C^1 DEFORMATIONS OF ALMOST GRASSMANNIAN STRUCTURES WITH STRONGLY ESSENTIAL SYMMETRY

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ABSTRACT. We construct a family of $(2,n)$-almost Grassmannian structure of regularity C^1, each admitting a one-parameter group of strongly essential automorphisms, and each not flat on any neighborhood of the higher-order fixed point. This provides a C^1 counterexample to Theorem 1.3 of [8, 2].

1. INTRODUCTION

Almost-Grassmannian structures belong to the class of *irreducible parabolic geometries*, which include projective and conformal structures, among many others. An (m,n)-*almost-Grassmannian structure* on an mn-dimensional manifold M comprises a vector bundle isomorphism of TM with $\mathcal{E}^* \otimes \mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ are vector bundles over M of respective ranks m and n, together with an isomorphism $\wedge^m \mathcal{E} \cong \wedge^n \mathcal{F}$; the latter corresponds to a volume form compatible with the tensor product. Denote $\mathrm{Gr}(m,n)$ the real Grassmannian variety of m-planes in $\mathbf{R}^{m+n}$, by $\mathcal{E}$ its tautological m-plane bundle, and by $\mathcal{F}$ the rank-n anti-tautological bundle. An (m,n)-almost-Grassmannian structure mimics the isomorphism of $T\mathrm{Gr}(m,n)$ with $\mathcal{E}^* \otimes \mathcal{F}$.

Almost-Grassmannian structures have been studied under the guise of *Segré structures*. The *Segré cone* $S(m,n)$ is the variety in $\mathbf{R}^{mn}$ comprising the rank-one elements under the identification with $\mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n)$. An (m,n)-Segré structure on M^{mn} is a bundle of Segré cones $S_x(m,n) \subset T_x M$. It is essentially equivalent to an (m,n)-almost Grassmannian structure (see [1]).

The $(2,n)$-almost-Grassmannian structures are connected to projective structures by twistor theory via *path geometries*. The latter are parabolic geometries that model systems of second-order ODEs on X^{n+1}. More precisely, they correspond to collections of unparametrized curves $\mathcal{C}$ obtained as the

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solutions of such a system; these lift to a foliation $\widetilde{\mathcal{C}}$ of $\mathbf{P}(TX)$. For one special class of path geometries, $\mathcal{C}$ are the unparametrized geodesics of an affine connection on M—that is, a projective structure ,—while for another class, the *path space* $\mathbf{P}(TX)/\widetilde{\mathcal{C}}$ locally carries a $(2,n)$-almost-Grassmannian structure . The intersection of the two classes is the *flat* path geometry, for which $\mathcal{C}$ comprises projective lines in $X = \mathbf{RP}^{n+1}$ (see [5, Secs 2 and 3]).

Irreducible parabolic geometries can admit certain very special automorphisms which fix a point and have trivial derivative at that point, which is then called a *higher-order fixed point.* Note that a semi-Riemannian metric or an affine connection admits no such nontrivial automorphisms. These *strongly essential automorphisms* occur in abundance on the homogeneous model spaces for each parabolic geometry. A structure that is locally equivalent to this model is said to be *flat.* (See Sections 2.2 and 2.3 below).

Many rigidity results say that only flat open submanifolds can support a strongly essential flow. Let x_0 be a higher-order fixed point.

- Nagano and Ochiai [9] proved for a torsion-free connection ∇ that existence of a strongly essential projective flow implies projective flatness of ∇ on a neighborhood of x_0.
- The second author and Neusser proved the analogous result for almost-c-projective structures and almost-quaternionic structures in [8, Thms 4.4, 1.2]. (See also [2, Thm 3.7] for a precursor result on almost-quaternionic structures.)

In conformal Lorentzian geometry, Frances smoothly deformed the Minkowski metric in a neighorhood of a point x_0 so that it retains a conformal flow with x_0 as higher-order fixed point. The resulting C^∞ metric is conformally flat inside the light cone of x_0, but nonflat outside [3, Sec 6]. Then came the following rigidity results:

- In semi-Riemannian geometry, Frances and the second author proved that existence of a strongly essential conformal flow implies conformal flatness on an open set U with $x_0 \in \overline{U}$ [4].
- In [8, Thm 1.3], the second author and Neusser proved that a $(2,n)$-almost-Grassmannian structure admitting a strongly essential flow is flat on an open set U with $x_0 \in \overline{U}$ (see also [2, Prop 3.5] for a partial result).

Kruglikov and The exhibited a C^ω homogeneous path geometry which is not flat and admits a strongly essential flow in [7, Prop 5.3.2]. Path geometries are not irreducible. The local path space in their example admits a $(2,n)$-almost-Grassmannian structure. The flow descends, but it is not strongly essential on the quotient.

The proofs of the rigidity theorems cited above, as well as the construction of [7], make use of the *Cartan geometry* canonically associated to the parabolic geometric structures in question, assuming a minimal order of regularity depending on the structure.

1.1. **Our examples.** In [10], the first author described the infinitesimal automorphisms and deformations of a parabolic geometry intrinsically in terms of the associated Cartan geometry, as certain sections of the adjoint tractor bundle. He indicated how to compute them in terms of operators on this bundle derived from BGG sequences. Motivated by this description of infinitesimal deformations, we explicitly construct a family, locally on $\mathrm{Gr}(2,n)$, that is invariant by a strongly essential flow and integrates to a family of deformed structures, all admitting this flow as automorphisms. They are counterexamples to the C^1 version Theorem 1.3 of [8].

An almost-Grassmannian structure is said to be C^k if M is at least C^{k+1}, $\mathcal{E}$ and $\mathcal{F}$ are at least C^{k+1}, and the isomorphisms $TM \cong \mathcal{E}^* \otimes \mathcal{F}$ are C^k. Such structures are said to be equivalent if they are C^k equivalent (see Section 2.3 below).

Theorem 1.1. *Let $n \geq 3$ and $x_0 \in Gr(2,n)$. There are a dense, open neighborhood U of x_0; a strongly essential flow $\{z^t\} < Aut\ Gr(2,n)$ with x_0 as higher order fixed point; and an $(n-1)$-parameter family of C^1 almost-Grassmannian structures of type $(2,n)$ on U, of which each:*

- *contains $\{z^t\}$ in its automorphism group;*
- *is not locally equivalent to $Gr(2,n)$ on any open set V with $x_0 \in \overline{V}$;*

The deformations are given in Section 4.2, and the precise claims about them are in Proposition 4.1.

Remark 1.2. *In fact, none of these deformed structures are locally equivalent to the path space of a path geometry; the harmonic torsion is the full obstruction to this property (see* [11, Props 4.4.3, 4.4.45]*).*

2. Background

2.1. Almost-Grassmannian structures as first-order G-structures. For almost-Grassmannian structures of low regularity, as we construct below, the description as Cartan geometries is not available. Thus we start by reviewing various descriptions of such structures with a special emphasis on the requirements on regularity.

Let us fix integers $m, n \geq 2$ as above, with the case $m = 2$, $n > 2$ being of primary interest. An almost-Grassmannian structure as defined above can be equivalently defined as a (first-order) G-structure for the Lie group

$$G_0 := \{(A, B) \in \mathrm{GL}(m, \mathbf{R}) \times \mathrm{GL}(n, \mathbf{R}) : \det(A)\det(B) = 1\}.$$

Under the identification $\mathbf{R}^{mn} \cong \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n)$, the natural representation of G_0 on $\mathbf{R}^{mn}$ is $\rho(A, B) \cdot X := BXA^{-1}$. Observe that the resulting homomorphism $G_0 \to \mathrm{GL}(mn, \mathbf{R})$ has two element kernel $\{(\mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, -\mathrm{Id})\}$ and thus is infinitesimally injective, so this indeed defines a family of first-order G-structures on manifolds of dimension mn.

Such a structure is given by a principal bundle $p_0 : \mathcal{G}_0 \to M$ with structure group G_0 together with a ρ-equivariant bundle morphism to the first order frame bundle $\mathcal{P}M$ of M. The structure is C^k provided $\mathcal{G}_0$ is a C^{k+1} principal bundle and the morphism to $\mathcal{P}M$ is C^k.

Proposition 2.1. *On a smooth manifold of dimension mn, a C^k first-order G_0-structure is equivalent to a C^k almost-Grassmannian structure of type (m, n).*

The bundles $\mathcal{E}^*$ and $\mathcal{F}$ are associated bundles to $\mathcal{G}_0$, while conversely $\mathcal{G}_0$ is obtained as a subbundle of the fibered product of the frame bundles of $\mathcal{E}^*$ and $\mathcal{F}$. This shows that a ρ-equivariant bundle morphism from $\mathcal{G}_0$ to $\mathcal{P}M$ is equivalent to an isomorphism $TM \xrightarrow{\sim} \mathcal{E}^* \otimes \mathcal{F}$, and the correspondence respects C^k regularity. We leave verification of details to the reader.

A ρ-equivariant bundle morphism $\mathcal{G}_0 \to \mathcal{P}M$ can be equivalently encoded as a one-form $\theta \in \Omega^1(\mathcal{G}_0, \mathbf{R}^{mn})$ which is G_0–equivariant and strictly horizontal. Denoting by $r^g : \mathcal{G}_0 \to \mathcal{G}_0$ the principal action of $g \in G_0$, equivariance means $(r^g)^*\theta = \rho(g) \circ \theta$. The second condition says that for each point $u \in \mathcal{G}_0$, the kernel of $\theta(u) : T_u\mathcal{G}_0 \to \mathbf{R}^{mn}$ is the vertical subspace in $T_u\mathcal{G}_0$. In this picture, C^k regularity means θ is C^k, in the sense that for each C^k vector field $\xi \in \mathfrak{X}(\mathcal{G}_0)$, the function $\theta(\xi) : \mathcal{G}_0 \to \mathbf{R}^{mn}$ is C^k.

2.2. **The homogeneous model—the Grassmann variety.** In this section we describe the (m, n)-almost-Grassmannian structure on $\mathrm{Gr}(m, n)$.

The group G_0 can be realized as the subgroup of $G := \mathrm{SL}(m+n, \mathbf{R})$ respecting the decomposition $\mathbf{R}^{m+n} \cong \mathbf{R}^m \oplus \mathbf{R}^n$. The Lie algebra $\mathfrak{g}_0$ is identified with the corresponding block diagonal subalgebra of $\mathfrak{g}$. Its adjoint action on $\mathfrak{g}$ preserves a decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are the subalgebras with nonzero entries only in the lower-left and upper-right blocks, respectively, complementary to $\mathfrak{g}_0$. The decomposition satisfies $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where we set $\mathfrak{g}_k = \{0\}$ for $|k| > 1$. Note that $\mathfrak{g}_{-1} \cong \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n)$, and the restriction of ad $\mathfrak{g}_0$ to $\mathfrak{g}_{-1}$ is the representation ρ from Section 2.1.

Next, let $P < G$ comprise the block-upper-triangular matrices, with Lie algebra $\mathfrak{p} := \mathfrak{g}_0 \ltimes \mathfrak{g}_1 \subset \mathfrak{g}$. It is the stabilizer of $\mathbf{R}^m \subset \mathbf{R}^{m+n}$, so G/P can be identified with $\mathrm{Gr}(m, n)$. As is well known, the tangent bundle $T(G/P)$ is the associated bundle $G \times_P (\mathfrak{g}/\mathfrak{p})$, where P acts via the adjoint representation, which factors on the quotient $\mathfrak{g}/\mathfrak{p}$ through projection to G_0. The vector space $\mathfrak{g}/\mathfrak{p}$ is moreover G_0-equivariantly isomorphic to $\mathfrak{g}_{-1}$. Consequently, $\mathrm{Gr}(m, n)$ carries an almost Grassmannian structure. Note that the auxiliarly bundles $\mathcal{E} \cong G \times_P \mathbf{R}^m$ and $\mathcal{F} \cong G \times_P (\mathbf{R}^{m+n}/\mathbf{R}^m)$ for this structure are exactly the tautological and the anti-tautological bundles.

2.3. **Automorphisms and flatness.** Let M be a C^{k+1} manifold of dimension mn with a C^k almost-Grassmannian structure comprising

- C^{k+1} vector bundles $\mathcal{E}$ and $\mathcal{F}$ of ranks m and n, respectively
- $\Phi : TM \xrightarrow{\sim} \mathcal{E}^* \otimes \mathcal{F}$ of regularity C^k
- $\nu : \Lambda^m \mathcal{E} \xrightarrow{\sim} \Lambda^n \mathcal{F}$ of regularity C^k

Definition 2.2. *An* automorphism *of the C^k almost-Grassmannian structure above is a C^{k+1} diffeomorphism h of M together with*

- *lifts $h^*_{\mathcal{E}}$ and $h_{\mathcal{F}}$ of h to automorphisms of $\mathcal{E}^*$ and $\mathcal{F}$, respectively,*
- *such that $h^*\Phi = (h^*_{\mathcal{E}} \otimes h_{\mathcal{F}}) \circ \Phi$, and*
- *such that $\nu \circ \Lambda^m h_{\mathcal{E}} = \Lambda^n h_{\mathcal{F}} \circ \nu$*

In the G-structure framework, $h \in \mathrm{Diff}^{k+1} M$ is an automorphism if it lifts to a principal bundle automorphism of $\mathcal{G}_0$ which is semi-conjugate via the ρ-equivariant bundle morphism $\mathcal{G}_0 \to \mathcal{P}M$ to the natural lift of h to $\mathcal{P}M$.

Isomorphisms of almost-Grassmannian structures are defined by the obvious extension of Definition 2.2. Local isomorphisms are isomorphisms between connected open subsets, with their restricted structures.

Consider $g \in G$ as a diffeomorphism of $\mathrm{Gr}(m, n) \cong G/P$. It naturally acts by automorphisms $g^*_{\mathcal{E}}$ and $g_{\mathcal{F}}$ of the vector bundles $\mathcal{E}^* \cong G \times_P \mathbf{R}^{m*}$ and $\mathcal{F} \cong G \times_P (\mathbf{R}^{m+n}/\mathbf{R}^m)$, respectively. The P-equivariant isomorphism of $\mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^{m+n}/\mathbf{R}^m)$ with $\mathfrak{g}/\mathfrak{p}$ gives a G-equivariant isomorphism of $T\mathrm{Gr}(m, n) \cong G \times_P \mathfrak{g}/\mathfrak{p}$ with $\mathcal{E}^* \otimes \mathcal{F}$, on which g acts by $g^*_{\mathcal{E}} \otimes g_{\mathcal{F}}$. Any $g \in G$ is thus an automorphism of the almost-Grassmannian structure on $\mathrm{Gr}(m, n)$.

Let P_+ be the connected, unipotent, normal subgroup of P with Lie algebra $\mathfrak{g}_1$. For $g \in P_+$, the linear isomorphisms $(g^*_{\mathcal{E}})_{[\mathrm{Id}_G]}$ and $(g_{\mathcal{F}})_{[\mathrm{Id}_G]}$ are trivial, so the derivative of g on $T_{[\mathrm{Id}_G]}(G/P)$ is trivial. The G-conjugates of P_+ furnish nontrivial strongly essential automorphisms at every point of $\mathrm{Gr}(m, n)$.

Each $g \in G \backslash \{\pm \mathrm{Id}_{\mathbf{R}^{m+n}}\}$ is a nontrivial transformation of G/P, because there is no larger G-normal subgroup in P. Thanks to the canonical Cartan connection associated to an almost-Grassmannian structure, presented in Section 2.6 below, the automorphism group of $\mathrm{Gr}(m, n)$ is precisely $G/\{\pm \mathrm{Id}\}$. On any almost-Grassmannian manifold (of sufficient regularity), the Cartan connection underlies the fact that the automorphism group is a Lie group of dimension at most $(m+n)^2 - 1 = \dim G$, with equality only if the structure is locally isomorphic to $\mathrm{Gr}(m, n)$, in which case it is said to be *flat*. Deciding whether an almost Grassmannian structure is flat thus is a fundamental question in the theory.

2.4. **The harmonic torsion.** The description as a G_0-structure $(p_0 : \mathcal{G}_0 \to M, \theta)$ directly leads to the first fundamental invariants of almost Grassmannian structures. We first choose a C^{k-1} principal connection $\gamma \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$. If θ is at least C^1, we can define the *torsion* of γ as the covariant exterior derivative $d^\gamma\theta \in \Omega^2(\mathcal{G}_0, \mathbf{R}^{mn})$; explicitly, for $\xi, \eta \in \mathfrak{X}(\mathcal{G}_0)$,

$$d^\gamma\theta(\xi, \eta) = d\theta(\xi, \eta) + \gamma(\xi)(\theta(\eta)) - \gamma(\eta)(\theta(\xi)). \tag{1}$$

If θ, ξ and η are C^k, then the above is a C^{k-1} function. From the properties of θ, it follows readily that $d^\gamma\theta$ is horizontal and G_0-equivariant and thus descends to a form $T^\gamma \in \Omega^2(M, TM)$, which is the usual interpretation of the torsion.

We next compute the dependence of T^γ on γ. First note that, at a point $u \in \mathcal{G}_0$, (1) depends only on $\gamma_u : T_u\mathcal{G}_0 \to \mathfrak{g}_0$. As discussed in 2.2, we can view θ as having values in $\mathfrak{g}_{-1}$. For any other principal connection $\hat{\gamma}$, the difference $\hat{\gamma}_u - \gamma_u = f_u \circ \theta_u$ for some linear map $f_u : \mathfrak{g}_{-1} \to \mathfrak{g}_0$. The first differential in the cochain complex of $\mathfrak{g}_{-1}$ with coefficients in ad $\mathfrak{g}$ restricts on $\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$ to the following G_0-equivariant linear map:

$$\partial_1 : \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0 \to \Lambda^2\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1} \qquad (\partial_1 f)(u,v) = f(u)v - f(v)u \tag{2}$$

For all u,

$$T^{\hat{\gamma}}_u - T^\gamma_u = (\partial_1 f_u) \circ \theta_u. \tag{3}$$

The image of ∂_1 determines a smooth subbundle $\mathcal{A} \subset \Lambda^2 T^*M \otimes TM$. The projection of T^γ to $(\Lambda^2 T^*M \otimes TM)/\mathcal{A}$ is thus independent of the choice of connection (see [11, Secs 3.1.10–3.1.13]). This invariant of the almost Grassmannian structure is called the *intrinsic torsion.*

For $\mathrm{Gr}(m,n)$ it is easy to see that locally there always are torsion-free connections preserving the structure, so the intrinsic torsion of the homogeneous model vanishes identically. Then nonzero intrinsic torsion is an obstruction to local isomorphism of a given almost Grassmannian structure to $\mathrm{Gr}(m,n)$. For a C^1-structure, it is an obstruction to local C^1-isomorphism to $\mathrm{Gr}(m,n)$ (for which the corresponding map between the underlying manifolds would be a local C^2-diffeomorphism).

We now explicitly describe the subbundle $\mathcal{A} \subset \Lambda^2 T^*M \otimes TM$ as a $\mathfrak{g}_0$ representation when $m = 2$. Recall that the representation corresponding to the tangent bundle TM is $\mathfrak{g}_{-1} \cong \mathbf{R}^{2*} \boxtimes \mathbf{R}^n$, where the exterior tensor product corresponds to the direct sum decomposition of $\mathfrak{g}_0$. Next we have the decomposition into irreducible components

$$\Lambda^2(\mathfrak{g}^*_{-1}) \cong (\Lambda^2\mathbf{R}^2 \boxtimes S^2\mathbf{R}^{n*}) \oplus (S^2\mathbf{R}^2 \boxtimes \Lambda^2\mathbf{R}^{n*}). \tag{4}$$

We tensor these with $\mathfrak{g}_{-1}$ and decompose into irreducibles. For $k \geq 2$, the representation $S^2\mathbf{R}^k \otimes \mathbf{R}^{k*}$ splits into a trace-free component, denoted $(S^2\mathbf{R}^k \otimes \mathbf{R}^{k*})_0$, and a trace component, isomorphic to $\mathbf{R}^k$, and similarly for the dual. There is an analogous decomposition of $\Lambda^2\mathbf{R}^k \otimes \mathbf{R}^{k^*}$, but here the trace-free part is trivial when $k = 2$.

For $m = n = 2$, the map ∂_1 from (2) is surjective, so no intrinsic torsion is available. In our case when $m = 2$, $n > 2$,

$$(\Lambda^2\mathbf{R}^2 \otimes \mathbf{R}^{2*}) \boxtimes (S^2\mathbf{R}^{n*} \otimes \mathbf{R}^n) \subset \mathrm{Im}(\partial_1).$$

The intersection of $\mathrm{Im}(\partial_1)$ with the other irreducible components of $\Lambda^2(\mathfrak{g}_{-1}^*)\otimes \mathfrak{g}_{-1}$ is the trace component, which can be written

$$\mathbf{R}^2 \boxtimes (\Lambda^2\mathbf{R}^{n*}\otimes\mathbf{R}^n) + (S^2\mathbf{R}^2\otimes\mathbf{R}^{2*})\boxtimes\mathbf{R}^{n*}, \tag{5}$$

where the factors $\mathbf{R}^2$ and $\mathbf{R}^{n*}$ are embedded via a tensor product with Id followed by a symmetrization and an alternation, respectively. Hence the harmonic torsion corresponds to a section of the bundle associated to the remaining irreducible component

$$\mathbb{T} = (S^2\mathbf{R}^2\otimes\mathbf{R}^{2*})_0\boxtimes(\Lambda^2\mathbf{R}^{n^*}\otimes\mathbf{R}^n)_0.$$

To verify non-vanishing harmonic torsion in the example we are going to construct, we need the following result.

Lemma 2.3. *Let $\xi,\eta\in Hom(\mathbf{R}^2,\mathbf{R}^n)$ both have kernel spanned by $0\neq v\in\mathbf{R}^2$, and let T belong to the trace component of $\Lambda^2(\mathbf{R}^2\boxtimes\mathbf{R}^{n*})\otimes(\mathbf{R}^{2*}\boxtimes\mathbf{R}^n)$. Then $T(\xi,\eta):\mathbf{R}^2\to\mathbf{R}^n$ maps v into the span of the images of ξ and η.*

Proof: Take $0\neq\alpha\in\mathbf{R}^{2*}$ with $\alpha(v)=0$, so we can write $\xi=\alpha\otimes w_1$ and $\eta=\alpha\otimes w_2$ for elements $w_1,w_2\in\mathbf{R}^n$. Let $T=T_1+T_2$ be the decomposition of T corresponding to (5) as T_1+T_2 (in a non-unique way). Given $\tilde{v}\in\mathbf{R}^2$, embedded in the trace component of $S^2\mathbf{R}^2\otimes\mathbf{R}^{2*}\cong\mathrm{Hom}(\mathbf{R}^2,S^2\mathbf{R}^2)$, it sends v to a multiple of the symmetric product $v\odot\tilde{v}$. Since $\xi(v)=\eta(v)=0$, we conclude that $T_1(\xi,\eta)(v)=0$. On the other hand, $\tau\in\mathbf{R}^{n^*}$, embedded in the trace component of $\Lambda^2\mathbf{R}^{n*}\otimes\mathbf{R}^n\cong\mathrm{Hom}(\Lambda^2\mathbf{R}^n,\mathbf{R}^n)$, sends (w_1,w_2) to a multiple of $\tau(w_1)w_2-\tau(w_2)w_1$, so all values of $T_2(\xi,\eta)$ belong to the span of the images of ξ and η. The desired conclusion follows. $\diamondsuit$

2.5. **Deformations of almost-Grassmannian structures.** Given an almost-Grassmannian structure with $\theta: TM\xrightarrow{\sim}\mathcal{E}^*\otimes\mathcal{F}$, we will construct deformations by post-composing with a continuous family $\{\Phi_t\}$ of linear automorphisms of $\mathcal{E}^*\otimes\mathcal{F}$. To construct this family, we will first construct endomorphisms, that is, a section Φ of $\mathrm{End}\ \mathcal{E}^*\otimes\mathrm{End}\ \mathcal{F}$ and then show that this exponentiates to a one-parameter family of automorphisms.

At a given point $x\in M$, write $\mathcal{E}_x\cong E$ and $\mathcal{F}_x\cong F$. The vector space automorphisms of $E^*\otimes F$ respecting the tensor product are those of the form $\Psi_{E^*}\otimes\Psi_F$, for $\Psi_{E^*}\in\mathrm{Aut}\ E^*$ and $\Psi_F\in\mathrm{Aut}\ F$. Given a one-parameter group of such automorphisms $\Psi^t_{E^*}\otimes\Psi^t_F$, the generating endomorphism has the form $\psi_{E^*}\otimes\mathrm{Id}_F+\mathrm{Id}_{E*}\otimes\psi_F$ for $\psi_{E*}\in\mathrm{End}\ E^*$ and $\psi_F\in\mathrm{End}\ F$. The condition $\det\Psi^t_{E^*}\cdot\det\Psi^t_F\equiv 1$ is equivalent to $\mathrm{tr}\ \psi_{E*}+\mathrm{tr}\ \psi_F=0$.

The sections of $\mathrm{Aut}(\mathcal{E}^* \otimes \mathcal{F})$ arising from automorphisms of the almost-Grassmannian structure are those of the form $\Psi_{\mathcal{E}^*} \otimes \Psi_{\mathcal{F}}$, for $\Psi_{\mathcal{E}^*} \in \mathrm{Aut}\ \mathcal{E}^*$ and $\Psi_{\mathcal{F}} \in \mathrm{Aut}\ \mathcal{F}$, with $\nu \circ \Lambda^m \Psi_{\mathcal{E}^*} = \Lambda^n \Psi_{\mathcal{F}} \circ \nu$. The generator of a nontrivial deformation is thus nontrivial modulo $\mathrm{End}\ \mathcal{E}^* \otimes \mathrm{Id}_{\mathcal{F}} + \mathrm{Id}_{\mathcal{E}^*} \otimes \mathrm{End}\ \mathcal{F}$. A pointwise complementary subbundle is given by the tensor product of trace-free endomorphisms $\mathrm{End}_0\ \mathcal{E}^* \otimes \mathrm{End}_0\ \mathcal{F}$. We will construct a section of this bundle in Sections 4.1 and 4.2 below.

The results of [10] apply to almost-Grassmannian structures of sufficient regularity to define a Cartan connection (see Section 2.6). Here infinitesimal automorphisms and deformations are described as the kernel and cokernel, respectively, of BGG operators acting on sections of the adjoint tractor bundle, with a certain "twisted" linear connection. The infinitesimal change of harmonic torsion and harmonic curvature (for the latter, see Section 2.6) resulting from a given infinitesimal deformation can also be described in general from these operators and this connection. This point of view was the inspiration for the concrete deformations we construct below.

2.6. **Prolongation and the canonical Cartan connection.** We will verify in Section 4.4 that the results of [2] apply to C^k almost-Grassmannian structures with $k \geq 2$, so any example of this regularity admitting a strongly essential flow by automorphisms has vanishing harmonic curvature on an open set containing the higher-order fixed point in its closure. We explain in this section that $(2,n)$-almost Grassmanian structures of regularity C^k with $k \geq 2$ determine a canonical C^0 Cartan connection as well as C^0 harmonic curvature. In low regularity, general existence results do not apply, so we briefly sketch the explicit constructions, following [12].

2.6.1. *Construction of the prolongation.* Given a C^k almost Grassmannian structure $(p_0 : \mathcal{G}_0 \to M, \theta)$ of type $(2,n)$, we will prolong $\mathcal{G}_0$ to a C^{k-1} principal P-bundle $\mathcal{G} \to M$. To this end, we view $\mathfrak{g}_0$ as a subalgebra of $\mathrm{End}\ \mathfrak{g}_{-1}$. This kernel of the codifferential from (2) is the subspace $\ker(\partial_1)$ of $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0 \subset \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$ of elements symmetric in $\mathfrak{g}_{-1}^*$, which is precisely the *first prolongation* of $\mathfrak{g}_0$ (see [6, I.1]). The kernel is in fact a Lie algebra, isomorphic to $\mathfrak{g}_1$, embedded into $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$ via the adjoint representation.

The bundle $\mathcal{G}$ is constructed as a $\mathfrak{g}_1$-bundle over $\mathcal{G}_0$. Note that $P \cong G_0 \ltimes \mathfrak{g}_1$. Given $u_0 \in \mathcal{G}_0$, denote by $V_{u_0}\mathcal{G}_0 \subset T_{u_0}\mathcal{G}_0$ the vertical subspace, so θ_{u_0} defines a linear isomorphism $T_{u_0}\mathcal{G}_0 / V_{u_0}\mathcal{G}_0 \to \mathbf{R}^{2n}$. Recall from Section 2.4 that (1) depends at u_0 only on the value γ_{u_0} of a chosen principal connection. Let $u : T_{u_0}\mathcal{G}_0 \to \mathfrak{g}_0$ be any linear map recognizing fundamental vector fields—that is $u(\zeta_A(u_0)) = A \in \mathfrak{g}_0$ for $\zeta_A(u_0) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 u_0.e^{tA}$. Now $d\theta_{u_0} + [u, \theta_{u_0}]$ vanishes when either input is in $V_{u_0}\mathcal{G}_0$, so it equals $\theta^*_{u_0} T_u$ for a unique map $T_u \in \Lambda^2\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$.

Varying the choice of u yields, as in Section 2.4, the affine subspace $T_u + \mathrm{Im}(\partial_1)$; moreover, the trace-free subspace $\mathbb{T} = (S^2\mathbf{R}^{2*} \otimes \mathbf{R}^2)_0 \boxtimes (\Lambda^2\mathbf{R}^n \otimes \mathbf{R}^{n*})_0$ is a G_0-invariant complement to $\mathrm{Im}(\partial_1)$. Thus for each point $u_0 \in \mathcal{G}_0$, the linear map u can be chosen such that T_u is totally trace-free. For a fixed u_0 and $T \in \mathbb{T}$, the space of linear maps u giving rise via θ_{u_0} to T is, according to (3), an affine space modeled on $\ker \partial_1 \cong \mathfrak{g}_1$. Explicitly, any two such maps differ according to $\hat{u} - u = \mathrm{ad}\, Z \circ \theta_{u_0} \in \mathfrak{g}_0$, for a unique element $Z \in \mathfrak{g}_1$.

Now $\mathcal{G}$ is the family of maps u as above for which $T_u \in \mathbb{T}$, as u_0 varies over $\mathcal{G}_0$. Denote q the projection $\mathcal{G} \to \mathcal{G}_0$. It can be realized as a subspace of the vector bundle $T^*\mathcal{G}_0 \otimes \mathfrak{g}_0 \to \mathcal{G}_0$ defined as above in terms of θ and $d\theta$, which is C^{k-1}; it follows that $\mathcal{G}$ is a C^{k-1} submanifold here. Elementary representation theory gives a G_0-equivariant linear map $S : \Lambda^2\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1} \to \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$ which vanishes on $\mathbb{T}$ and such that $\partial_1 \circ S$ is the projection to $\mathrm{Im}(\partial_1)$. From a C^{k-1} section of $T^*\mathcal{G}_0 \otimes \mathfrak{g}_0$, one can use S to produce a C^{k-1} principal connection γ on $\mathcal{G}_0$ with $\gamma_{u_0} \in \mathcal{G}$ for all $u_0 \in \mathcal{G}_0$; these correspond to local C^{k-1} sections of q. Now $q : \mathcal{G} \to \mathcal{G}_0$ is a C^{k-1} principal P_+-bundle, and $p := p_0 \circ q : \mathcal{G} \to M$ is a C^{k-1} principal P-bundle (see [12] for more details).

2.6.2. *Harmonic curvature and Cartan connection.* Construction of the Cartan connection on $\mathcal{G}$ entails, by analogy with the prolongation process of the previous section, finding canonical $\mathfrak{g}_1$-valued one-forms on $\mathcal{G}$, which turn out to be unique. There are tautological forms $\theta_{-1} + \theta_0$, where $\theta_{-1} := q^*\theta$ and $(\theta_0)_u := q^*u$, viewing $u \in \mathrm{Hom}(T_{q(u)}\mathcal{G}_0, \mathfrak{g}_0)$. It is easy to see that $(\theta_0)_u(\zeta_A) = A$ for all $A \in \mathfrak{g}_0$, and that $\theta_{-1} + \theta_0$ is P-equivariant once $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ is identified with $\mathfrak{g}/\mathfrak{g}_1$.

Assuming that θ is at least C^2, the form θ_0 is at least C^1, so we can form its exterior derivative $d\theta_0 \in \Omega^2(\mathcal{G}, \mathfrak{g}_0)$. Let $\phi : T_u\mathcal{G} \to \mathfrak{g}_1$ be a linear map satisfying $\phi(\zeta_{A+Z}(u)) = Z$ on the fundamental vector fields for $A + Z \in \mathfrak{g}_0 \ltimes \mathfrak{g}_1 \cong \mathfrak{p}$. As before,

$$(d\theta_0)_u + \frac{1}{2}[(\theta_0)_u, (\theta_0)_u] + [\phi, (\theta_{-1})_u] \tag{6}$$

vanishes if either input is in $V_u\mathcal{G}$ (the vertical bundle for $p : \mathcal{G} \to M$), so it equals $(\theta_{-1})^* K_\phi$ for a linear map $K_\phi : \Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_0$.

For another choice $\hat{\phi}$, the difference is $\hat{\phi} - \phi = f \circ (\theta_{-1})_u$ for some linear map $f \in \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_1$. Another grading component of our Lie algebra codifferential, $\partial_2 : \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_1 \to \Lambda^2\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$, is a G_0–equivariant linear map for which $K_{\hat{\phi}} - K_\phi = \partial_2(f)$. Projection modulo the subbundle $\mathcal{B} \subset \Lambda^2 T^*\mathcal{G} \otimes \mathfrak{g}_0$ determined by $\mathrm{Im}(\partial_2)$ yields another invariant: given a local principal connection γ on $\mathcal{G}_0$ for which $\gamma_{u_0} \in \mathcal{G}$ for all u_0 in the domain, and a locally defined $\phi \in \Omega^1(\mathcal{G}, \mathfrak{g}_1)$, smooth of class C^{k-1}, one obtains a C^{k-1} section of $(\Lambda^2 T^*\mathcal{G} \otimes \mathfrak{g}_0)/\mathcal{B}$ called the *harmonic curvature* of the geometry. This can be recovered as a component of the curvature of any adapted connection.

Now, it can be shown that ∂_2 is injective, and that there is a natural G_0–invariant complement $\mathbb{K}$ to $\mathrm{Im}(\partial_2)$. Hence for each $u \in \mathcal{G}$, there is a unique ϕ such that $K_\phi \in \mathbb{K}$. We obtain $\theta_1 \in \Omega^1(\mathcal{G}, \mathfrak{g}_1)$ of class C^{k-2}, and a C^{k-2} Cartan connection $\omega = \theta_{-1} \oplus \theta_0 \oplus \theta_1 \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Note that when $k \geq 3$, the Cartan curvature can be defined by $K = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathcal{G}, \mathfrak{g})$, and the harmonic torsion and the harmonic curvature are components of K. For the homogeneous model, ω is the Maurer-Cartan form, so K vanishes identically, as does the harmonic curvature (see [12] for more details).

Proposition 2.4. *For $k \geq 2$, a C^k almost-Grassmannian structure of type $(2, n)$, $n \geq 3$, on M determines a C^{k-1} principal P-bundle $\mathcal{G} \to M$ equipped with a C^{k-2} Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. A C^k morphism between two such almost-Grassmannian structures canonically lifts to a morphism of the associated Cartan geometries.*

Proof: It remains only to verify the second statement. Let h be a local C^{k+1} diffeomorphism between open subsets of M and $\widetilde{M}$, lifting to a G_0-equivariant local C^k-diffeomorphism $\Phi_0 : \mathcal{G}_0 \to \widetilde{\mathcal{G}}_0$ with $\Phi_0^*\tilde{\theta} = \theta$. Given $u \in q^{-1}(u_0) \subset \mathcal{G}$, let $\Phi(u) := (\Phi_0^{-1})^* u$. It is easy to check that $T_{\Phi(u)} = T_u$, so $\Phi(u) \in (\tilde{q})^{-1}(\Phi_0(u_0)) \subset \tilde{\mathcal{G}}$. This evidently defines a P-equivariant C^{k-1} lift $\Phi : \mathcal{G} \to \tilde{\mathcal{G}}$ of Φ_0. The construction implies that $\Phi^*\tilde{\theta}_i = \theta_i$ for $i = -1, 0$. Then repeating this argument with the harmonic curvature allows us to conclude that $\Phi^*\tilde{\omega} = \omega$, so Φ is a morphism of Cartan geometries. $\diamondsuit$

As a corollary, we note that for a structure of class at least C^2, nonvanishing harmonic curvature is an obstruction to local isomorphism to $\mathrm{Gr}(2, n)$.

3. Description of the strongly essential flow in coordinates

By homogeneity of $\mathrm{Gr}(2,n)$, we may assume the point x_0 in Theorem 1.1 is the standard 2-plane spanned by the first two coordinate vectors in $\mathbf{R}^{2+n}$. The deformation will be constructed on the open subset U comprising the orbit of x_0 under all transformations $\mathrm{Id}+X$, with $X \in \mathrm{Hom}(\mathbf{R}^2, \mathbf{R}^n)$. This set is the domain of an affine chart in the Plücker coordinates on $\mathrm{Gr}(2,n)$, and equals the top cell in the standard Schubert decomposition. Identify U with $\mathrm{Hom}(\mathbf{R}^2, \mathbf{R}^n)$, and represent an element of U in coordinates

$$X = (x_{ij})_{i=1,j=1}^{i=n,j=2}$$

The principal P-bundle $G \to \mathrm{Gr}(2,n) \cong G/P$ restricted to U is smoothly equivalent to the trivial bundle $U \times P$. The quotient bundle G/G_1 restricted to U is $\mathcal{G}_0|_U$, which is smoothly equivalent to $U \times G_0$.

We will use the following explicit trivializations over U of the tautological and anti-tautological bundles, together with their duals. Denote by $\{e_1, \ldots, e_{n+2}\}$ the standard basis of $\mathbf{R}^{2+n}$ with dual basis $\{e^1, \ldots, e^{n+2}\}$. For $j' = 1', 2'$, define a section of $\mathcal{E}|_U$ by $E_{j'}(X) = (\mathrm{Id}+X)e_{j'}$, and let $E^{j'}$ be the sections $e^{j'}$ of $\mathcal{E}^*$. Next let $E_i(X)$ equal the image of e_{i+2} in $\mathbf{R}^{2+n}/X$ for $i = 1, \ldots, n$, and $E^i(X) = e^{i+2} - x_{i1}e^1 - x_{i2}e^2$, which are well-defined on $\mathbf{R}^{2+n}/X$. We will henceforth denote the restrictions of these bundles to U simply by $\mathcal{E}$, $\mathcal{E}^*$, $\mathcal{F}$, and $\mathcal{F}^*$.

In the above trivializations, $\theta \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-1})$ becomes, for $(u,v) \in T_XU \oplus T_gG_0$,

$$\theta_{(X,g)}(u,v) = g^{-1}\left(\sum_{i',j} u_{i'j}E^{i'}(X) \otimes E_j(X)\right)$$

We will henceforth restrict θ to $TU \oplus \{0\}_{\mathrm{Id}}$ and view it as as a vector bundle isomorphism $TU \xrightarrow{\sim} \mathcal{E}^* \otimes \mathcal{F}$.

3.1. **The strongly essential flow.** Let Z be a rank-one element of $\mathfrak{g}_1 \cong \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^2)$. Let $\{z^t\}$ be the one-parameter subgroup of $P < G$ generated by Z; it is just the group of matrices $\{\mathrm{Id} + tZ\} < \mathrm{SL}(2+n, \mathbf{R})$. Theorem 1.1 applies to any strongly essential flow generated by a rank-one element of $\mathfrak{g}_1$. After conjugation in G, we may assume $Z = e^1 \otimes e_{1'}$, where now $\{e^1, \ldots, e^n\}$ is the standard basis of $\mathbf{R}^{n*}$ and $\{e_{1'}, e_{2'}\}$ the standard basis of $\mathbf{R}^2$. Then $\mathrm{im}\, Z = \mathbf{R}e_{1'}$, and $\ker Z = \mathrm{span}\{e_2, \ldots, e_n\}$.

For $X \in U \cong \mathfrak{g}_{-1}$, denote e^X the corresponding lower-triangular unipotent matrix in G. Then compute the image in $U \times P$

$$\begin{aligned} z^t.e^X &= \begin{pmatrix} \mathrm{Id}_2 + tZX & tZ \\ X & \mathrm{Id}_n \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Id}_2 & 0 \\ X(\mathrm{Id}_2 + tZX)^{-1} & \mathrm{Id}_n \end{pmatrix} \begin{pmatrix} \mathrm{Id}_2 + tZX & tZ \\ 0 & \mathrm{Id}_n - X(\mathrm{Id}_2 + tZX)^{-1}tZ \end{pmatrix}, \end{aligned}$$

assuming that $\mathrm{Id}_2 + tZX$ is invertible. A formal inverse is given by the series $\mathrm{Id}_2 - tZX + \cdots + (-t)^k (ZX)^k + \cdots$.

The following two subspaces are fixed by $\{z^t\}$:

$$F_1 = \{X \ : \ XZ = 0\}$$

and

$$F_2 = \{X \ : \ ZX = 0\}$$

The intersection $F_1 \cap F_2$ will be called the *strongly fixed set* and denoted SF. Note that $X \in SF$ if and only if $[X, Z] = 0$ and, in coordinates,

$$SF = \{X \ : \ x_{12} = 0 = x_{i1} \text{ for all } i = 1, \ldots, n\}$$

Let $H_0 = \{X \ : \ x_{11} = 0\}$. For $X \notin H_0$, decompose X as

$$X_f + X_d = \begin{pmatrix} 0 & 0 \\ & x_{22} - \frac{x_{12}}{x_{11}} \cdot x_{21} \\ \vdots & \vdots \\ 0 & x_{n2} - \frac{x_{12}}{x_{11}} \cdot x_{n1} \end{pmatrix} + \begin{pmatrix} x_{11} & \frac{x_{12}}{x_{11}} \cdot x_{11} \\ & \\ \vdots & \vdots \\ x_{n1} & \frac{x_{12}}{x_{11}} \cdot x_{n1} \end{pmatrix}$$

with $X_f \in SF$ and X_d rank 1. If $X \notin H_0$, then $z^t.X = X_f + z^t.X_d$, which equals

$$X_f + \begin{pmatrix} \frac{x_{11}}{1+tx_{11}} & \frac{\frac{x_{12}}{x_{11}} \cdot x_{11}}{1+tx_{11}} \\ \vdots & \vdots \\ \frac{x_{n1}}{1+tx_{11}} & \frac{\frac{x_{12}}{x_{11}} \cdot x_{n1}}{1+tx_{11}} \end{pmatrix} \tag{7}$$

Let $H_+ = \{X \in U \ : \ x_{11} > 0\}$ and $H_- = \{X \in U \ : \ x_{11} < 0\}$. The formula (7) above yields $z^t.X \to X_f$ as $t \to \pm\infty$ for $X \in H_\pm$, respectively.

Note that if $X \in H_0$, then $ZXZ = 0$, and $(\mathrm{Id} + tZX)^{-1} = \mathrm{Id} - tZX$. Then $z^t X = X(\mathrm{Id} - tZX)$ which equals X as t varies if and only if $XZX = 0$; the latter holds only when ZX or $XZ = 0$. We conclude that $F_1 \cup F_2$ equals the fixed set of z^t in U.

3.2. **Action on associated vector bundles.** The matrix in P

$$p_t(X) = \begin{pmatrix} \mathrm{Id}_2 + tZX & tZ \\ 0 & \mathrm{Id}_n - X(\mathrm{Id}_2 + tZX)^{-1}tZ \end{pmatrix}$$

from above encodes the action of z^t on $\mathcal{E}$ and $\mathcal{F}$, and, in turn, on TU. For $X \notin H_0$,

$$p_t(X) = p_t(X_d) = \begin{pmatrix} 1+tx_{11} & tx_{12} & t & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \frac{1}{1+tx_{11}} & & \\ & & \frac{-tx_{21}}{1+tx_{11}} & 1 & \\ & & \vdots & \ddots & \\ & & \frac{-tx_{n1}}{1+tx_{11}} & & 1 \end{pmatrix} \tag{8}$$

For $X \in H_0$, straightforward calculation gives the formula (8), with $x_{11} = 0$. On $\mathcal{E} = (G \times_P \mathbf{R}^2)\big|_U \cong U \times \mathbf{R}^2$ the action of $\{z^t\}$ is

$$(z^t_{\mathcal{E}})_X = \begin{pmatrix} 1+tx_{11} & tx_{12} \\ 0 & 1 \end{pmatrix} \text{ with respect to } \{E_{1'}, E_{2'}\} \tag{9}$$

and on $\mathcal{E}^*$,

$$(z^{-t}_{\mathcal{E}})^*_{z^t.X} = \begin{pmatrix} \frac{1}{1+tx_{11}} & \frac{-tx_{12}}{1+tx_{11}} \\ 0 & 1 \end{pmatrix} \text{ with respect to } \{E^{1'}, E^{2'}\} \tag{10}$$

On $\mathcal{F}$, the action is

$$(z^t_{\mathcal{F}})_X = \begin{pmatrix} \frac{1}{1+tx_{11}} & & & \\ \frac{-tx_{21}}{1+tx_{11}} & 1 & & \\ \vdots & & \ddots & \\ \frac{-tx_{n1}}{1+tx_{11}} & & & 1 \end{pmatrix} \text{ w.r.t. } \{E_1, \ldots, E_n\} \tag{11}$$

and, finally, on $\mathcal{F}^*$,

$$(z^{-t}_{\mathcal{F}})^*_{z^t.X} = \begin{pmatrix} 1+tx_{11} & & & \\ tx_{21} & 1 & & \\ \vdots & & \ddots & \\ tx_{n1} & & & 1 \end{pmatrix} \text{ w.r.t. } \{E^1, \ldots, E^n\} \tag{12}$$

4. The invariant deformation and non-flatness

4.1. **Eigen-sections of associated bundles.** Here we define sections of $\mathcal{E}$ and $\mathcal{F}$ and of the dual bundles, which are invariant by z^t up to multiplication by a function on U. For any X where the decompositions

$$\mathcal{E}_X = \ker X_d \oplus \operatorname{im} Z \qquad \text{and} \qquad \mathcal{F}_X = \ker Z \oplus \operatorname{im} X_d \tag{13}$$

are valid, each section will have values in one factor or its dual. The sections together will span the fibers over X in $\mathcal{E}, \mathcal{F}$, or their duals.

Define

$$\begin{aligned} v(X) &= -x_{12}E_{1'} + x_{11}E_{2'} \qquad & \iota(X) &= E_{1'} \\ \tilde{v}(X) &= E^{2'} \qquad & \tilde{\iota}(X) &= x_{11}E^{1'} + x_{12}E^{2'} \end{aligned}$$

These pairs of sections are smooth on U and independent on $U \setminus H_0$. We compute from (9) and (10)

$$\begin{aligned} z^t_{\mathcal{E}}(v(X)) &= (1 + tx_{11}) \cdot v(z^t.X) \qquad & z^t_{\mathcal{E}}(\iota(X)) &= (1 + tx_{11}) \cdot \iota(z^t.X) \\ (z^{-t}_{\mathcal{E}})^*(\tilde{v}(X)) &= \tilde{v}(z^t.X) \qquad & (z^{-t}_{\mathcal{E}})^*(\tilde{\iota}(X)) &= \tilde{\iota}(z^t.X) \end{aligned}$$

Now define sections of $\mathcal{F}$ and $\mathcal{F}^*$ for $i = 2, \dots, n$ by

$$\begin{aligned} w(X) &= x_{11}E_1 + \cdots + x_{n1}E_n \qquad & \kappa_i(X) &= E_i \\ \tilde{w}(X) &= E^1 \qquad & \tilde{\kappa}^i(X) &= x_{11}E^i - x_{i1}E^1 \end{aligned}$$

These sections transform, for $i = 2, \dots, n$, according to (11) and (12) by

$$\begin{aligned} z^t_{\mathcal{F}}(w(X)) &= w(z^t.X) \qquad & z^t_{\mathcal{F}}(\kappa_i(X)) &= \kappa_i(z^t.X) \\ (z^{-t}_{\mathcal{F}})^*(\tilde{w}(X)) &= (1 + tx_{11}) \cdot \tilde{w}(z^t.X) \qquad & (z^{-t}_{\mathcal{F}})^*(\tilde{\kappa}^i(X)) &= (1 + tx_{11}) \cdot \tilde{\kappa}^i(z^t.X) \end{aligned}$$

4.2. **Invariant section of the endomorphism bundle.** Now consider the sections

$$\varphi' = v \otimes \tilde{\iota} \qquad \text{and} \qquad \varphi_i = \tilde{\kappa}^i \otimes w$$

of $\operatorname{End} \mathcal{E}^*$ and $\operatorname{End} \mathcal{F}$, respectively, for $i = 2, \dots, n$. These preserve the decompositions (13). They are each nilpotent endomorphisms of order two for any X: $(\varphi'_X)^2 = 0$ and $((\varphi_i)_X)^2 = 0$ for all i; in particular, they are trace-free.

The tensor $\varphi' \otimes \varphi_i$ is a section of the subbundle $\operatorname{End}_0 \mathcal{E}^* \otimes \operatorname{End}_0 \mathcal{F} \subset \operatorname{End} TU$, as in Section 2.5, corresponding to nontrivial deformations of the structure.

The flow acts on this section by

$$(z^t)_*(\varphi'(X)\otimes\varphi_i(X)) = (1+tx_{11})^2\cdot\varphi'(z^t.X)\otimes\varphi_i(z^t.X)$$

Define $q(X) = x_{12}^2+x_{11}^2+\cdots+x_{n1}^2$. Note that $q(z^t.X) = (1+tx_{11})^{-2}\cdot q(X)$. Then the section

$$\Phi_i = \frac{1}{q}\varphi'\otimes\varphi_i$$

is z^t-invariant. The coefficients of the components of Φ are rational functions in the variables $x_{12}, x_{11}, \ldots, x_{n1}$ with numerator degree four and denominator degree two. They are smooth on $U\backslash SF$ and C^1 on SF, in particular at the origin.

Denote by $E^{i'}_{j'}$ the elementary endomorphism of $\mathcal{E}^*$ sending $E^{j'}$ to $E_{i'}$, and by E^i_j the elementary endomorphism of $\mathcal{F}$ sending E_i to E_j. The coefficients of Φ_i are given by

$$\frac{1}{q(X)}\cdot\left(-x_{11}x_{12}E^{1'}_{1'} - x_{12}^2E^{2'}_{1'} + x_{11}^2E^{1'}_{2'} + x_{11}x_{12}E^{2'}_{2'}\right)$$
$$\otimes\left(\sum_{k=1}^n x_{11}x_{k1}E^i_k - \sum_{k=1}^n x_{i1}x_{k1}E^1_k\right)$$

which expands further, denoting $E^{i'}_{j'}\otimes E^\ell_k$ by $E^{i'\ell}_{j'k}$, as

$$\sum_k \frac{x_{12}x_{11}x_{i1}x_{k1}}{q(X)}E^{1'1}_{1'k} + \sum_k \frac{-x_{12}x_{11}^2x_{k1}}{q(X)}E^{1'i}_{1'k} + \sum_k \frac{x_{12}^2x_{i1}x_{k1}}{q(X)}E^{2'1}_{1'k}$$
$$+\sum_k \frac{-x_{12}^2x_{11}x_{k1}}{q(X)}E^{2'i}_{1'k} + \sum_k \frac{-x_{11}^2x_{i1}x_{k1}}{q(X)}E^{1'1}_{2'k} + \sum_k \frac{x_{11}^3x_{k1}}{q(X)}E^{1'i}_{2'k}$$
$$+\sum_k \frac{-x_{12}x_{11}x_{i1}x_{k1}}{q(X)}E^{2'1}_{2'k} + \sum_k \frac{x_{12}x_{11}^2x_{k1}}{q(X)}E^{2'i}_{2'k}$$

Of course, for any constants $c_2, \ldots, c_n$, the endomorphism field

$$\Phi = \sum_{i=2}^n c^i\Phi_i$$

will be z^t-invariant and C^1.

Proposition 4.1. *Let $\theta : TU \xrightarrow{\sim} Hom(\mathbf{R}^2, \mathbf{R}^n)$ be the flat almost-Grassmannian structure on U. Then for any $\mathbf{c}$, $(Id+\Phi)\circ\theta$ is a $\{z^t\}$-invariant, C^1 almost-Grassmannian structure on U not C^1 equivalent to θ.*

We first show that $(\mathrm{Id}+\Phi)\circ\theta$ is an almost-Grassmannian structure on U. Recall from above that $(\varphi')^2=0$ and $(\varphi_i)^2=0$ for all $i=2,\dots,n$; note that moreover, $\varphi_i\circ\varphi_j=0$ for all $i,j=2,\dots,n$, so that $c_2\varphi_2+\cdots+c_n\varphi_n$ is also nilpotent of order two for any nonzero $\mathbf{c}$. It follows that Φ_X is a nilpotent endomorphism of $(\mathcal{E}^*\otimes\mathcal{F})_X$ of order two for all $X\in U$, so the matrix exponential of Φ_X in $\mathrm{SL}(\mathcal{E}^*\otimes\mathcal{F})_X$ is simply $\mathrm{Id}+\Phi_X$. We conclude that $\mathrm{Id}+\Phi_X$ is an isomorphism for all $X\in U$, so $(\mathrm{Id}+\Phi)\circ\theta$ is an almost-Grassmannian structure on U for all $\mathbf{c}$. The $\{z^t\}$-invariance holds by construction.

The derivatives of the rational coefficients in Φ are undefined on SF, the zero set of q. The numerators are homogeneous polynomials in $x_{12},x_{11},\dots,x_{n1}$ of degree five, with denominators all equal q^2. Such functions extend continuously to 0 on SF.

The final claim of the proposition is proved in the following section.

4.3. **Calculation of nonzero harmonic torsion terms.** Recall from Section 2.4 that the harmonic torsion can be computed from any principal connection $\gamma\in\Omega^1(\mathcal{G}_0,\mathfrak{g}_0)$. Such a connection determines a connection on $\mathcal{E}^*\otimes\mathcal{F}$ of the form $\nabla_{\mathcal{E}^*}\otimes\mathrm{Id}_{\mathcal{F}}+\mathrm{Id}_{\mathcal{E}^*}\otimes\nabla_{\mathcal{F}}$. Given an isomorphism $\Psi: TU\xrightarrow{\sim}\mathcal{E}^*\otimes\mathcal{F}$, conjugation by Ψ gives a connection which can be used to compute the harmonic torsion directly on TU, provided Ψ is at least C^1; namely, we compute the usual torsion on TU, conjugate by Ψ^{-1}, and, according to Section 2.4, project to the bundle associated to $(S^2\mathbf{R}^2\otimes\mathbf{R}^{2*})_0\boxtimes(\Lambda^2\mathbf{R}^{n*}\otimes\mathbf{R}^n)_0$. Finally, we conjugate by Ψ back to TU—but nonvanishing of either tensor is an obstruction to C^1 flatness.

As $\mathcal{E}^*$ and $\mathcal{F}$ are trivial bundles over U, we choose the trivial connections on each to obtain a connection ∇ on TU via our deformed almost-Grassmannian structure corresponding to $\mathrm{Id}+\Phi$. There is a framing of TU by parallel vector fields; the torsion will be given by the brackets of these.

Let, for $s>1$,

$$\tilde{E}^{2'}_s=(\mathrm{Id}+\Phi)^{-1}(E^{2'}_s)=E^{2'}_s-c_s\cdot\sum_{p'=1,k=1}^{p'=2,k=n}\frac{x_{1p'}x_{11}^2x_{k1}}{q(X)}E^{p'}_k$$

and let

$$\tilde{E}^{2'}_1=(\mathrm{Id}+\Phi)^{-1}(E^{2'}_1)=E^{2'}_1+\sum_{p'=1,k=1,i=2}^{p'=2,k=n,i=n}c_i\cdot\frac{x_{1p'}x_{11}x_{k1}x_{i1}}{q(X)}E^{p'}_k$$

Each of these vector fields is parallel for ∇ on U. Then the torsion is

$$\tilde{T}(\tilde{E}^{2'}_s, \tilde{E}^{2'}_1) = [\tilde{E}^{2'}_s, \tilde{E}^{2'}_1] = \frac{c_s x_{11}^2}{q(X)} \cdot \sum_k \left(x_{k1} E^{2'}_k - \frac{2x_{k1}x_{12}}{q(X)} \sum_{p'} x_{1p'} E^{p'}_k \right)$$

Next we must apply $\mathrm{Id} + \Phi$ to $\tilde{E} = \tilde{T}(\tilde{E}^{2'}_s, \tilde{E}^{2'}_1)$. This gives, in order of increasing *net degree*—degree of numerator minus degree of denominator,

$$\begin{aligned} T^{s1}_{2'2'} := (\mathrm{Id} + \Phi)\tilde{E} \quad &= \quad \frac{c_s x_{11}^2}{q(X)} \cdot \sum_k \left(x_{k1} E^{2'}_k - \frac{2x_{k1}x_{12}}{q(X)} (x_{11} E^{1'}_k + x_{12} E^{2'}_k) \right. \\ &+ \quad \left. x_{k1} \Phi(E^{2'}_k) - \frac{2x_{k1}x_{12}}{q(X)} \left((x_{11}\Phi(E^{1'}_k) + x_{12}\Phi(E^{2'}_k) \right) \right) \end{aligned}$$

This is a continuous vector field on U. At each $X \in U$, it corresponds to a component of $T_X := (\mathrm{Id} + \Phi_X) \circ \tilde{T}_X \circ \Lambda^2(\mathrm{Id} + \Phi_X)^{-1}$ belonging to $(\mathrm{Sym}^2\mathbf{R}^2 \otimes \mathbf{R}^{2*}) \boxtimes (\Lambda^2\mathbf{R}^{n*} \otimes \mathbf{R}^n)$; as such, its projection to $\mathbb{T}$ is nontrivial provided it does not belong to the trace component identified in Section 2.4 formula (5). Observe that $E^{2'}_1$ and $E^{2'}_s$ share the kernel $\mathbf{R}E_{1'}$. By Lemma 2.3, T_X has nontrivial projection on $\mathbb{T}$ if $(T^{s1}_{2'2'})_X(E_{1'})$ is nontrivial modulo $\mathrm{span}\{\mathrm{Im}(E^{2'}_1), \mathrm{Im}(E^{2'}_s)\} = \mathrm{span}\{E_1, E_s\}$. Then compute the net-degree one terms

$$T^{s1}_{2'2'}(E_{1'}) = \frac{c_s x_{11}^2}{q(X)} \cdot \sum_k \frac{-2x_{k1}x_{12}x_{11}}{q(X)} E_k + \text{ higher-order terms}$$

where the higher-order terms have net degree three. The result thus has nontrivial projection modulo $\mathrm{span}\{E_1, E_s\}$ on an open, dense subset of any neighborhood of 0 in U, provided $c_s \neq 0$.

We conclude that the harmonic torsion of the deformed structure given by $\mathrm{Id} + \Phi$ is nontrivial on an open, dense subset of any neighborhood of 0. This structure is thus inequivalent to $\mathrm{Gr}(2, n)$ on any open subset containing 0 in its closure.

4.4. **Vanishing of harmonic curvature for C^2 structures.** Let $(\mathcal{G}_0 \to M, \theta)$ be a $(2, n)$ almost-Grassmannian structure of regularity C^2 admitting a strongly essential flow $\{z^t\}$ with higher-order fixed point x_0. In closing, we note that the proof of [2, Thm 3.1] applies to such a structure, so it has vanishing harmonic curvature on an open set containing the higher order fixed point in its closure.

Let $(p : \mathcal{G} \to M, \omega)$ be the C^1 prinicpal P-bundle and C^0 Cartan connection, respectively, given by Proposition 2.4. There is a C^1 exponential map exp :

$\mathcal{G} \times \mathfrak{g} \to \mathcal{G}$, giving C^1 exponential curves $\tilde{\gamma}_X(s) = \exp(u, sX)$ as in [2, Def 1.4]. This differentiability is sufficient to apply all of the holonomy calculations of [2, Sec 2].

Let ξ be the vector field generating $\{z^t\}$. For any $u \in p^{-1}(x_0)$, the value $\omega_u(\xi) \in \mathfrak{g}_1$ ([2, Sec 1.2]); let Z be the value for a particular choice of u. The rank of Z as an element of $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^2)$ can be two or one. In either case, Z defines a subset $\mathcal{T}(X) \subset \mathfrak{g}_{-1}$ comprising elements generating an $\mathfrak{sl}_2$-triple $\{X, A = [Z, X], Z\}$ (see [2, Def 2.11]). Along exponential curves $\tilde{\gamma}_X$ for $X \in \mathcal{T}(Z)$, the harmonic curvature and harmonic torsion must belong to the *stable subspaces* $\mathbb{K}^{[st]}$ and $\mathbb{T}^{[st]}$, respectively, determined by A ([2, Def 2.13]). This restriction appears in Corollary 2.14 (1) of [2], which in turn rests on Proposition 2.9 of the same; the required property here is that the harmonic curvature and torsion are given by *continuous* P-equivariant functions on $\mathcal{G}$.

When rk $Z = 2$, then $|\mathcal{T}(Z)| = 1$, and $\mathbb{K}^{[st]} = 0$. The harmonic curvature vanishes not only along the curve $\gamma_X = p \circ \tilde{\gamma}_X$, but on a neighborhood of $\gamma_X \backslash \{x_0\}$, as given by [2, Prop 3.3].

When rk $Z = 1$, as in the examples constructed above, then the *strongly stable subspace* $\mathbb{K}^{[ss]}$ determined by A is trivial. Together with other purely algebraic features of the $\mathfrak{sl}_2$-triple and its representation on $\mathbb{K}$, this property suffices to again prove vanishing of the harmonic curvature on a neighborhood of $\gamma_X \backslash \{x_0\}$, as shown in [2, Prop 3.5 (4a)].

The proof of [8, Thm 1.3] that $(2, n)$-almost-Grassmannian manifolds with strongly essential automorphisms have vanishing harmonic torsion on an open set containing the higher-order fixed point in its closure is more involved, and requires higher regularity; in particular, the full Cartan curvature and a *fundamental derivative* (see [8, Sec 3.4]) of it must be continuous. Thus it remains uncler whether a deformation with the properties in the conclusion of Theorem 1.1 could have higher regularity than C^1.

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