Compact Central WENO Schemes for Hyperbolic Problems

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Overview

1. Hyperbolic Conservation Laws
2. Central Schemes
3. Compact Weighted Essentially Non-Oscillatory Schemes
4. Examples
Hyperbolic Conservation Laws

Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. In one space dimension,

$$\frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} f(u) = 0 \quad (1)$$

is a hyperbolic conservation law if the flux Jacobian $J_{ij} = \frac{\partial f_i}{\partial x_j}$ is real-diagonalizable.

Because solutions can contain discontinuities, (1) is usually expressed in integral form:

$$\frac{d}{dt} \int_b^a u \, dx + [f(u(b, t)) - f(u(a, t))] = 0 \quad (2)$$
Let \( u : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}^n \) and \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \). In one space dimension, \( \frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} f(u) = 0 \) (1) is a hyperbolic conservation law if the flux Jacobian \( J_{ij} = \frac{\partial f_i}{\partial x_j} \) is real-diagonalizable. Because solutions can contain discontinuities, (1) is usually expressed in integral form:

\[
\frac{d}{dt} \int_a^b u \, dx + [f(u(b, t)) - f(u(a, t))] = 0
\] (2)
Numerical Solutions

The integral form directly suggests a discretization based on evolving averages of the solution over cells:

\[ \bar{u}_j^{n+1} = \bar{u}_j^n - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} [f(u(x_{j+1/2}, \tau)) - f(u(x_{j-1/2}, \tau))] d\tau \]  

(3)
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\]
How to Evolve the Solution?

To evolve the cell-averages exactly, what information is needed?

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The Riemann Problem

Because the reconstructed solution is discontinuous at cell interfaces, there is not a single solution vector that can be plugged into the flux function in order to evaluate the flux at an interface.
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Because the reconstructed solution is discontinuous at cell interfaces, there is not a single solution vector that can be plugged into the flux function in order to evaluate the flux at an interface. Each interface can be considered as an independent Riemann problem:
Developing a Riemann solver requires knowledge of the flux Jacobian.

Riemann solvers for nonlinear systems are often computationally expensive and/or approximate.

Results depend on the choice of Riemann solver.

Riemann solvers technically are only valid for piecewise-constant reconstructions, which are too inaccurate for practical problems.

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Central schemes sidestep these deficiencies.
The key concept underlying central schemes is a staggered grid:

\[
\bar{u}_{n+1}^{j+1/2} = \bar{u}_{n}^{j+1/2} - \Delta x \int_{t_n}^{t_{n+1}} \left[ f(u(x_{j+1}, \tau)) - f(u(x_j, \tau)) \right] d\tau
\]
The key concept underlying central schemes is a staggered grid:

This setup requires the update equation to be applied to the staggered cells:

\[
\bar{u}_{j+1/2}^{n+1} = \bar{u}_{j+1/2}^n - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} [f(u(x_{j+1}, \tau)) - f(u(x_j, \tau))] \, d\tau
\]  

(4)
Avoiding Riemann Solvers

Reconstruction viewed on the staggered grid

= Riemann solution
Steps:

1. For each cell, calculate the average of the solution over the left and right subcells and the value of the solution at the midpoint.
2. From the subcell averages, compute the staggered-cell averages at the current time step.
3. Evolve the staggered cell averages, evaluating the (smooth) flux integral by any quadrature method.
4. Repeat the process with the staggered grid shifted in the opposite direction to return to the original grid

\[
\bar{u}_{j+1/2}^{n+1} = \bar{u}_{j+1/2}^n - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} [f(u(x_{j+1}, \tau)) - f(u(x_j, \tau))] \, d\tau
\]  

(5)
Compact WENO Schemes

How can we compute the point values and subcell averages from the cell averages, given that the underlying solution might be discontinuous?

Consider three polynomial reconstruction stencils for the left subcell average, each with accuracy $O(\Delta x^3)$:

1. $5 \bar{u}_{Lj} - 1 + 3 \bar{u}_{Lj} = 16 \bar{u}_j - 2 + 13 \bar{u}_j$ + $5 \bar{u}_j$
2. $3 \frac{1}{16} \bar{u}_{Lj} - 1 + 5 \bar{u}_{Lj} + 3 \frac{1}{16} \bar{u}_{Lj} + 1 = 32 \bar{u}_j - 1 + 5 \bar{u}_j + 13 \bar{u}_j$
3. $3 \frac{1}{8} \bar{u}_{Lj} + 5 \bar{u}_{Lj} + 1 = 64 \bar{u}_j - 2 + 5 \bar{u}_j + 13 \bar{u}_j$

From these schemes a sixth-order accurate scheme can be formed by convex combination with ideal weights $d_k$:
Compact WENO Schemes

How can we compute the point values and subcell averages from the cell averages, given that the underlying solution might be discontinuous? Consider three polynomial reconstruction stencils for the left subcell average, each with accuracy $O(\Delta x^3)$

\[
(1) \quad \frac{5}{8} \bar{u}_{j-1}^L + \frac{3}{8} \bar{u}_j^L = \frac{1}{64} \bar{u}_{j-2} + \frac{13}{32} \bar{u}_{j-1} + \frac{5}{64} \bar{u}_j
\]

\[
(2) \quad \frac{3}{16} \bar{u}_{j-1}^L + \frac{5}{8} \bar{u}_j^L + \frac{3}{16} \bar{u}_{j+1}^L = \frac{5}{32} \bar{u}_{j-1} + \frac{5}{16} \bar{u}_j + \frac{1}{32} \bar{u}_{j+1}
\]

\[
(3) \quad \frac{3}{8} \bar{u}_{j-1}^L + \frac{5}{8} \bar{u}_{j+1}^L = \frac{19}{64} \bar{u}_j + \frac{7}{32} \bar{u}_{j+1} - \frac{1}{64} \bar{u}_j
\]
Compact WENO Schemes

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3. \[ \frac{3}{8} \bar{u}_j^L + \frac{5}{8} \bar{u}_{j+1}^L = \frac{19}{64} \bar{u}_j + \frac{7}{32} \bar{u}_{j+1} - \frac{1}{64} \bar{u}_j \]

From these schemes a sixth-order accurate scheme can be formed by convex combination with \textit{ideal weights} $d_k$: \[ \sum_{k=1}^{3} d_k \ast (k) \]
The WENO procedure replaces the ideal weights $d_k$ with non-oscillatory weights $\omega^{(k)}$:

$$
\sum_{k=1}^{3} d_k \ast (k) \rightarrow \sum_{k=1}^{3} \omega_k \ast (k)
$$

$$
\omega_k = \frac{\alpha_k}{\alpha_1 + \alpha_2 + \alpha_3}, \quad \alpha_k = \frac{d_k}{(\epsilon + \beta_k)^2}
$$
WENO Weights

The WENO procedure replaces the ideal weights $d_k$ with *non-oscillatory* weights $\omega^{(k)}$:

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\]

where $\beta_k$ are formed from undivided differences that reflect the local smoothness of the solution:

\[
\beta_1 = \frac{13}{12} (u_{j-2} - 2u_{j-1} + u_j)^2 + \frac{1}{4} (u_{j-2} - 4u_{j-1} + 3u_j)^2 \sim \mathcal{O}(\Delta x^2)
\]

\[
\beta_2 = \frac{13}{12} (u_{j-1} - 2u_j + u_{j+1})^2 + \frac{1}{4} (u_{j-1} - u_{j+1})^2 \sim \mathcal{O}(\Delta x^2)
\]

\[
\beta_3 = \frac{13}{12} (u_j - 2u_{j+1} + u_{j+2})^2 + \frac{1}{4} (3u_j - 4u_{j+1} + u_{j+2})^2 \sim \mathcal{O}(\Delta x^2)
\]
Compact schemes require solving a system, but resolve small-scale features of the solution more efficiently. However*: 

- One system per quantity (subcell average, point value) must be solved. This cost increases with the dimension of the problem because more subcell averages are required.
- The ideal weights and LHS coefficients might differ for each quantity. Therefore a new system must be assembled for each quantity.
- If the subcell-average reconstruction is not diagonally dominant, the solution will display oscillations.
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*I have solved these problems.
Amazing! How Did You Solve Them?

\[
\frac{1}{2} \tilde{u}_{j-1}^{L} + \frac{1}{2} \tilde{u}_{j}^{L} = \frac{3}{8} \tilde{u}_{j-1} + \frac{1}{8} \tilde{u}_{j}
\]

\[
\frac{3}{17} \tilde{u}_{j-1}^{L} + \frac{11}{17} \tilde{u}_{j}^{L} + \frac{3}{17} \tilde{u}_{j+1}^{L} = \frac{41}{272} \tilde{u}_{j-1} + \frac{11}{34} \tilde{u}_{j} + \frac{7}{272} \tilde{u}_{j+1}
\]

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\]

\[
d_1 = \frac{3}{40}, d_2 = \frac{17}{20}, d_3 = \frac{3}{40}
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\frac{1}{2} u_j + \frac{1}{2} u_{j+1} = \frac{3}{8} u_j + \frac{1}{8} u_{j+1}
\]

\[
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\]
Examples

A 2D Riemann problem
A 2D Riemann problem with a shock and rarefaction