Energy-Stable WENO Schemes

Kilian Cooley

University of Maryland

March 11, 2017
Overview

1 Motivation
2 The WENO Approach
3 Energy-Stable Modification
4 Preservation of Accuracy
5 Preservation of Non-Oscillatory Property
Continuous Problem

We want a numerical approximation to the solution of:

\[ u_t + f(u)_x = 0 \]  \hspace{1cm} (1)

For simplicity, let \( f(u) = au \) with \( a > 0 \) constant. Discretize in space using conservative finite differences:

\[ \frac{d u_j}{dt} + a \frac{\hat{u}_{j+1/2} - \hat{u}_{j-1/2}}{\Delta x} = 0 \]  \hspace{1cm} (2)
Interpolation of the Solution

How to compute the solution at half-points from known grid values, knowing that the true solution may be discontinuous?

Polynomial interpolation:

\[ \hat{u}_{j+1/2} = \sum_{k} a_k u_{j+k} \]
Interpolation of the Solution

How to compute the solution at half-points from known grid values, knowing that the true solution may be discontinuous?

- Polynomial interpolation: \( \hat{u}_{j+1/2} = \sum_k a_k u_{j+k} \)
Consider three polynomial interpolation stencils, each with accuracy $O(\Delta x^3)$

\[
\hat{u}^{(1)}_{j+1/2} = \frac{1}{3} u_{j-2} - \frac{7}{6} u_{j-1} + \frac{11}{6} u_j \\
\hat{u}^{(2)}_{j+1/2} = -\frac{1}{6} u_{j-1} + \frac{5}{6} u_j + \frac{1}{3} u_{j+1} \\
\hat{u}^{(3)}_{j+1/2} = \frac{1}{3} u_j + \frac{5}{6} u_{j+1} - \frac{1}{6} u_{j+2}
\]
WENO Interpolation

Consider three polynomial interpolation stencils, each with accuracy $\mathcal{O}(\Delta x^3)$

\[
\hat{u}_{j+1/2}^{(1)} = \frac{1}{3} u_{j-2} - \frac{7}{6} u_{j-1} + \frac{11}{6} u_j
\]

\[
\hat{u}_{j+1/2}^{(2)} = -\frac{1}{6} u_{j-1} + \frac{5}{6} u_j + \frac{1}{3} u_{j+1}
\]

\[
\hat{u}_{j+1/2}^{(3)} = \frac{1}{3} u_j + \frac{5}{6} u_{j+1} - \frac{1}{6} u_{j+2}
\]

From these schemes a fifth-order accurate scheme can be formed by convex combination:

\[
\hat{u}_{j+1/2}^{5th} = \sum_{k=1}^{3} d_k \hat{u}_{j+1/2}^{(k)} + \mathcal{O}(\Delta x^5)
\]

\[
d_1 = \frac{1}{10}, \quad d_2 = \frac{6}{10}, \quad d_3 = \frac{3}{10}
\]
WENO Interpolation

The WENO procedure replaces the ideal weights $d_k$ with non-oscillatory weights $\omega^{(k)}$:

$$\hat{u}_{j+1/2} = \sum_{k=1}^{3} \omega^{(k)} \hat{u}^{(k)}_{j+1/2}$$

$$\omega^{(k)} = \frac{\alpha_k}{\alpha_1 + \alpha_2 + \alpha_3}, \quad \alpha_k = d_k \left( 1 + \frac{\tau}{\epsilon + \beta_k} \right)$$

\hspace{1cm} (5)
The WENO procedure replaces the ideal weights $d_k$ with non-oscillatory weights $\omega^{(k)}$:

$$\hat{u}_{j+1/2} = \sum_{k=1}^{3} \omega^{(k)} \hat{u}_{j+1/2}$$

$$\omega^{(k)} = \frac{\alpha_k}{\alpha_1 + \alpha_2 + \alpha_3}, \quad \alpha_k = d_k \left( 1 + \frac{\tau}{\epsilon + \beta_k} \right)$$

where $\tau, \beta_k$ are formed from undivided differences:

$$\tau = (u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2})^2$$

$$\beta_1 = \frac{13}{12} (u_{j-2} - 2u_{j-1} + u_j)^2 + \frac{1}{4} (u_{j-2} - 4u_{j-1} + 3u_j)^2$$

$$\beta_2 = \frac{13}{12} (u_{j-1} - 2u_j + u_{j+1})^2 + \frac{1}{4} (u_{j-1} - u_{j+1})^2$$

$$\beta_3 = \frac{13}{12} (u_j - 2u_{j+1} + u_{j+2})^2 + \frac{1}{4} (3u_j - 4u_{j+1} + u_{j+2})^2$$
Non-Oscillatory Interpolation

WENO Reconstruction Across Shock

-0.2 0 0.5 1 1.5 2
0 0.2 0.4 0.6 0.8 1 1.2

- Step
- WENO
WENO interpolation as described does not lead to an energy estimate similar to those available in the continuous case.

The aim of this talk (based on [Yamaleev & Carpenter, 2009]) is to modify the WENO procedure so that becomes stable.
The continuous advection equation is neutrally stable in the $L^2$ norm:

$$\frac{d}{dt} \|u\|^2 = 0$$
The continuous advection equation is neutrally stable in the $L^2$ norm:

$$\frac{d}{dt} \|u\|^2 = 0$$

Consider a perturbed advection equation ($\mu_n \geq 0$ are $C^\infty$ functions of $u$):

$$u_t + au_x = \sum_{n=1}^{N} (-1)^{n-1} \partial_x^n (\mu_n \partial_x^n u)$$  (7)
Stability of the PDE

The continuous advection equation is neutrally stable in the $L^2$ norm:

$$\frac{d}{dt} \| u \|^2 = 0$$

Consider a perturbed advection equation ($\mu_n \geq 0$ are $C^\infty$ functions of $u$):

$$u_t + a u_x = \sum_{n=1}^{N} (-1)^{n-1} \partial_x^n (\mu_n \partial_x^n u) \quad (7)$$

Integration by parts shows that the perturbations are dissipative:

$$\frac{d}{dt} \| u \|^2 = -2 \sum_{n=1}^{N} \int_0^1 \mu_n (\partial_x^n u)^2 \, dx \leq 0 \quad (8)$$
Define a $p$-th order approximation to the derivative operator as

$$
\frac{\partial \vec{u}}{\partial x} = D\vec{u} + \mathcal{O}(\Delta x^p)
$$

(9)
Define a $p$-th order approximation to the derivative operator as

$$\frac{\partial \mathbf{u}}{\partial x} = D\mathbf{u} + \mathcal{O}(\Delta x^p) \quad (9)$$

Express the matrix $D$ as $D = P^{-1}(Q + R)$ where:

- $Q$ is skew-symmetric
- $R$ is symmetric positive semidefinite
- $P$ is symmetric positive definite (e.g. $\Delta x I$)

$R$ mimics the continuous dissipation operator:

$$R = \sum_{n=1}^{N} D_n D_n^T + \Lambda_n \begin{bmatrix} D_n & 0 \end{bmatrix}$$

where $D_n$ is the forward difference matrix and $\Lambda_n$ are diagonal positive semidefinite.
Define a $p$-th order approximation to the derivative operator as

$$\frac{\partial \vec{u}}{\partial x} = D\vec{u} + \mathcal{O}(\Delta x^p)$$  \hspace{1cm} (9)$$

Express the matrix $D$ as $D = P^{-1}(Q + R)$ where:

- $Q$ is skew-symmetric
- $R$ is symmetric positive semidefinite
- $P$ is symmetric positive definite (e.g. $\Delta x I$)

$R$ mimics the continuous dissipation operator:

$$R = \sum_{n=1}^{N} D_+^n \Lambda_n [D_+^n]^T$$ \hspace{1cm} (10)$$

where $D_+$ is the forward difference matrix and $\Lambda_n$ are diagonal positive semidefinite.
Write the discrete equation in terms of $D$ and premultiply by $u^T P$:

$$u_t + aP^{-1}Qu = -aP^{-1}Ru = -a \sum_{n=1}^{N} P^{-1} D_n \Lambda_n (D_n^T)^T u$$  \hspace{1cm} (11)

$$\frac{d}{dt} \frac{1}{2} \|u\|_P^2 + au^TQu = -a \sum_{n=1}^{N} u^T D_n \Lambda_n (D_n^T)^T u \leq 0$$  \hspace{1cm} (12)
The Discrete Operator is Dissipative

Write the discrete equation in terms of $D$ and premultiply by $u^T P$:

$$u_t + aP^{-1}Qu = -aP^{-1}Ru = -a \sum_{n=1}^{N} P^{-1}D^n \Lambda_n (D^n)^T u \quad (11)$$

$$\frac{d}{dt} \frac{1}{2} \|u\|_P^2 + au^T Qu = -a \sum_{n=1}^{N} u^T D^n \Lambda_n (D^n)^T u \leq 0 \quad (12)$$

Remarks:

- $P, Q, \Lambda_n$ may depend on $u$
The Discrete Operator is Dissipative

Write the discrete equation in terms of $D$ and premultiply by $u^T P$:

$$ u_t + a P^{-1} Qu = -a P^{-1} Ru = -a \sum_{n=1}^{N} P^{-1} D_n \Lambda_n (D_n^T)^T u $$ \hspace{1cm} (11)

$$ \frac{d}{dt} \frac{1}{2} \| u \|_P^2 + a u^T Qu = -a \sum_{n=1}^{N} u^T D_n \Lambda_n (D_n^T)^T u \leq 0 $$ \hspace{1cm} (12)

Remarks:

- $P, Q, \Lambda_n$ may depend on $u$
- $u$ is not assumed to be smooth
Write the discrete equation in terms of $D$ and premultiply by $u^T P$:

$$u_t + aP^{-1}Qu = -aP^{-1}Ru = -a \sum_{n=1}^{N} P^{-1}D_n^\pm \Lambda_n (D_n^\pm)^T u$$  \hspace{1cm} (11)$$

$$\frac{d}{dt} \frac{1}{2} \|u\|_P^2 + au^T Qu = -a \sum_{n=1}^{N} u^T D_n^\pm \Lambda_n (D_n^\pm)^T u \leq 0$$  \hspace{1cm} (12)$$

Remarks:

- $P, Q, \Lambda_n$ may depend on $u$
- $u$ is not assumed to be smooth
- Stability does not preclude spurious oscillations
Why is WENO Unstable?

The differentiation matrix $D$ formed by the WENO process may have $\Lambda_n$ that are not positive semidefinite:

$$(\Lambda_3)_{j,j} = \frac{1}{6} \omega^{(1)}_{j+5/3}$$

$$(\Lambda_2)_{j,j} = \frac{1}{12} \left[ \omega^{(1)}_{j+3/2} - 4\omega^{(1)}_{j+5/2} + \omega^{(2)}_{j+3/2} - \omega^{(3)}_{j+1/2} \right]$$

$$(\Lambda_1)_{j,j} = \frac{1}{12} \left[ 3\omega^{(1)}_{j+1/2} - 5w^{(0)}_{j+3/2} + 2\omega^{(1)}_{j+5/2} + \omega^{(2)}_{j+1/2} 
- \omega^{(2)}_{j+3/2} + \omega^{(3)}_{j-1/2} - \omega^{(3)}_{j+1/2} \right]$$

$$(\Lambda_0)_{j,j} = \frac{1}{2} \left[ -\omega^{(1)}_{j-1/2} + \omega^{(2)}_{j-1/2} - \omega^{(3)}_{j-1/2} + \omega^{(1)}_{j+1/2} + \omega^{(2)}_{j+1/2} + \omega^{(3)}_{j+1/2} \right] = 0$$
Smoothly modify the matrices $\Lambda_n$ to be positive.

1. $D = D_{\text{skew}} + D_{\text{sym}}$

The stability conditions are met by construction. Is the design order of accuracy retained?
Smoothly modify the matrices $\Lambda_n$ to be positive.

1. $D = D_{skew} + D_{sym}$
2. $D_{sym} = P^{-1} \sum_{i=0}^{s} D^i \Lambda_i (D^i)^T$ where $2s + 1$ is the bandwidth of $D$
Smoothly modify the matrices $\Lambda_n$ to be positive.

1. $D = D_{skew} + D_{sym}$
2. $D_{sym} = P^{-1} \sum_{i=0}^{s} D_i^T \Lambda_i (D_i^T)^T$ where $2s + 1$ is the bandwidth of $D$
3. $(\Lambda_i)_{j,j} \rightarrow (\bar{\Lambda}_i)_{j,j} = \frac{1}{2} \left[ \sqrt{(\Lambda_i)_{j,j}^2 + \delta_i^2} - (\Lambda_i)_{j,j} \right]$
Energy-Stable Modification

Smoothly modify the matrices $\Lambda_n$ to be positive.

1. $D = D_{skew} + D_{sym}$
2. $D_{sym} = P^{-1} \sum_{i=0}^{s} D_{+}^i \Lambda_i (D_{+}^i)^T$ where $2s + 1$ is the bandwidth of $D$
3. $(\Lambda_i)_{j,j} \rightarrow (\bar{\Lambda}_i)_{j,j} = \frac{1}{2} \left[ \sqrt{(\Lambda_i)_{j,j}^2 + \delta_i^2} - (\Lambda_i)_{j,j} \right]$
4. $\bar{D}_{ad} = P^{-1} \sum_{i=0}^{s} D_{+}^i \bar{\Lambda}_i (D_{+}^i)^T$
5. $\bar{D} = D + \bar{D}_{ad}$

The stability conditions are met by construction.
Smoothly modify the matrices $\Lambda_n$ to be positive.

1. $D = D_{skew} + D_{sym}$
2. $D_{sym} = P^{-1} \sum_{i=0}^{s} D_i^T \Lambda_i (D_i^T)^T$ where $2s + 1$ is the bandwidth of $D$
3. $(\Lambda_i)_{j,j} \rightarrow (\bar{\Lambda}_i)_{j,j} = \frac{1}{2} \left[ \sqrt{(\Lambda_i)_{j,j}^2 + \delta_i^2} - (\Lambda_i)_{j,j} \right]$
4. $\bar{D}_{ad} = P^{-1} \sum_{i=0}^{s} D_i^T \bar{\Lambda}_i (D_i^T)^T$
5. $\bar{D} = D + \bar{D}_{ad}$

The stability conditions are met by construction.

Is the design order of accuracy retained?
Recall that the diagonal elements of $\Lambda_3$ are given by $\frac{1}{6} \omega_{j+5/3} = O(1)$ and those of $\Lambda_2$ by:

$$(\Lambda_2)_{j,j} = \frac{1}{12} \left[ \omega_{j+3/2}^{(1)} - 4\omega_{j+5/2}^0 + \omega_{j+3/2}^{(2)} - \omega_{j+1/2}^{(3)} \right]$$

(13)

Replacing the non-oscillatory weights by their ideal values ($d_1 = 0.1, d_2 = 0.6, d_3 = 0.3$) we find:

$$\frac{1}{12} \left[ d_1 - 4d_1 + d_2 - d_3 \right] = 0$$

(14)
Convergence of $\tilde{D}_{ad}$

Recall that the diagonal elements of $\Lambda_3$ are given by $\frac{1}{6}\omega_{j+5/3}^{(1)} = O(1)$ and those of $\Lambda_2$ by:

$$(\Lambda_2)_{j,j} = \frac{1}{12} \left[ \omega_{j+3/2}^{(1)} - 4\omega_{j+5/2}^{0} + \omega_{j+3/2}^{(2)} - \omega_{j+1/2}^{(3)} \right]$$  \hspace{1cm} (13)

Replacing the non-oscillatory weights by their ideal values ($d_1 = 0.1$, $d_2 = 0.6$, $d_3 = 0.3$) we find:

$$\frac{1}{12} [d_1 - 4d_1 + d_2 - d_3] = 0$$  \hspace{1cm} (14)

$$(\Lambda_2)_{j,j} = \frac{1}{12} \left[ (\omega_{j+3/2}^{(1)} - d_1) - 4(\omega_{j+5/2}^{(1)} - d_1) 
+ (\omega_{j+3/2}^{(2)} - d_2) - (\omega_{j+1/2}^{(3)} - d_3) \right]$$

$$= O(\Delta x^3)$$  \hspace{1cm} (15)
Convergence of $\bar{D}_{ad}$

Similarly, $(\Lambda_1)_{j,j} = O(\Delta x^4)$. To evaluate $\bar{\lambda}_i = \frac{1}{2} \left( \sqrt{\lambda_i^2 + \delta_i^2} - \lambda_i \right)$ consider three cases:

1. $|\lambda_i| \gg \delta_i > 0$
2. $\lambda_i = O(\delta_i)$
3. $|\lambda_i| \ll \delta_i$
Similarly, $(\Lambda_1)_{j,j} = \mathcal{O}(\Delta x^4)$. To evaluate $\bar{\lambda}_i = \frac{1}{2} \left( \sqrt{\lambda_i^2 + \delta_i^2} - \lambda_i \right)$ consider three cases:

1. $|\lambda_i| \gg \delta_i > 0$
2. $\lambda_i = \mathcal{O}(\delta_i)$
3. $|\lambda_i| \ll \delta_i$

In case 1, expand:

$$\bar{\lambda}_i = \frac{|\lambda_i| - \lambda_i}{2} + \frac{\delta_i^2}{4|\lambda_i|} \approx \begin{cases} |\lambda_i| & \text{if } \lambda_i < 0 \\ \delta_i & \text{if } \lambda_i = 0 \\ \frac{\delta_i^2}{4|\lambda_i|} & \text{if } \lambda_i > 0 \end{cases}$$

At worst, $\bar{\lambda}_i = |\lambda_i| + \mathcal{O}(\delta_i^2/\lambda_i)$ so only $\bar{\lambda}_i = |\lambda_i|$ need be considered.
Convergence of $\tilde{D}_{ad}$

With $\tilde{\lambda}_i = |\lambda_i|$, the artificial dissipation operator becomes:

$$\tilde{D}_{ad} = P^{-1} \left[ D_+ \Lambda_1 D_+^T + D_+^2 \Lambda_2 (D_+^2)^T + D_+^3 \Lambda_3 (D_+^3)^T \right]$$

$$= O(\Delta x^3) D_+ D_+^T + O(\Delta x^2) D_+^2 (D_+^2)^T + O(\Delta x^{-1}) D_+^3 (D_+^3)^T \quad (16)$$

$$= O(\Delta x^5)$$

since $D_+ = O(\Delta x)$. In case 2 ($\lambda_i = O(\delta_i)$, and 3 which is similar),

$$\tilde{D}_{ad} = \frac{O(\delta_1)}{\Delta x} D_+ D_+^T + \frac{O(\delta_2)}{\Delta x} D_+^2 (D_+^2)^T + \frac{O(\delta_3)}{\Delta x} D_+^3 (D_+^3)^T \quad (17)$$

$$= O(\delta_1 \Delta x) + O(\delta_2 \Delta x^3) + O(\delta_3 \Delta x^5)$$

For $p$th-order accuracy:

$$\delta_1 = O(\Delta x^{p-1}), \delta_2 = O(\Delta x^{p-3}), \delta_3 = O(\Delta x^{p-5}).$$
Does the energy-stable modification destroy the scheme’s ability to resolve shocks without spurious oscillations?
Still Non-Oscillatory?

Does the energy-stable modification destroy the scheme’s ability to resolve shocks without spurious oscillations?

Numerical tests with the quasi-1D Euler equations:

$$\frac{\partial q}{\partial t} + \frac{\partial F}{\partial x} = G$$

$$q = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ (E + P)u \end{bmatrix}, \quad G = -\frac{A_x}{A} \begin{bmatrix} \rho u \\ \rho u^2 \\ (E + P)u \end{bmatrix} \quad (18)$$

$$P = (\gamma - 1) \left( E + \frac{\rho u^2}{2} \right), \quad \gamma = 1.4$$
Numerical Test Cases

1. Steady quasi-1D nozzle flow (inflow Mach number = 1.5)

\[ A(x) = 1.398 + 0.347 \tanh(0.8x - 4), \quad 0 \leq x \leq 10 \]
1. Steady quasi-1D nozzle flow (inflow Mach number = 1.5)

\[ A(x) = 1.398 + 0.347 \tanh(0.8x - 4), \ 0 \leq x \leq 10 \]

2. Sod shock tube \((A_x = 0)\)

\[
(\rho, u, P) = \begin{cases} 
(1, 0, 1) & \text{if } -0.5 \leq x < 0 \\
(0.125, 0, 0.1) & \text{if } 0 \leq x \leq 0.5
\end{cases}
\]
Numerical Test Cases

1. Steady quasi-1D nozzle flow (inflow Mach number = 1.5)

\[ A(x) = 1.398 + 0.347 \tanh(0.8x - 4), \quad 0 \leq x \leq 10 \]

2. Sod shock tube ($A_x = 0$)

\[ (\rho, u, P) = \begin{cases} (1, 0, 1) & \text{if } -0.5 \leq x < 0 \\ (0.125, 0, 0.1) & \text{if } 0 \leq x \leq 0.5 \end{cases} \]

3. Shock-density wave interaction ($A_x = 0$)

\[ (\rho, u, P) = \begin{cases} (3.857, 2.629, 10.333) & \text{if } -5 \leq x < -4 \\ (1 + 0.2 \sin(5x), 0, 1) & \text{if } -4 \leq x < 5 \end{cases} \]

Figures from [Yamaleev & Carpenter, 2009].
Quasi-1D Nozzle, 51 points

The graph shows the density variation with a comparison to the exact solution. Three different schemes are compared:

- **5th-order WENO**
- **5th-order ESWENO**
- **6th-order ESWENO**

The graph indicates that the 6th-order ESWENO scheme provides the closest approximation to the exact solution.
Sod Shock Tube at $t = 0.16, 201$ points

WENO: $\alpha^{(k)} = \frac{d_k}{(\epsilon + \beta_k)^2}$, WENO-NW: $\alpha^{(k)} = d_k \left( 1 + \frac{\tau}{\epsilon + \beta_k} \right)$
“Exact” solution computed with conventional 5th-order WENO on 4000 points.
A nonlinear artificial dissipation was added to a high-order WENO scheme to create an energy-stable WENO scheme.
A nonlinear artificial dissipation was added to a high-order WENO scheme to create an energy-stable WENO scheme. The artificial dissipation does not degrade the formal order of accuracy, under easily satisfied conditions.
A nonlinear artificial dissipation was added to a high-order WENO scheme to create an energy-stable WENO scheme.

The artificial dissipation does not degrade the formal order of accuracy, under easily satisfied conditions.

The artificial dissipation preserves the non-oscillatory nature of WENO schemes.
A nonlinear artificial dissipation was added to a high-order WENO scheme to create an energy-stable WENO scheme.

The artificial dissipation does not degrade the formal order of accuracy, under easily satisfied conditions.

The artificial dissipation preserves the non-oscillatory nature of WENO schemes.

ESWENO schemes outperform conventional WENO schemes on several test problems.
A Systematic Methodology for Constructing High-Order Energy Stable WENO Schemes