applied to an R^2-patching along with a subdivision of the cone point patch $M(I_R \circ I(P_\infty))$ into guaranteed patches. Notice that a priori there is no way of knowing if $M(I_R \circ I(P_\infty))$ is guaranteed, and in most cases it isn't. Instead we choose a subdivision into smaller patches which are guaranteed. By abuse of notation, call this guaranteed patching P_∞ also. There will be no ambiguity, P_∞ is simply a guaranteed patching which covers $\infty \in \mathcal{H}$.

Define an **N-hat** to be the patching $I_R \circ I(P_\infty)$ for $R = \sqrt{N}$.

Another way of looking at this is that for any N-patching of a cone Ω, we can complete the patching to the cone point by adding on an N-hat.

Suppose Ω_1 and Ω_2 are two cones with chains C_1 and C_2 in standard form in $\mathcal{H}$, and M_1 and M_2 are elements of $PU(2,1)$. We wish to check the disjointness of $M_1(\Omega_1)$ and $M_2(\Omega_2)$. Our algorithm follows:

1. For $i = 1, 2$, let N_i be the a small number satisfying:

    (a) The cylinder in $\mathcal{H}$ of radius N_i contains the chain C_i.

    (b) Let H_i be the N_i-hat. The two patchings $M_i(H_i)$ are guaranteed and ball disjoint.

    In practice this is done simply by picking N_i to be slightly bigger than necessary for the first criteria, and then doubling it until it satisfies the second.

2. Let P_i be a guaranteed N_i-patching of Ω_i for $i = 1, 2$.

3. Throw out any patch in P_1 which is ball disjoint from any patch in $P_2 \cup H_2$.

4. Throw out any patch in P_2 which is ball disjoint from any patch in $P_1 \cup H_1$.

5. If H_1 is not ball disjoint from P_2, increase N_1 and recompute H_1. Extend P_2 out to a guaranteed N_1 patching of Ω_1.

6. if H_2 is not ball disjoint from P_1, increase N_2 and recompute H_2. Extend P_1 out to a guaranteed N_2 patching of Ω_2.

If the algorithm halts, the cones are disjoint. Only one thing could go wrong which would cause the algorithm to run indefinitely. That would be if there are a string of non-guaranteed patches $P_1, P_2, ...$ such that P_i is in the refinement of P_{i-1}.

2.5.4 Practical Implementation

The major problem with implementing this algorithm on a present-day machine is roundoff error. There are two approaches to dealing with this. The best and most complicated approach would be to build the code around a computational engine with arbitrary precision. If the sizes of objects become so small that the error in the calculations interferes (we can test this by testing object size against current precision), we merely scrap the current attempt and run the entire algorithm with more precision. Assuming the two cones are in fact disjoint, there is a minimum distance between them which provides us with safe knowledge that eventually we shall have enough precision to do the job (assuming we don't run out of machine memory).

In practicality, though, this is an exceedingly difficult approach, rife with programming issues, the major one being that the precision checking would multiply the running time by a rather large factor. We take the other track, which is to compute using standard machine precision. Under this approach, if the objects we are dealing with become to small, the algorithm simply fails to work.

We still have to bound our error, though, and make sure it doesn't get out of hand. We use interval arithmetic with slight modifications. One problem that may still arise is that after numerous calculations, the intervals get larger in size, hence decreasing accuracy. We write the code in a fashion that if the computer runs up against precision issues, the code simply fails to halt. In this manner we guarantee that we will not get an erroneous halting.

Our implementation begins with two pairs of data. Each pair is a directed hybrid cone, which is defined to be the data (p_i, M_i), where p_i is the polar vector for a chain linking V_0, and M_i is an element of $PU(2,1)$. The implementation will halt if the cones $M_1(\Omega(V_0, \infty; p_1))$ and $M_2(\Omega(V_0, \infty; p_2))$ are disjoint. It will fail to halt if the cones are either not disjoint, or if the size of the intervals in the interval arithmetic outgrow the size of the data we are working with.

We assume, to minimize error, that the vectors and matrices which are fed to the program are at least precise to 15 digits. We compute ours with Mathematica, but the choice is up to the user. The code itself is written in Turbo C++, running the ANSI C standard. The data types are doubles, which themselves are precise to 15 digits. Interval arithmetic is implemented in the following manner. Doubles in C are represented by 64 bits. Denoting these bits $se_1...e_{11}b_1...b_{52}$, let E be the decimal number between 0 and 2047 represented by the binary $e_1...e_{11}$, and we have a representation of the number

$$(-1)^s 2^{E-1023}(1+\sum_{j=1}^{52} b_j 2^{-j}).$$

This is a fairly fascinating way to represent a number. Disregarding the sign, the first part of the representation can give us every power of two between 2^{-1023} and 2^{1024}. For each interval between a power of two and its neighbors, the second part (the part in parenthesis) essentially breaks the interval into $2^{52}-1$ further values. In other words, there are $2^{52}-1$ representable numbers strictly between $1/4$ and $1/2$, between 1 and 2, between 2 and 4, and so on.

Note that the number 0 is *not* representable! Instead, there are two approximations of equal precision, namely 2^{-1023} and -2^{-1023}. In fact, the only natural numbers which are exactly representable are those between -2^{53} and 2^{53} (except 0). Note also that working with numbers that are smaller in absolute value is more precise than working with numbers that are larger, however the accuracy is relative to the magnitude.

Assigning a value x to a double in C gives that double the value closest to x which the computer can represent. Denote this by $\tilde{x}$. For any number x that the computer CAN represent, there is a unique predecessor and a unique successor, which we will represent by x_- and x_+ respectively. For any number x which might not be representable, we take the value to be the interval $[\tilde{x}_-, \tilde{x}_+]$. Necessary operations may be redefined appropriately, being careful, for instance, to give an error when dividing by an interval containing 0. The net result is that the result of a calculation to a value x is bounded by the interval resulting from performing precisely the same calculation on $[\tilde{x}_-, \tilde{x}_+]$.

In summary, we redefine our necessary operations as follows:

1. $[x_1, x_2] * [y_1, y_2] = [(x_1 + y_1)_-, (x_2 + y_2)_+]$ for $* = +, -$.

2. $[x_1, x_2] \times [y_1, y_2] = [(\min\{x_i \times y_i\})_-, (\max\{x_i \times y_i\})_+]$

3. $[x_1, x_2]/[y_1, y_1] = [x_1, x_2] \times [1/y_2, 1/y_1]$ for $[y_1, y_2]$ not containing 0.

4. $\sqrt{[x_1, x_2]} = [(\sqrt{x_1})_-, (\sqrt{x_2})_+]$.

5. $\sin(I) = [(\min\{\sin(x)|x \in I\})_-, (\max\{\sin(x)|x \in I\})_+]$. Any other trig function may be defined from sin.

To be consistent with the computer's problematic representation of 0 (the issue is that any calculation resulting in 0 might be represented either way by the system), we consider both representations as identical. In other words, the successor of either representation is defined to be the successor of the higher, and the predecessor of either is the predecessor of the lower.

Whenever dealing with numbers where we have some choice, we choose the interval to be perfect. That is, reasonably sized natural numbers $n \neq 0$ are introduced as the interval $[n, n]$, since C++ can represent n precisely.

The angles that bound our initial foliated patches are taken to be representable sums of powers of 2, so they themselves are representable exactly, and dividing a patch in half angle-wise essentially yields no loss in precision. In a worst-case scenario, a patch will have bounding angles $[x, x]$ and $[x_+, x_+]$ for some x, and dividing this in half will yield two intervals with bounding angles being selections from the same pair. Any pair of bounding angles $[x, x]$ and $[x, x]$ is simply split into two of the same. All we can do then is cycle down into redundancy where the code will simply not stop running. Essentially this avoids the algorithm halting because errors have caused it to think it's done.

The radii are taken to be perfect intervals, so that when we refine a patch, we split not at half the distance between the radii, but at the perfect interval $[\tilde{x}, \tilde{x}]$, where $x = (r_1 + r_2)/2$. The issue is not that we split the patch exactly in quarters, but that we get pieces which are more likely to be guaranteed but which still cover the same area when put together.

2.6 Return to the Parabolic Ridge: Results

1. 20 February 2000: We run our code with $r = 0.3$ and appropriate z. The code takes approximately 690 minutes to run and halts without error. We conclude that the triangle group is discrete for this r and z.

2. 23 February 2000: We run our code, after some optimizations, with $r = 0.35$. The code takes approximately 438 minutes to run and halts without error.

3. 7 March 2000: We run our code with $r = 0.41$. The code halts after approximately 344 minutes.

4. 8 March 2000: We run our code with $r = 0.4$. The code halts after approximately 319 minutes.

5. 8 March 2000: We run our code with $r = 0.39$. The code halts after approximately 333 minutes.

Notice that more or less as r shrinks, the time requirement increases. This is a result of the fact that while the two spheres Σ_+ and Σ_- are smaller and less convoluted, the complementary spheres $i_0(\Sigma_+)$ and $i_0(\Sigma_-)$ are larger and more twisted. This results in the need for more refinement during the guarantee process. On the other hand, more initial refinement leads to less division during the disjointness part of the algorithm. These aspects do not balance out, however, and the time requirement increases as r decreases.

At this point we have not entirely proven that the triangle groups are discrete along the entire parabolic ridge, we have only built an algorithm which can prove for us that for any given parameter along the ridge, the group is discrete. In order to finish the job, we would need to show that it suffices to prove discreteness for a finite number of parameters. This can be done by proving that the topology for the entire family may be approximated by a finite set of parameters. We have not done this yet.

2.7 The General $[m, l, k]$ Case

The method applied in §2.3 may be extended somewhat to yield a condition that is sufficient for some general $[m, l, k]$ triangle groups to be discrete. The method in §2.2.2 may be extended by the same method, but the calculations are slightly more difficult. The advantage of the former is the construction of the fundamental domains and the quotient spaces.

The approach we took was to start with chains C_+, C and C_-, construct $D_+ = i_c(C_+)$ and $D_- = i_c(C_-)$, then look at the perpendicular bisectors $C_\pm$ and

$D_{\pm}$. Since there is an inherent symmetry gained from construction via i_c, we need only look at the fundamental domain for $C_{\pm}$. Within this complex geodesic, there was a symmetry gained from the fact that we were inspecting isosceles ultra-ideal or ideal triangle groups.

For the general $[m, l, k]$ case, the former symmetry remains, a fact which can be easily verified computationally, but the latter symmetry vanishes. We can, however, tweak the approach slightly to coax out some results. Given a (z_+, r_+)-chain and a (z_-, r_-)-chain with inversions i_+ and i_- respectively, we still examine the perpendicular bisector $C_{\pm}$, resulting in both our chains reducing to points p_+ and p_- in $C_{\pm}$, and having the i's project to inversions in those points in $C_{\pm}$. The only difference here is that p_+ and p_- are no longer conjugate, that was gained only from the isoscelarity. Note that expressions for the p's were explicitly computed in §2.3 to be

$$p(z, r) = \frac{-1 + r^2 - iz}{-1 - r^2 + iz}.$$

As we saw before, the Euclidean norms of p_+ and p_- are greater than or equal to $\sqrt{1/2}$. What changes is that the Euclidean norms of $i_-i_+(0)$ and $i_+i_-(0)$ are different from one another and both less straightforward to compute. Instead then we take a more constructive approach.

Definition 2.7.0.1. *Suppose γ is a geodesic. Define the* ***angle of*** *γ, denoted $\angle\gamma$, to be the unique angle satisfying $-\pi/2 < \angle\gamma \leq \pi/2$ such that rotation of γ by $\angle\gamma$ radians puts the image perpendicular to the x-axis.*

Denote by γ the geodesic connecting p_+ and p_-. In the $[l, l, 2l]$-case, $\angle\gamma = 0$

since p_+ and p_- were complex conjugates, but it is not necessarily so in this case.

Theorem 2.7.0.2. *Given an $[m, l, k]$ triangle group as above, suppose $\angle\gamma = 0$, though p_+ and p_- may not be complex conjugates, and suppose both p_+ and p_- each satisfies the curve condition, then the triangle group is discrete.*

Proof. Suppose without loss of generality that the Euclidean norm of p_+ is less than that of p_-. Since γ_+ and γ_- lie entirely outside of the circle of radius $\sqrt{1/2}$, it is sufficient to show that γ_{+-} and γ_{-+} do.

Let p'_- be the conjugate of p_+, which lies on γ, and let i'_- be inversion in p'_-. Let γ'_{-+} be the geodesic equidistant from 0 and $i'_- i_+(0)$. It is clear that chains projecting to p'_- and p_+ will yield a discrete group since the two are complex conjugates and satisfy Condition D.

To show that γ_{-+} is further away from the origin than γ'_{-+}, recall one of the cosine rules from hyperbolic geometry. If we have a triangle with sides A, B, and C with opposite angles a, b, and c, then

$$\cosh C = \cosh A \cosh B - \sinh A \sinh B \cos c.$$

Letting $p = p'_-$ with i being inversion in p, consider the triangle with sides (and lengths) as follows: $A = \rho(0, p)$, $B = \rho(p, i_0 i_+(0))$ and $C = \rho(i_0 i_+(0), 0)$. Now then, $c > \pi/2$ and as p moves away from p'_- towards p_-, c increases but does not reach π. It follows then that $\cos c < 0$ at all points along this journey. Both of A and B increase, and so it follows from the cosine rule above that $\cosh C$ and hence C increases. Thus in the final case when $p = p_-$, the distance from 0 to

$i_-i_+(0)$ is greater than that from 0 to $i'_-i_+(0)$. Hence γ_{-+} is further away from the origin than γ'_{-+} as claimed.

Substituting the triple $A = \rho(0, p_+)$, $B = \rho(p_+, i_+i(0))$ and $C = \rho(i_+i(0), 0)$ in the above proof shows that in the final case, the distance from 0 to $i_+i_-(0)$ is greater than that from 0 to $i_+i'_-(0)$. Defining γ'_{+-} in the obvious way, it follows that γ_{+-} is further away from the origin than γ'_{+-}.

It follows that the triangle group is discrete via fundamental domain construction.

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Suppose now that $\angle\gamma \neq 0$. What we need only do is rotate our picture of Δ by $\angle\gamma$ to see whether Condition D is satisfied by the two chains. Consider the version of Condition D given in polar coordinates in §2.2.3. Replacing θ by $\theta + \angle\gamma$ suffices for a discreteness condition. We have thus proved:

Theorem 2.7.0.3. *Suppose we have a triangle group with chains projecting to points as given in the calculations above, then the group is discrete if the points in polar coordinates* (θ, s) *lie outside the curve given by*

$$1 - 5s^2 + 4s^4 - 5s^6 + s^8 + s^8 + (s + s^3)^2 \cos 2(\theta + \angle\gamma) = 0.$$

Let us analyze these fundamental domains. Assuming without loss of generality that the norm of p_+ is less than or equal to that of p_- and $\angle\gamma = 0$, we see that

$$\begin{aligned}\rho(0, i_+(0)) &\geq \rho(0, i_-(0)) \\ &= \rho(i_+(0), i_+ i_-(0))\end{aligned}$$

so that

$$i_+(0) \notin \mathcal{H}(0, i_+ i_-(0)).$$

It follows that since $i_+(0)$ is outside of $\mathcal{H}(0, i_+(0))$ by definition, so is the most clockwise end of γ_{+-}. Thus γ_{+-} intersects γ_+. Applying the intersection argument derived from the face identifications we used in §2.3.1 gives us that γ_{-+} intersects γ_-.

It follows that the fundamental domains and quotient spaces of these $[m, l, k]$ triangle groups are identical to those for the $[l, l, 2l]$ cases.

Chapter 3

Let's Go Inside - Some (p, q, r) Triangle Groups

3.1 Preliminaries

Consider now what happens if the complex geodesics which generate the triangle group meet in $H^2_{\mathbb{C}} \cup \partial H^2_{\mathbb{C}}$. Denote by x_0, x_1, and x_2 the points of intersection.

Given two complex geodesics that intersect inside the ball, we have seen how to define the angle between them in a nice manner and which extends continually to be zero if the geodesics meet at the boundary.

Definition 3.1.0.4. *For $p, q, r \in \mathbb{Z} \cup \{\infty\}$ we may define the* $(\mathbf{p}, \mathbf{q}, \mathbf{r})$***-triangle group*** *to be the representation $\rho \colon \Gamma \to PU(2,1)$ given by taking a generator to each of three reflections in complex geodesics meeting at angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, and $\frac{\pi}{r}$ respectively.*

When $p = q = r = \infty$, we get the (ideal) triangle groups discussed in Goldman-Parker. One thing we immediately notice is that if the images of the three generators are given by i_0, i_1 and i_2, we have the relations $(i_0 i_1)^r = (i_0 i_2)^q = (i_1 i_2)^p = id$.

This summary of results deals with the $(4,4,\infty)$ and (n,n,∞) triangle groups. We begin by finding a parameter which is an extension, of sorts, of the Cartan Angular Invariant.

Definition 3.1.0.5. *An (n,n,∞) triangle group Γ will be said to be in* ***regular position*** *if there exists a $z \in \mathbb{C}$ so that Γ is generated by reflections in the $(0,1)$-chain, the z-chain, and the $(-\bar{z})$-chain.*

Proposition 3.1.0.6. *Any (n,n,∞) triangle group is $PU(2,1)$-equivalent to a unique triangle group in regular position, with $|z| = \cos\frac{\pi}{n}$.*

Lemma 3.1.0.7. *Any (m,n,∞) triangle group is $PU(2,1)-$equivalent to one generated by inversions in the $(0,1)$-chain and in two vertical chains.*

Proof. Let the original generators be i_0, i_1 and i_2, where i_i is inversion in c_i and $(i_0 i_1)^m = (i_0 i_2)^n = id$. Let r be the (unique) point where c_1 and c_2 intersect and let C be the complex geodesic containing c_0 Let p and q be two points in C which are endpoints of a real geodesic containing $\Pi_C(r)$, where $\Pi_C\colon H^2_{\mathbb{C}} \cup \partial H^2_{\mathbb{C}} \to C$ is orthogonal projection. Then $\mathbb{A}(p,q,r) = 0$ by the area equivalence given in [Go]. Consider that $\mathbb{A}((1,0,1),(-1,0,1),(0,-1,1)) = 0$ also, and thus by [Go], there exists an element T (unique up to the choice of p and q) of $PU(2,1)$ taking $p,q,$ and r to $(1,0,1),(-1,0,1)$, and $(0,-1,1)$ respectively. The latter three of these correspond in $\mathcal{H}$ to the points in $(1,0)$, $(-1,0)$, and ∞. Thus T takes c_0 to the unit circle and r to ∞. Since the only chains through infinity are vertical, c_1 and c_2 must be mapped by T to vertical chains.

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Lemma 3.1.0.8. *If i_z denotes inversion in the z-chain, i_0 denotes inversion in the unit circle, and the order n of $i_0 i_z$ is finite, then $|z| = \cos\frac{\pi}{n}$.*

Proof. The inversion i_0 is represented by the polar vector $p_0 = (0,1,0)$ while i_z is represented by $p_z = (1, -\bar{z}, \bar{z})$. Since $p_0 \boxtimes p_z$ is negative, we find that the angle between the two complex geodesics satisfies $\cos\theta = |< p_0, p_z >| = |z|$. For $\theta = \frac{\pi}{n}$ we have our claim.

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The first part of the proposition now follows from the following elementary fact: The elliptic element R_μ of $PU(2,1)$ given by the diagonal matrix $(\mu, 1, 1)$, with μ a unit complex number, rotates $\mathcal{H}$ around the 0-chain by an angle of $\text{Arg}(\mu)$ radians. Thus if z_1, z_2 and w_1, w_2 are two pairs of complex numbers with $\measuredangle(z_1, z_2) = \measuredangle(w_1, w_2)$, and if each pair, each along with the unit circle, represents a triangle group, then there is a μ such that R_μ takes one to the other.

Lastly we prove uniqueness. It suffices to show that any element T of $PU(2,1)$ which preserves the property of regular position preserves the angle between the vertical chains. Let C be the complex geodesic corresponding to the $(0,1)$-chain. T must leave C invariant and hence commutes with projection to C.

Projection from $\mathcal{H}$ to C, with the latter considered as the unit disk with the Poincaré model is given by

$$(z, y) \mapsto \frac{2z}{1+|z|^2}$$

$$\infty \mapsto 0$$

so that the "unit disk" in $\mathcal{H}$, that is, the set of points $\{(z, 0) : |z| \leq 1\}$, is mapped homeomorphically onto C in a manner which sends circles centered at the origin to circles centered at the origin, preserving the angle between pairs of nonzero points.

Lastly, note that T also fixes the point at infinity. It follows that T projects down to an action on C which fixes the origin, preserves norm, and acts on angles between points in exactly the same manner as angles between vertical chains. Such an action is an elliptic element of $PU(1,1)$ fixing the origin and is hence a rotation, preserving angles. We thus have uniqueness.

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In order to facilitate neat computations, we shall use the parameter $t = \tan\theta$, where $0 \leq \theta \leq \frac{\pi}{2}$ is the argument of z. We only concern ourselves with $0 < t \leq \infty$, since negative values of t will yield the same results by relabelling of chains. We have defined $\tan(\pi/2)$ to be ∞ for convenience, as once we write down matrices for the inversions, we see we can simply take the limit as $t \to \infty$ of these matrices as the matrix for $t = \infty$, that is, for $\theta = \pi/2$.

The polar vectors corresponding to a triangle group in regular position using the parameters $t = \tan(\arg z)$ and $r = |z|$ are then given by

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \sqrt{t^2+1} \\ -r + rti \\ r - rti \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sqrt{t^2+1} \\ r + rti \\ -r - rti \end{bmatrix},$$

while the inversions themselves are given by

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -\frac{2r+2rti}{\sqrt{1+t^2}} & -\frac{2r+2rti}{\sqrt{1+t^2}} \\ \frac{-2r+2rti}{\sqrt{1+t^2}} & 2r^2-1 & 2r^2 \\ \frac{2r-2rti}{\sqrt{1+t^2}} & -2r^2 & -2r^2-1 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & \frac{2r-2rti}{\sqrt{1+t^2}} & \frac{2r-2rti}{\sqrt{1+t^2}} \\ \frac{2r+2rti}{\sqrt{1+t^2}} & 2r^2-1 & 2r^2 \\ -\frac{2r+2rti}{\sqrt{1+t^2}} & -2r^2 & -2r^2-1 \end{bmatrix}.$$

The correspondence with Cartan's Angular Invariant is evident for $t = \infty$. In this case, the triangle group is ideal, and letting α be the angle between z and $-\bar{z}$, we have

$$t = \tan\theta = \tan(\frac{\pi}{2} - \frac{\alpha}{2}) = \tan\frac{1}{2}(\pi - \alpha) = \tan\mathbb{A}$$

where $\mathbb{A}$ is the angular invariant of the points where the pairs of geodesics meet.

3.2 The $(4,4,\infty)$ case.

Looking specifically at the $(4,4,\infty)$ triangle group, we have $|z| = \sqrt{1/2}$, so we have parametrized all $(4,4,\infty)$ triangle groups by $t = \tan\theta$ where θ is the argument of z. Note that for $t = 0$ the triangle group thus generated is $\mathbb{R}$-fuchsian. That is, it fixes an $\mathbb{R}$-circle, namely the one-point compactification of the horizontal line $\mathbb{R} \times \{0\}$. For $t > 0$, the representation is never again $\mathbb{R}$-fuchsian, nor is it ever $\mathbb{C}$-fuchsian.

Our three generators are denoted i_0, $i_1(t)$, and $i_2(t)$, giving inversions in the $(0,1)$-chain, z-chain, and $(-\bar{z})$-chain respectively.

We find the matrix representation for inversion in the unit circle chain (as an element of $PU(2,1)$) is:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The matrix for $i_1(t)$ is given by

$$\begin{bmatrix} 1 & -\dfrac{i\sqrt{2}(t-i)}{\sqrt{1+t^2}} & -\dfrac{i\sqrt{2}(t-i)}{\sqrt{1+t^2}} \\ \dfrac{i\sqrt{2}(t+i)}{\sqrt{1+t^2}} & 0 & 1 \\ \dfrac{\sqrt{2}(1-ti)}{\sqrt{1+t^2}} & -1 & -2 \end{bmatrix},$$

while the matrix for $i_2(t)$ is given by

$$\begin{bmatrix} 1 & \frac{\sqrt{2}(1-it)}{\sqrt{1+t^2}} & \frac{\sqrt{2}(1-it)}{\sqrt{1+t^2}} \\ \frac{\sqrt{2}(1+it)}{\sqrt{1+t^2}} & 0 & 1 \\ -\frac{\sqrt{2}(t-i)}{\sqrt{1+t^2}} & -1 & -2 \end{bmatrix}.$$

Consider the triangle group Γ with generators i_0, $i_1(t)$ and $i_2(t)$.

Recall the discriminant function $\bar{f}$ given in Section 1.2. For real z, $\bar{f}(z)$ factors into

$$\bar{f}(z) = (z+1)(z-3)^3,$$

which yields the same categorization of element types as

$$f(z) = (z+1)(z-3).$$

It is clear therefore that an element α is hyperbolic iff $\text{Tr}(\alpha) > 3$ or $\text{Tr}(\alpha) < -1$, regular elliptic iff $-1 < \text{Tr}(\alpha) < 3$, and all other cases when $\text{Tr}(\alpha) = -1$ or 3.

The following two propositions are almost identical:

Proposition 3.2.0.9. *Consider the element $\xi_t = i_0 i_1 i_0 i_2(t)$. If Γ is discrete then either $t^2 \le 3$ or $f(Tr(\xi_t^n)) = 0$ for some n.*

Proof. The eigenvalues of ξ_t are

$$\left\{1, \frac{7-t^2+4t\sqrt{3-t^2}}{t^2+1}, \frac{7-t^2-4t\sqrt{3-t^2}}{t^2+1}\right\}.$$

The trace of any power of ξ_t is real, so then a straightforward calculation with f above shows that ξ_t is hyperbolic for $t^2 < 3$, unipotent for $t^2 = 3$, and regular elliptic for $t^2 > 3$. Suppose Γ is discrete and $t^2 > 3$, then ξ_t must have finite order, since a regular elliptic element of infinite order generates a nondiscrete subgroup. Since for $t^2 > 3$ the latter two eigenvalues above are norm 1 nonreal complex conjugates of one another, to have finite order n they must be n^{th} roots of unity. Thus ξ_t^n has all eigenvalues equal to 1 and therefore trace 3, and hence $f(\mathrm{Tr}(\xi_t^n)) = 0$.

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Proposition 3.2.0.10. *Consider the element $\eta_t = i_0 i_1 i_2 i_1(t)$. If Γ is discrete then either $t^2 \leq 7$ or $f(Tr(\eta_t^n)) = 0$ for some n.*

Proof. The eigenvalues of η_t are

$$\left\{1, \frac{8+\sqrt{(7-t^2)(t^2+9)}}{t^2+1}, \frac{8-\sqrt{(7-t^2)(t^2+9)}}{t^2+1}\right\}.$$

The remainder of the proof is precisely the same as the previous proposition, mutatis mutandi.

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It is important to take note that it is not sufficient to show that some element is regular elliptic to destroy discreteness unless we also show it is of infinite order, since we are asking that the representation be discrete but not necessarily faithful. A regular elliptic element of finite order still generates a discrete group.

Theorem 3.2.0.11. *The triangle groups with* $t^2 = 7, 15$ *are discrete.*

Proof. We will be more explicit for the former case, the latter case is similar. Let J be the matrix which puts the element element $i_1 i_2$ into Jordan canonical form. We compute J to be

$$\begin{bmatrix} 0 & 1 & -\frac{1}{2}+\frac{1}{2}\sqrt{-7} \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Conjugating each of i_0, i_1, and i_2 by J yields generators

$$J^{-1} i_0 J = \begin{bmatrix} -1 & 0 & 0 \\ -1+\sqrt{-7} & -1 & 1-\sqrt{-7} \\ -2 & 0 & 1 \end{bmatrix},$$

$$J^{-1} i_1 J = \begin{bmatrix} -1 & \frac{1}{2}-\frac{1}{2}\sqrt{-7} & \frac{1}{2}+\frac{1}{2}\sqrt{-7} \\ 0 & 1 & -\frac{3}{2}+\frac{1}{2}\sqrt{-7} \\ 0 & 0 & -1 \end{bmatrix},$$

and

$$J^{-1} i_2 J = \begin{bmatrix} -1 & -\frac{1}{2}-\frac{1}{2}\sqrt{-7} & 1 \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}\sqrt{-7} \\ 0 & 0 & -1 \end{bmatrix}.$$

All of the entries of these matrices are in the ring

$$\left\{ \frac{\alpha}{2} + \frac{\beta}{2}\sqrt{-7} \;\middle|\; \alpha + \beta = \text{even} \right\}$$

which is discrete as a subring of $\mathbb{C}$. Thus the group of matrices with elements in this ring is also discrete, so that the triangle group is conjugate to a discrete group and hence is discrete.

For the $t^2 = 15$ case, the same type of conjugation yields generators all of whose matrix entries are in the ring

$$\left\{ \frac{\alpha}{2} + \frac{\beta}{2}\sqrt{-15} \;\middle|\; \alpha + \beta = \text{even} \right\},$$

hence this triangle group is also discrete for the same reason.

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Theorem 3.2.0.12. *The triangle group with $t^2 = 3$ is discrete.*

Proof. This is not so clear as the $t^2 = 7$ and $t^2 = 15$ cases, but it is not difficult. First conjugate by the appropriate J as in the previous two proofs. We assume this has been done. Under this conjugation, the group is generated by the three matrices $m_0 = J^{-1} i_0 J$, $m_1 = J^{-1} i_1 J$ and $m_2 = J^{-1} i_2 J$. Denote by G the index two subgroup generated by the four elements m_1, m_2, $m_0 m_1 m_0$ and $m_0 m_2 m_0$. The appropriate matrices are

$$m_1 = \begin{bmatrix} -1 & 0 & 0 \\ -2 + 2\sqrt{-3} & -1 & 1 - \sqrt{-3} \\ -4 & 0 & 1 \end{bmatrix},$$

$$m_2 = \begin{bmatrix} -1 & \frac{1}{2} - \frac{1}{2}\sqrt{-3} & \frac{1}{2}\sqrt{-3} \\ 0 & 1 & \frac{3}{2} + \frac{1}{2}\sqrt{-3} \\ 0 & 0 & -1 \end{bmatrix},$$

$$m_0 m_1 m_0 = \begin{bmatrix} -3 & \frac{1}{2} - \frac{1}{2}\sqrt{-3} & 1 + \frac{1}{2}\sqrt{-3} \\ -6 + 2\sqrt{-3} & -1 - 2\sqrt{-3} & \frac{9}{2} + \frac{1}{2}\sqrt{-3} \\ -8 & -2 - 2\sqrt{-3} & 3 + 2\sqrt{-3} \end{bmatrix},$$

and

$$m_0 m_2 m_0 = \begin{bmatrix} -3 & -\frac{1}{2} - \frac{1}{2}\sqrt{-3} & \frac{3}{2} \\ -2 + 2\sqrt{-3} & -3 & \frac{3}{2} - \frac{3}{2}\sqrt{-3} \\ -8 & -2 - 2\sqrt{-3} & 5 \end{bmatrix}$$

It suffices to show that the group generated by these four elements is discrete.

Denote by R the ring composed of elements

$$\left\{ \frac{\alpha}{2} + \frac{\beta}{2}\sqrt{-3} \;\middle|\; \alpha + \beta = \text{even} \right\},$$

and by S the set

$$\left\{ \frac{\alpha}{2} + \frac{\beta}{2}\sqrt{-3} \;\middle|\; \alpha, \beta \in \mathbb{Z} \right\}.$$

Denote by Z the ring of elements

$$\left\{ \alpha + \beta\sqrt{-3} \;\middle|\; \alpha, \beta \in \mathbb{Z} \right\},$$

and by $2Z$ and $4Z$ the sets of multiples of elements in Z by 2 and 4 respectively.

The following are either obvious from ring structure or follow from straightforward calculations:

1. $ZZ \subset Z$
2. $RR \subset R$
3. $2R \subset Z$
4. $2S \subset Z$
5. $ZR \subset R$
6. $ZS \subset S$
7. $Z + Z \subset Z$
8. $R + R \subset R$
9. $S + S \subset S$
10. $Z + R \subset R$
11. $Z + S \subset S$
12. $R + S \subset S$
13. $4Z \subset 2Z \subset Z \subset R \subset S$

Denote by M the set of matrices which have entry (m, n) being an element of the set at position (m, n) in the matrix

$$\begin{bmatrix} Z & R & S \\ 2Z & Z & R \\ 4Z & 2Z & Z \end{bmatrix}.$$

Observe that all four generators above and the identity matrix are contained in M and also

$$\begin{bmatrix} Z & R & S \\ 2Z & Z & R \\ 4Z & 2Z & Z \end{bmatrix} \begin{bmatrix} Z & R & S \\ 2Z & Z & R \\ 4Z & 2Z & Z \end{bmatrix}$$

$$= \begin{bmatrix} ZZ+2ZR+4ZS & ZR+ZR+2ZS & ZS+RR+ZS \\ 2ZZ+2ZZ+4ZR & 2ZR+ZZ+2ZR & 2ZS+ZR+ZR \\ 4ZZ+4ZZ+4ZZ & 4ZR+2ZZ+2ZZ & 4ZS+2ZR+ZZ \end{bmatrix}$$

$$\subset \begin{bmatrix} Z & R & S \\ 2Z & Z & R \\ 4Z & 2Z & Z \end{bmatrix}.$$

from whence it follows that M is actually a group. Since M is a discrete set, the group generated by the four elements above is discrete as well.

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It is actually worth noting that this last proof also works for the cases $t^2 = 7$ and $t^2 = 15$, though we leave the alternate method in for a little variety.

Theorem 3.2.0.13. *The triangle group with parameter t is indiscrete for $t^2 > 7$ and $t^2 \neq 15$.*

The proof of this uses the following interesting proposition. This proof is by Larry Washington.

Proposition 3.2.0.14. *Suppose α and β are rational multiples of π (so that $e^{i\alpha}$ and $e^{i\beta}$ are roots of unity) and $\cos\alpha + \cos\beta = 1$, then putting $\alpha = 2\pi c/n$ with $gcd(c, n) = 1$, we must have $n \in \{1, 2, 3, 4, 6\}$.*

Lemma 3.2.0.15. *Suppose α and β are rational multiples of π (so that $e^{i\alpha}$ and $e^{i\beta}$ are roots of unity) and $\cos\alpha + \cos\beta = 1$. Then $\cos\alpha$ is rational.*

Proof. Assume $\cos\alpha$ is irrational. We will find a Galois conjugate taking $\cos\alpha$ to a negative number.

Write $\alpha = 2\pi c/n$ with $gcd(c,n) = 1$. Let $z = e^{2\pi i/n}$. Apply the inverse of the Galois automorphism $z \mapsto z^c$. Call this inverse g. This maps $(e^{2\pi ic/n} + e^{-2\pi ic/n})/2 = \cos(2\pi c/n)$ to $(e^{2\pi i/n} + e^{-2\pi i/n})/2 = \cos 2\pi/n$. We want to find a Galois conjugate of z in the 2nd or 3rd quadrant. The conjugates of z are z^w with $gcd(w,z) = 1$. We therefore need to show there exists w with $n/4 < w < 3n/4$ and $gcd(w,z) = 1$. Bertrand's postulate states that for $x > 1$ there is always a prime between x and $2x$. Therefore for $n > 4$, there is always a prime p with $n/4 < p < n/2$. If $gcd(p,n) = 1$, let $w = p$ and we're done. If not, then $n = kp$. Since $n/4 < n/k < n/2$ we must have $k = 3$. Since $p \neq 2$ (since this is the $n = 6$ case) n is odd and there is a power of 2, call it w, between $n/4$ and $n/2$ with $gcd(w,n) = 1$.

Now then, observe that since the cosine of an angle corresponding to a root of unity is equal to 1/2 of that root plus its complex conjugate, and since Galois conjugation takes roots of unity to roots of unity and complex conjugates to complex conjugates, it follows that cosines of such angles are taken to cosines of such angles. This follows from the fact that the cosine may be expressed as $(\alpha + \bar{\alpha})/2$, where α is the root of unity. Thus applying g followed by $z \mapsto z^w$ to the equation $\cos\alpha = 1 - \cos\beta$ yields $0 > \cos\alpha' = 1 - \cos\beta' \geq 0$, a contradiction.

$\triangledown$

Proof. (of proposition.)

By hypothesis, $e^{i\alpha}$ is a primitive n-th root of unity. Since $e^{i\alpha}$ is a root of $x^2 - (2\cos\alpha)x + 1$, which is a polynomial over $\mathbb{Q}$ since $\cos\alpha \in \mathbb{Q}$ by the lemma, the field extension containing all n-th roots of unity has degree at most 2 over $\mathbb{Q}$. Such a field extension has degree $\phi(n)$, where ϕ is the Euler phi-function. Thus we must have $\phi(n) \leq 2$, so that $n \in \{1, 2, 3, 4, 6\}$.

$\bigtriangledown$

The theorem now follows from the following observation. For $t^2 > 7$, both ξ_t and η_t are both regular elliptic. ξ_t has an eigenvalue a in the 2nd quadrant and η_t has one, call it b, in the 1st. These were computed earlier to be

$$a = \frac{7 - t^2 + 4t\sqrt{3 - t^2}}{t^2 + 1}$$

and

$$b = \frac{8 + \sqrt{(7 - t^2)(t^2 + 9)}}{t^2 + 1}.$$

Assume that both are roots of unity, and specifically that $a = 2\pi c/n$ with $gcd(c, n) = 1$. Notice that Re(a)+1=Re(b). Let α be the angle between the negative x-axis and a and let β be the angle between the positive x-axis and b. Then $\cos\beta = \mathrm{Re}(b) = \mathrm{Re}(a) + 1 = -\cos\alpha + 1$ so that $\cos\alpha + \cos\beta = 1$. Thus by the proposition $n \in \{1, 2, 3, 4, 6\}$. The corresponding values of t^2 are then $\{\mathrm{No\ Soln}, 3, 13/3, 7, 15\}$. Since $t^2 > 7$ and $t^2 \neq 15$, we have a contradiction. Thus both cannot simultaneously be roots of unity, so one must have infinite order. Thus the group is indiscrete.

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Conjecture 3.2.0.16. *The triangle group with parameter t is discrete for $t^2 < 3$ and discrete for infinitely many t satisfying $3 \leq t^2 \leq 7$.*

The basic rationale behind this conjecture is that it seems that in the region $3 \leq t^2 \leq 7$, only ξ is regular elliptic with infinitely many parameters for which it has finite order, and for $t^2 < 3$, there are no elliptic elements of infinite order, and perhaps no elliptic elements other than the generators.

We present some preliminary computer data that supports this conjecture.

We find the of roots of the discriminant function for various powers of the element ξ, and then start testing these values of t. By testing, we mean we have taken all words of up to a certain length, and started raising them to powers. What we have found is that in all cases for which the root R satisfies $3 \leq R^2 \leq 7$, all words become non-regular elliptic (and non ellipto-parabolic) by a certain power. That is, they become either the identity or unipotent. Both of these are good for discreteness.

Here are the roots for powers up to 6:

Power	Roots
1	$\pm\sqrt{3}$
2	$\pm\sqrt{3}, \pm\sqrt{7}$
3	$\pm\sqrt{3}, \pm\sqrt{\frac{13}{3}}, \pm\sqrt{15}$
4	$\pm\sqrt{3}, \pm\sqrt{7}, \pm\sqrt{\frac{97}{15+8\sqrt{2}}}, \pm\sqrt{\frac{97}{15-8\sqrt{2}}}$
5	$\pm\sqrt{3}, \pm\sqrt{\frac{209}{23+8\sqrt{5}}}, \pm\sqrt{\frac{209}{23-8\sqrt{5}}}, \pm\sqrt{\frac{181}{35+8\sqrt{5}}}, \pm\sqrt{\frac{181}{35-8\sqrt{5}}}$
6	$\pm\sqrt{3}, \pm\sqrt{\frac{13}{3}}, \pm\sqrt{15}, \pm\sqrt{7}, \pm\sqrt{\frac{193}{31+16\sqrt{3}}}, \pm\sqrt{\frac{193}{31-16\sqrt{3}}}$

Next we have a table of results for each of the above roots. We check all words of up to length listed in the "Length" column, and raised each element to all powers less than or equal to"Power". For the cases where a specific element is not listed, then all the elements were either not regular elliptic to begin with, or became non-regular elliptic by the time they were raised to that power. Where an element is listed, we have found that element to not satisfy this criterion; that is, the element is regular elliptic for all powers less than or equal to the power listed. It is important to note that the existence of such an element does not destroy the chance of discreteness, as some power of that element might be the identity, which would save the day. This does not seem to happen, however, as we shall illustrate shortly. The final column lists the power n such that the element ξ_t is the identity or, as in the first line, simply notes that it is unipotent to begin with.

t^2 value	$\approx$	Length	Power	n so $\xi_t^n = id$
3	3	10	16	n.a. (unipotent)
7	7	10	16	4
$\frac{13}{3}$	4.333333	10	16	6
15	15	10	16	6
$\frac{97}{15+8\sqrt{2}}$	3.686292	10	16	8
$\frac{97}{15-8\sqrt{2}}$	27.31371	6	$i_0i_1i_2i_1$ is r.e.	
$\frac{209}{23+8\sqrt{5}}$	5.111456	10	16	10
$\frac{209}{23-8\sqrt{5}}$	40.88854	6	$i_0i_1i_2i_1$ is r.e.	
$\frac{181}{35+8\sqrt{5}}$	3.422291	10	16	10
$\frac{181}{35-8\sqrt{5}}$	10.57771	6	$i_0i_1i_2i_1$ is r.e.	
$\frac{193}{31+16\sqrt{3}}$	3.287187	10	16	12
$\frac{193}{31-16\sqrt{3}}$	58.71281	6	$i_0i_1i_2i_1$ is r.e.	

3.3 An Interesting Attempted Approach

Let p be the vector polar to the geodesic which is perpendicular to c_0 and $i_1(c_2)$. Note that p exists only for $t^2 < 7$, where $i_0i_1i_2i_1$ is hyperbolic so c_0 and $i_1(c_2)$ are disjoint.

Conjecture 3.3.0.17. *For* $3 < t^2 < 7$, *We have* $< w(p), p >\geq (t^2+1)/(7-t^2)$ *for all* $w \in \Gamma / < i_0i_1i_0i_2 >$.

Note that this will yield discreteness for all the isolated triangle groups we have conjectured to be as such. This follows from the fact that for $t^2 > 3$,

$(t^2+1)/(7-t^2) > 1$, so for such t, the geodesic given by p is taken off itself for all w, and there is a lower bound on the distance away that it may be. Thus every nontrivial element is away from the element by a nontrivial amount independent of the element. Since $i_0 i_1 i_0 i_2$ has finite order in these cases, the discreteness of $\Gamma / < i_0 i_1 i_0 i_2 >$ proves the discretenss of Γ.

This conjecture is supported by overwhelming computer evidence. The lower bound was found for various values of t for all word lengths up to 30.

3.4 The General (n, n, ∞) Triangle Group

In [Gp] and [Sc], it was shown that in the ideal case, Γ is indiscrete iff $i_0 i_1 i_2$ is elliptic iff $t^2 \geq 125/3$. It seems sensible then that for some N perhaps very large, the behavior of the (n, n, ∞) triangle group for $n \geq N$ is influenced by this element. We clarify precisely the manner in which this occurs with the following proposition.

Proposition 3.4.0.18. *There is a "wedge" of parameter pairs (n, t) for which the (n, n, ∞) triangle group with parameter t is indiscrete as a result of $i_0 i_1 i_2$ being elliptic of infinite order.*

In the proof of this proposition we shall clarify what we mean by a "wedge".

We compute the trace of $i_0 i_1 i_2(r, t)$, to be

$$\frac{i(1+16r^2+it)}{t-i} = -\frac{1+16r^2+t^2}{t^2+1} + \frac{16r^2 t}{t^2+1} i.$$

Plugging this in $f(x)$ and setting equal to 0 yields solutions

$$t^2 = \sqrt{\frac{2 + 11r^2 - 80r^4 + 64r^6 + r^2(8r^2 - 7)^{3/2}\sqrt{8r^2 + 1}}{2(r^2 - 1)}}$$

and

$$t^2 = \sqrt{\frac{2 + 11r^2 - 80r^4 + 64r^6 - r^2(8r^2 - 7)^{3/2}\sqrt{8r^2 + 1}}{2(r^2 - 1)}}.$$

Since $t \geq 0$ is real, t only exists if $8r^2 \geq 7$ in which case the two resulting values of t are identical. It follows that the solution set (r, t) is the edge along which the element $i_0 i_1 i_2$ may go into and out of regular ellipticity.

The first of these solution sets shall be called the outer curve while the second shall be called the inner curve. The outer curve intersects the $n = \infty$ $(r = 1)$ case at $t = \infty$ while the inner curve intersects at $t = \sqrt{125/3}$. A simple test of an inside and outside element gives us $i_0 i_1 i_2$ regular elliptic inside this wedge and hyperbolic outside.

Notice that $\cos(\pi/8) < \sqrt{7/8} < \cos(\pi/9)$, so the first triangle group which has a regular elliptic $i_0 i_1 i_2$ is the $(9, 9, \infty)$ case. For lower n, the element is always hyperbolic.

Figures 3.1 and 3.2 show pictures of the wedge and a close up, along with circles of appropriate distance for the $n = 8$, $n = 9$, and $n = 10$ triangle groups.

We only need now show that $i_0 i_1 i_2$ has infinite order on the triangle-group radii inside the wedge. To do this, we appeal to Richard Schwartz. The following is a (very slight) modification of a proof found in [Sc].

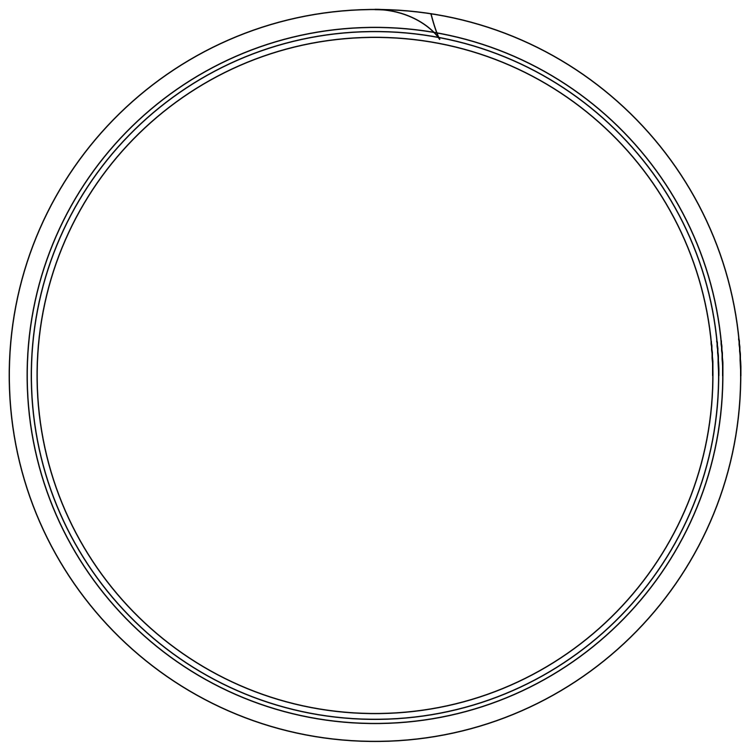

Figure 3.1: $i_0 i_1 i_2$ Ellipticity Wedge

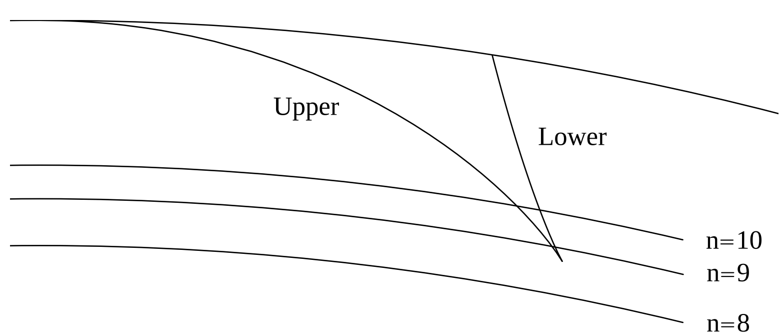

Figure 3.2: Close-Up of $i_0 i_1 i_2$ Ellipticity Wedge

Lemma 3.4.0.19. *Suppose T the trace of $i_0 i_1 i_2(n,t)$ satisfies*

$$(T+9)(\bar{T}+9) \leq 64; \quad T \neq -1, \tag{3.1}$$

then $i_0 i_1 i_2$ has infinite order.

Proof. Suppose $i_0 i_1 i_2$ has finite order, then denoting by ω_n the n-th root of unity, we have

$$T = \omega_n^p + \omega_n^q + \omega_n^r; \quad p+q+r=0. \tag{3.2}$$

Where n is taken as small as possible.

For k relatively prime to n, let $\sigma_k : \mathbb{Q}[\omega_n] \to \mathbb{Q}[\omega_n]$ be the Galois automorphism defined by $\sigma_k(\omega_n) = \omega_n^k$. Equation (3.1) is defined in $\mathbb{Q}[\omega_n]$, so it remains true if we replace T by $\sigma_k(T)$. In particular, (3.1) gives us that $Re(\sigma_k(T)) < -1$. Restricting k to $\{1, ..., n-1\}$ and summing we have

$$\sum_k \sigma_k(T) < -\phi(n)$$

where ϕ denotes the Euler phi function. Hence

$$\left| \sum_k \sigma_k(T) \right| > \phi(n). \tag{3.3}$$

For $m = p$,q, or r, let (m,n) denote the GCD of m and n. Let $d_m = n/(m,n)$. Note that ω_n^m is a primitive d_m-th root of unity, and the sum of all such roots is one of -1, 0, or 1. The map $\mathbb{Z}/n \to \mathbb{Z}/d_m$ induces a map $(\mathbb{Z}/n)^* \to (\mathbb{Z}/d_m)^*$ which is onto, with multiplicity $\phi(n)/\phi(d_m)$. This gives the bound

$$\left|\sum_k \sigma_k(\omega_n^m)\right| \leq \frac{\phi(n)}{\phi(d_m)}. \tag{3.4}$$

Combining equations (3.1),(3.2),(3.3),(3.4), we get

$$\frac{1}{\phi(d_p)} + \frac{1}{\phi(d_q)} + \frac{1}{\phi(d_r)} > 1.$$

There is no positive integer k such that $\phi(k) = 3$, and $\phi(k) = 2$ only for $k \in \{2,3,4,6\}$. We conclude that all three roots are at most 12-th roots of unity. A rapid computer search rules out this possibility.

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It is only necessary therefore to show (3.1). We compute T to be

$$\frac{9 + it + 8\cos(2\pi/n)}{-1 - it}$$

so that

$$(T+9)(\bar{T}+9) = \frac{32(2 + t^2 + t^2\cos(4\pi/n))}{t^2+1}$$

which is ≤ 64 provided that $n \geq 8$. Since the triangle group radii intersect the wedge only for $n \geq 9$, this criteria is satisfied, and $i_0 i_1 i_2$ has infinite order in the wedge. It follows that the representation is indiscrete there.

$\bigtriangledown$

One may ask whether or not the behavior of the pair $i_0 i_1 i_0 i_2$ and $i_0 i_1 i_2 i_1$ might also occur for the general case.

We have shown that every (n, n, ∞) triangle group is represented by the unit circle chain in $\mathcal{H}$ and two vertical chains through points z and $-\bar{z}$ where $|z| = \cos\frac{\pi}{n}$. Defining i_1 to be inversion in the $(re^{i\tan^{-1} t})$-chain and i_2 to be inversion in the $(re^{i(\pi - \tan^{-1} t)})$-chain, with i_0 as before, and putting $\xi_{r,t} = i_0 i_1 i_0 i_2$, we compute the trace of $\xi_{r,t}$ to be

$$\frac{3(t^2+1) + 16r^4(t^2+1) - 16r^2(t^2-1)}{t^2+1}.$$

Thus, since this is real and using our discriminant function, we find $\xi_{r,t}$ is regular elliptic iff

$$t < \sqrt{\frac{1+r^2}{1-r^2}}$$

and hyperbolic iff

$$t > \sqrt{\frac{1+r^2}{1-r^2}}.$$

An interesting thing to note is the simplicity of the solution set for

$$t = \sqrt{\frac{1+r^2}{1-r^2}},$$

it is the lemniscate $r = \sqrt{-\cos 2\theta}$ (recall $t = \tan\theta$). We must be careful in noting that if r is not $\cos\frac{\pi}{n}$ for some n, the elements $i_0 i_1$ and $i_0 i_2$ are elliptic of infinite order, obstructing discreteness. Thus for all the following, we assume r to satisfy this criterion.

The first thought is whether the embedding is discrete for values of t less than

$$\sqrt{\frac{1+r^2}{1-r^2}},$$

though this is shown to be false by noting that from the calculations related to the $i_0i_1i_2$ ellipticity wedge, for $n \geq 14$, the element $i_0i_1i_2$ becomes regular elliptic of infinite order for values of t less than the above.

The obvious conjecture is then

Conjecture 3.4.0.20. *For $n \leq 13$, the embedding is discrete for*

$$t \leq \sqrt{\frac{1+r^2}{1-r^2}},$$

and for $n \geq 14$, the embedding is discrete for values of t less than or equal to the critical value where $i_0i_1i_2$ goes regular elliptic.

In making an analogy to the $(4, 4, \infty)$ case, we offer the additional following conjecture:

Conjecture 3.4.0.21. *For $n \leq 13$, there are infinitely many t's in the range*

$$0 \leq t \leq \sqrt{\frac{1+r^2}{1-r^2}}$$

such that the embedding is discrete.

Chapter 4

Some $[m, m, 0]$ Triangle Groups

Glancing at the arrangement of the chains for the (n, n, ∞) triangle groups, the vertical chains link the $(0, 1)$-chain. An obvious question would be: What sorts of triangle groups do we get if we place the vertical chains *outside* the $(0, 1)$-chainΓ Since they will then not link this chain, the triangle group we get is ultraideal, with the vertical chains at distance 0. If the vertical chains are placed symmetrically (like the (n, n, ∞) cases), we get $[m, m, 0]$ groups. If not, we get $[m, l, 0]$ groups.

The proof of the 1-1 correspondance between the value $t = \tan\theta$ and the (n, n, ∞) triangle group given in proposition 3.1.0.6 extends in the canonical fashion to the $[m, m, 0]$ groups. In the $[m, m, 0]$ case, if the distance between a vertical chain and the $(0, 1)$-chain is to be m, we must choose z such that this is so. For ultraparallel geodesics, we use the notation of lemma 3.1.0.8 and the fact that $|z| = |< p_0, p_z >| = \cosh(m/2)$ so that $m = 2\ \mathrm{arccosh}|z|$.

Here is a quick proof that most of these $[m, m, 0]$ triangle groups are discrete. We use the compressing criteria described in definition 2.2.1.1 and the fact that a compressing triangle group is discrete.

Let U be the unit spinal sphere described earlier. That is, the spinal sphere with vertices $(0, \pm 1)$ in $\mathcal{H}$. Note that U is not a Euclidean sphere inside $\mathcal{H}$, instead it is the set of points $\{(z, y) \in \mathcal{H} : |z|^4 + y^2 = 1\}$. It thus contains the $(0, 1)$-chain and in fact, inversion in the $(0, 1)$ chain preserves the spinal sphere and switches its inside and outside.

Let i_0 denote inversion in the $(0, 1)$ chain, let i_1 and i_2 denote inversion in the z-chain and the $-\bar{z}$ chains respectively. Let Γ be the group generated by all three inversions, and let Λ be the group generated by i_1 and i_2.

Let U_1 be the part of $\mathcal{H} - U$ outside U and U_2 be the part inside. If we can find a $V \subsetneq U_1$ such that $i(U_2) \subsetneq V$ for all non-identity $i \in \Lambda$, we have proven that Γ is compressing and hence discrete.

Inversion in a vertical chain acts on all the other vertical chains simply by a rotation of π radians. Some vertical translation of any individual chain will generally occur, but the chain is setwise sent to the vertical chain rotated π radians around the chain we are inverting in.

Let Y be the set of vertical chains through all w with $|w| = 1$. Let Y_2 be the component of $\mathcal{H} - Y$ containing the origin, and let Y_1 the other component. We have $U_2 \subset Y_2$ and so $i(U_2) \subset i(Y_2)$ for all non-identity $i \in \Lambda$.

Now then, we can look at the images $i(Y_2)$ merely by looking at the intersection of these images with $\mathbb{C} \times \{0\} \subset \mathcal{H}$. Since these intersections are all circles, Λ simply moves the unit circle around in $\mathbb{C}$ by rotations of π radians around z and $-\bar{z}$. Provided that the unit circle is mapped off itself by all non-identity $i \in \Lambda$, the same is true for Y_2 and hence for U_2. We can then let V be the union of all the images of U_2. We are sure that $V \neq U_1$ since V is missing all images of $Y_2 - U_2$.

Since the radii of the circle is preserved, it suffices to show that the origin is moved to a distance of at least 2 by all non-identity $i \in \Lambda$. Clearly we must have $|z| \geq 1$ so that $|i_1(0)| = 2$. In general, rotating a point $\alpha \in \mathbb{C}$ in another point $\beta \in \mathbb{C}$ yields the point $2\beta - \alpha$. Consider applying i_1 then i_2 then i_1 and so on to the origin 0. i_1 takes 0 to $2z$, so then i_2 takes $2z$ to $2(-\bar{z}) - 2z = -4\text{Re}(z)$. i_1 gets us to $2z + 4\text{Re}(z)$, and then another i_2 gets us to $-8\text{Re}(z)$. The pattern is clear, we have $(i_2 i_1)^n(0) = -4n\text{Re}(z)$ and $i_1(i_2 i_1)^n(0) = 2z + 4n\text{Re}(z)$.

In order for $|-4n\text{Re}(z)| \geq 2$ for all n, we must have $\text{Re}(z) \geq 1/2$. If this criteria is met, then automatically since $4n\text{Re}(z) \geq 2$ we have $|2z + 4n\text{z}| \geq 2$ as long as $n > 0$. Provided $|z| \geq 1$, we have it always.

It follows then that Γ, the $[m, m, 0]$ triangle group, is discrete for generators having $\text{Re}(z) \geq 1/2$. A little arithmetic restates this in a more analogous fashion to the (n, n, ∞) groups: Given an m, the $[m, m, 0]$ ultra-ideal group is discrete for $t \leq \sqrt{4\cosh^2(m/2) - 1}$.

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