THE LORENTZIAN LICHNEROWICZ CONJECTURE FOR REAL-ANALYTIC, THREE-DIMENSIONAL MANIFOLDS

CHARLES FRANCES AND KARIN MELNICK

Contents

1. Introduction

The Lichnerowicz Conjecture in conformal Riemannian geometry was proved simultaneously by J. Ferrand and M. Obata. Recall that the conformal transformations of a semi-Riemannian manifold (M, g) form the group

$$\mathrm{Conf}(M, [g]) = \{f \in \mathrm{Diff}(M) \ : \ f^* g = e^{2\lambda} g, \ \lambda \in C^\infty(M)\}$$

and that this is a Lie group provided $\dim M \geq 3$. A subgroup $H \leq \mathrm{Conf}(M, [g])$ is called *essential* if it does not act isometrically with respect any $g' = e^{2\lambda} g$ in the conformal class $[g]$ of g. The identity component of a Lie group H is denoted by H^0.

Theorem 1.1 (Ferrand '71 [26]/ Obata '71 [33])**.** *Let (M, g) be a compact, Riemannian manifold with dimension $n \geq 2$. If $Conf^0(M, [g])$ is essential, then (M, g) is conformally diffeomorphic to the round sphere $\mathbf{S}^n$.*

The second author was partially supported by a Joan and Joseph Birman Fellowship for Women Scientists and NSF Award DMS-2109347.

The first attempt to characterize the round sphere by this property of its conformal group seems to have been by A. Lichnerowicz [28] around 1964. Ferrand actually proved a stronger result for all $n \geq 2$, that the above conclusion holds whenever $\mathrm{Conf}(M, [g])$ is essential. (In dimension 2, the theorem is a straightforward consequence of the uniformization theorem for Riemann surfaces.) She later proved a statement for noncompact M in [27]. The reader will find in [12] a nice account by her of the subject.

Obata's proof is based on techniques from differential geometry and transformation groups, while Ferrand's is based on quasiconformal analysis. Two more, totally different proofs, also covering the noncompact case, were given in 1995 by Schoen [37], based on geometric PDEs, in particular, scalar curvature theory, and in 2007 by the first author [15], using Cartan connections and dynamical techniques.

The question whether there is a higher-signature analogue of theorem 1.1 has been around for about thirty years (see [9, Sec 6.7]). Note that essentiality of a conformal action on a compact Riemannian manifold is equivalent to noncompactness of the group. In higher-signature, there is a wide, largely uncharted array of compact pseudo-Riemannian manifolds with noncompact isometry group. Accordingly, there is not such a simple characterization of which conformal groups can act essentially.

Even compact Lorentzian manifolds with essential conformal group occur in a wide variety of global geometries. The first author found infinitely-many topological types of compact manifolds, for each $n \geq 3$, supporting infinitely-many nonequivalent Lorentzian conformal structures admitting an essential conformal flow, in [13]. Locally, however, they are all conformally equivalent to Minkowski space—that is, all known essential Lorentzian examples are *conformally flat*. The conjecture is:

> **Lorentzian Lichnerowicz Conjecture (LLC).** *Let (M^n, g) be a compact Lorentzian manifold with $n \geq 3$. If $\mathrm{Conf}(M, [g])$ is essential, then (M, g) is conformally flat.*

For pseudo-Riemannian metrics of type (p, q) with $p, q \geq 2$, there are rather simple, polynomial deformations g of the flat, (p, q)-Minkowski metric such that a compact quotient of $(\mathbf{R}^n, g)\backslash\{0\}$, diffeomorphic to $S^1 \times S^{n-1}$ for $n = p + q$, is not conformally flat and admits an essential flow [18]. Thus it seems there is no version of the Lichnerowicz Conjecture true in signature higher than Lorentzian.

In this article, we prove the Lichnerowicz Conjecture for 3-dimensional, real-analytic Lorentzian manifolds:

Theorem 1.2. *Let (M, g) be a 3-dimensional, compact, real analytic, Lorentzian manifold. If $\mathrm{Conf}^0(M, [g])$ is essential, then (M, g) is conformally flat.*

1.1. **Previous work on the conjecture.** It is nearly understood which connected Lie groups can act conformally and essentially on compact Lorentzian manifolds. The expectation is that any such group admits a local monomorphism into $\mathrm{O}(2, n)$.

Let (M, g) be a compact, pseudo-Riemannian manifold of type (p, q), $p+q \geq 3$, and assume $p \leq q$. Let $H \leq \mathrm{Conf}(M, [g])$ be a connected subgroup.

For H semisimple, Zimmer proved that $\mathrm{rk}_{\mathbf{R}} H \leq p+1$ [43]; moreover, if this rank is attained, then H necessarily acts essentially. Bader–Nevo proved that if H is simple and attains the maximal $\mathbf{R}$-rank, then it is locally isomorphic to $\mathrm{O}(p+1, k+1)$, for $p \leq k \leq q$ [2]. Under the same assumptions, the first author and Zeghib subsequently proved that M is conformally flat, and in fact conformally equivalent to a certain compact, conformally homogeneous model space, up to covering spaces [23]. For g Lorentzian, Pecastaing has shown that if H is noncompact, simple, and essential, then (M, g) is conformally flat [35].

For H nilpotent, the authors proved in [21] that the nilpotence degree of H is at most $2p + 1$, and that, when this maximal degree is attained, (M, g) is conformally flat and again equivalent to the homogeneous model, up to covering spaces. Moreover, if H has the maximal nilpotence degree, it necessarily acts essentially.

A recent result of the second author and Pecastaing [31], supporting the LLC, does not assume any structure on the group, as above, but rather topological properties of the space. The theorem states that the conformal group of a compact, simply connected, analytic Lorentzian manifold is compact. The proof shows that noncompactness of H implies conformal flatness. By D'Ambra's Theorem [8], H noncompact is equivalent to H essential for such spaces. Conformal flatness leads to a contradiction of the simple connectedness assumption. The proof reduces to the case that the group is abelian.

1.2. **Compact three-dimensional Lorentzian manifolds.** One of our motivations for theorem 1.2 was the thorough understanding of isometries of compact, 3-dimensional Lorentzian manifolds. In [40], Zeghib classified all such spaces admitting an unbounded isometric flow. The first author recently improved this classification to all such spaces admitting any noncompact isometry group—including in particular the case where the isometry group is infinite and discrete [19].

There are moreover many useful classifications of homogeneous models for 3-dimensional Lorentzian manifolds (eg, [4, 7, 38, 36]). In [10], Dumitrescu and Zeghib classified all metrically homogeneous Lorentzian spaces $\mathbf{X}$ such that there is a compact, 3-dimensional Lorentzian manifold locally modeled on $\mathbf{X}$, and they proved that these are all complete.

1.3. **Overview of proof.** The proof rests on the approach we have developed to conformal Lorentzian transformations in our previous papers, which in turn is based on techniques involving the Cartan connection associated to a conformal structure and on Gromov's results on automorphisms of rigid geometric structures. We moreover draw on some of the work specific to 3-dimensional Lorentzian manifolds referenced above as well as the recent advance in [31], from which we draw two major parts of our proof.

In section 2, we use Zeghib's classification of unbounded 3-dimensional Lorentzian flows [40] to show that (M, g) has an essential conformal vector field. Denoting such a vector field by X, we ultimately focus on the Lie algebra $\mathfrak{z}_X$ of local conformal vector fields commuting with X.

In section 4 we gather local results yielding conformal flatness, based on our previous work and Gromov's theory, applied in this 3-dimensional, analytic context. These are used throughout the paper, and they immediately imply that the dimension of $\mathfrak{z}_X$ is at most 4.

The remainder of the paper comprises four more or less distinct proofs, for each of the cases, $\dim \mathfrak{z}_X$ equals 4, 3, 2, or 1. The case $\dim \mathfrak{z}_X = 4$ corresponds to $(M, [g])$ being locally conformally homogenous, which quickly leads to the conclusion that it is conformally flat, or X is inessential, a contradiction.

When $\dim \mathfrak{z}_X = 3$, it can be $\mathbf{R}^3$, $\mathfrak{heis}(3)$, or $\mathfrak{aff}(\mathbf{R}) \oplus \mathbf{R}$. In the case of $\mathfrak{heis}(3)$, we explicitly find a coordinate chart exhibiting g as conformally flat. For $\mathfrak{aff}(\mathbf{R}) \oplus \mathbf{R}$, we find a complete $(G, \mathbf{X})$-structure on a closed, invariant surface, and use this to show that the flow along X on this surface gives rise to conformal flatness.

When $\dim \mathfrak{z}_X = 2$, it is isomorphic to $\mathbf{R}^2$. We show in section 7.2 that $\mathfrak{z}_X$ globalizes and integrates to a cylinder action on M. Then the situation strongly resembles that of [31, Sec 6]; in the remainder of section 7, we follow the outline of that proof to reach the desired conclusion.

Finally, when $\dim \mathfrak{z}_X = 1$, we use fixed points of the flow along X, guaranteed by Gromov's theory, to alternately reach a contradiction or conclude conformal flatness. The proof in this case follows section 5 of [31].

2. Existence of an essential vector field

The objective of this section is to prove that, under the hypothesis that $\mathrm{Conf}^0(M, [g])$ is essential, there exists an *essential conformal vector field*, namely, a vector field generating an essential conformal flow. We denote by $\mathcal{X}^{conf}(M)$ the space of all conformal vector fields on M. Our proof will be specific to 3 dimensions, but will not require analyticity.

Proposition 2.1. *Let (M, g) be a compact, smooth, 3-dimensional Lorentzian manifold. If $Conf^0(M, [g])$ is essential, then it contains an essential 1-parameter subgroup. In fact, every 1-parameter subgroup which is not relatively compact is essential.*

Proof: If $\mathrm{Conf}^0(M, [g])$ is essential, then it is necessarily noncompact. Let $\{\varphi_X^t\} < \mathrm{Conf}^0(M, [g])$ be an unbounded 1-parameter group. Suppose, for a contradiction, that $\{\varphi_X^t\}$ is inessential, that is, contained in $\mathrm{Isom}^0(M, g')$ for some $g' \in [g]$. In this case, Zeghib's classification of noncompact Lorentz-isometric flows on compact 3-dimensional manifolds [40, Thm. 2] gives two possibilities for (M, g'):

(1) (M, g') is flat and complete—that is, a compact quotient of Minkowski space.
(2) $M \cong G/\Gamma_\rho$ for G a finite cover of $\mathrm{PSL}_2(\mathbf{R})$, and Γ_ρ the image of a uniform lattice $\Gamma < G$ under a homomorphism $\mathrm{Id}_\Gamma \times \rho$ into $G \times G$; the image of ρ is in a 1-parameter hyperbolic or unipotent subgroup $\{h^t\} < G$. The metric g' lifts to a $G \times \{h^t\}$-invariant metric on G.

It is a general fact that the conformal group of a flat, complete, Lorentzian manifold (M, g') is inessential. Indeed, any $f \in \mathrm{Conf}(M)$, can be lifted to a conformal transformation of Minkowski space $\mathbf{R}^{1,2}$, namely, an element of $\mathrm{Sim}(\mathbf{R}^{1,2}) \cong (\mathbf{R}^* \times \mathrm{SO}(1,2)) \ltimes \mathbf{R}^3$. It follows that f is a *homothetic* transformation, one for which the conformal distortion is a constant λ. But a homothety on a compact manifold is necessarily an isometry (consider the formula $\int_M \mathrm{dvol}_{g'} = \int_M \mathrm{dvol}_{f^* g'}$).

To simplify the argument for case (2), we initially assume that $G = \mathrm{SL}_2(\mathbf{R})$. Denote by Λ the kernel of $\rho : \Gamma \to \{h^t\}$, and by Λ^Z the Zariski closure in G. Note that Λ is not solvable, because Γ, which is commensurable to a surface group, is not solvable. It follows that $\Lambda^Z = G$. Denote by $\tilde{M}$ the cover of M diffeomorphic to G. Because G acts isometrically on the left on $\tilde{M}$, it acts linearly on the finite-dimensional vector space V of global conformal Killing fields of $\tilde{M}$. This representation is given by an algebraic homomorphism $\alpha : \mathrm{SL}_2(\mathbf{R}) \to \mathrm{GL(V)}$. Denote by $\tilde{\mathcal{X}}^{conf}(\tilde{M})$ the subspace of V comprising lifts of vector fields in $\mathcal{X}^{conf}(M)$. The restriction of $\alpha(\Lambda)$ to this subspace is trivial, hence the same holds for $\Lambda^Z = G$.

When G is a quotient or a connected finite cover of $\mathrm{SL}_2(\mathbf{R})$, the previous arguments are easily adapted: we lift or project Λ to $\mathrm{SL}_2(\mathbf{R})$, as appropriate. The representation of G on $\tilde{\mathcal{X}}^{conf}(\tilde{M})$ lifts to, or factors through, a representation α of $\mathrm{SL}_2(\mathbf{R})$, for which the subgroup corresponding to Λ is trivial on $\tilde{\mathcal{X}}^{conf}(\tilde{M})$. The same holds for the Zariski closure, hence, this subspace is a trivial summand of α. We conclude that G centralizes $\tilde{\mathcal{X}}^{conf}(\tilde{M})$.

Now G commutes with all lifts of elements $h \in \mathrm{Conf}^0(M)$. Let $\tilde{h}$ be such a lift. Choose a lift $\tilde{x}_0$ of x_0 to $\tilde{M}$, and let $\tilde{h}^* g'_{\tilde{x}_0} = \lambda g'_{\tilde{h}.\tilde{x}_0}$. Given $\tilde{x} \in \tilde{M}$, let

$f \in G$ with $f.\tilde{x}_0 = \tilde{x}$. As f commutes with $\tilde{h}$,

$$\tilde{h}^* g'_{\tilde{x}} = \tilde{h}^* f^* g'_{\tilde{x}_0} = f^* \tilde{h}^* g'_{\tilde{x}_0} = f^* \lambda g'_{\tilde{h}.\tilde{x}_0} = \lambda g'_{f\tilde{h}.\tilde{x}_0} = \lambda g'_{\tilde{h}.\tilde{x}}$$

Thus $\tilde{h}$ is a homothety, and so is h. Because M is compact, h must be an isometry. We conclude that $\mathrm{Conf}^0(M)$ is inessential. $\diamondsuit$

Remark 2.2. *We expect proposition 2.1 to hold in any dimension, but this fact would obviously require a more general proof.*

By proposition 2.1, under the assumptions of theorem 1.2, there is an essential conformal vector field on M. We fix such a vector field and call it X. Because we assume $(M, [g])$ to be real-analytic, so will X be real-analytic. Indeed, the conformal Lorentzian structure can be viewed as an analytic reduction of the frame bundle of M to $\mathrm{CO}(1,2)$; Killing fields are determined by the analytic condition of preserving this reduction when lifted to the frame bundle. In what follows, we will work with the geometric structure defined by the pair $([g], X)$.

3. Local and infinitesimal symmetries of $([g], X)$

The conformal structure $[g]$ on M determines a *rigid geometric structure of algebraic type*, in the sense of Gromov (see [24]). It is also fruitful to consider the canonically associated Cartan geometry, modeled on the 3-dimensional Lorentzian Einstein space $\mathrm{Ein}^{1,2}$. The latter space can be obtained as

$$\mathrm{Ein}^{1,2} = (S^1 \times S^2 / \langle \iota \rangle, [-d\theta^2 \oplus g_{S^2}])$$

where ι is the antipodal map on both factors. It is a conformally homogeneous space $\mathrm{PO}(2,3)/P$, for P the stabilizer of a null line in $\mathbf{R}^{2,3}$. Denote $G = \mathrm{PO}(2,3)$ with corresponding Lie algebra $\mathfrak{g}$. The Cartan geometry comprises (see [39] Ch V, [6] Sec 1.6):

- a principal P-bundle $\pi : \hat{M} \to M$; and
- a Cartan connection $\omega \in \Omega^1(\hat{M}, \mathfrak{g})$ satisfying, for all $\hat{x} \in \hat{M}$,
  (1) $\omega_{\hat{x}} : T_{\hat{x}}\hat{M} \to \mathfrak{g}$ is a linear isomorphism
  (2) $\omega_{\hat{x}.g} \circ R_{g*} = \mathrm{Ad}\ g^{-1} \circ \omega_{\hat{x}} \quad \forall g \in P$
  (3) $\omega\left(\frac{d}{dt}\big|_0 (\hat{x}.e^{tY})\right) \equiv Y \quad \forall\ Y \in \mathfrak{p}$

The pair $([g], X)$, with X as in section 2, is also a rigid geometric structure of algebraic type. It is not quite a Cartan geometry, but rather an *enhanced* Cartan geometry, a notion which was studied in [34, Sec 4.4.1]. It is proved there that the properties of the local orbit structure which we will use are the same for enhanced Cartan geometries as for usual Cartan geometries.

3.1. **Local transformations and vector fields.** The conformal group of M lifts to a group of automorphisms of the Cartan bundle $\hat{M}$ preserving ω. Because ω gives a parallelization of $\hat{M}$, the action of $\mathrm{Conf}(M)$ on $\hat{M}$ is free and proper (see, eg, [25, Thm. I.3.2]).

We will denote by $\mathrm{Conf}^{loc}(M)$ the pseudogroup of local conformal transformations of M. As for global conformal transformations, any element of $\mathrm{Conf}^{loc}(M)$ defined on some open subset $U \subset M$ lifts to an embedding of $\pi^{-1}(U)$ into $\hat{M}$ commuting with the principal P-action and preserving ω. We will work below with the sub-pseudogroup $Z_X^{loc} \subset \mathrm{Conf}^{loc}(M)$ centralizing X, where defined.

Let $\mathrm{Is}^{loc}(x) \subset \mathrm{Conf}^{loc}(M)$ be the stabilizer of a point $x \in M$; it is a group. Any choice of $\hat{x} \in \pi^{-1}(x)$ gives a monomorphism $\iota_{\hat{x}} : \mathrm{Is}^{loc}(x) \to P$, the *isotropy monomorphism with respect to* $\hat{x}$, defined implicitly by

$$h.\hat{x} = \hat{x}.\iota_{\hat{x}}(h)$$

We will denote the image $\hat{I}_{\hat{x}}$. A different choice $\hat{x}' = \hat{x}.p$ gives $\hat{I}_{\hat{x}'} = p^{-1}\hat{I}_{\hat{x}}p$. Denote $\mathrm{Is}_X^{loc}(x)$ the subgroup of $\mathrm{Is}^{loc}(x)$ centralizing X. For $\hat{x} \in \pi^{-1}(x)$, denote $(\hat{I}_X)_{\hat{x}}$ the image of $\mathrm{Is}^{loc}(x)$ under $\iota_{\hat{x}}$.

A theorem of Amores [1] says that on real-analytic manifolds, germs of local conformal vector fields can be uniquely extended along paths. It follows that the algebra of germs of local conformal vector fields defined around a point $x \in M$ is independent of x, up to isomorphism. Moreover, local conformal vector fields on the universal cover of M extend to global ones; note that these may not necessarily be complete.

We will work below with the local conformal vector fields commuting with X. Amores' theorem also implies that these form a well-defined subspace of the local conformal vector fields on M, which we will denote $\mathfrak{z}_X$.

3.2. **Gromov's Frobenius theorem and isotropy.** Under the assumption that M is compact and C^ω, Gromov's Frobenius theorem [24] ensures that, at each point x, a finite number of infinitesimal conditions are sufficient for the production of local conformal transformations at x. In the setting of analytic Cartan geometries, the second author showed that the jets of the curvature κ (see section 4.2 for the definition) provide this sufficient condition [30]. The jet of order i can be captured by a P-equivariant map $D^{(i)}\kappa : \hat{M} \to \mathbb{U}^{(i)}$, where $\mathbb{U}^{(i)}$ is a finite-dimensional vector space derived from the curvature module, on which P acts linearly. For the enhanced Cartan structure $([g], X)$, there are a corresponding curvature κ_X and corresponding P-equivariant maps $D^{(i)}\kappa_X : \hat{M} \to \mathbb{U}_X^{(i)}$ [34, Sec 4.4.1]. In this setting, the Frobenius theorem says:

Theorem 3.1. [30, Prop 3.8], [24, 1.6.C and 1.7.A] *Let (M, g) be a compact, real-analytic, pseudo-Riemannian manifold. Let X be an analytic vector*

field on M. There exists an integer $i_0 \in \mathbf{N}$ such that two points x and y are related by an element of $Conf^{loc}(M)$ if and only if $D^{(i_0)}\kappa(\hat{x}) = D^{(i_0)}\kappa(\hat{y})$ for some $\hat{x} \in \pi^{-1}(x)$ and $\hat{y} \in \pi^{-1}(y)$. The same holds for Z_X^{loc} with κ_X in place of κ.

From theorem 3.1, the group $\hat{I}_{\hat{x}}$ coincides with the stabilizer of $D^{(i_0)}\kappa(\hat{x})$ for the representation of P on $\mathbb{U}^{(i_0)}$; similarly, $(\hat{I}_X)_{\hat{x}}$ is the stabilizer of $D^{(i_0)}\kappa_X(\hat{x})$. This leads to the following result on the structure of isotropy groups (which in fact holds more generally, for real-analytic rigid geometric structures of algebraic type).

Theorem 3.2 (Gromov [24] 3.4.A)**.** *Let (M, g) be a real-analytic pseudo-Riemannian manifold, and let X be a real-analytic vector field. With respect to any $\hat{x} \in \pi^{-1}(x)$, the isotropy images $\hat{I}_{\hat{x}}$ and $(\hat{I}_X)_{\hat{x}}$ are real-algebraic subgroups of P; in particular, $Is^{loc}(x)$ and $Is_X^{loc}(x)$ have finitely-many components.*

3.3. **Local orbit stratification.** In this section, we will focus on the structure of the $\mathrm{Conf}^{loc}(M)$-orbits in M. By the $\mathrm{Conf}^{loc}(M)$-orbit of a point $x \in M$, we mean all points that can be reached from x by applying a finite sequence of local conformal maps; it will be denoted $\mathcal{O}(x)$. The Z_X^{loc}-orbit of the point x is defined analogously, and denoted $\mathcal{O}_X(x)$.

Here are consequences of Gromov's stratification theorem which will be used below (see also [30, Thm 4.1], [34, Thm 4.19]). This theorem stems from the Frobenius theorem 3.1 and properties of orbits for algebraic actions.

Theorem 3.3 (Gromov [24] 3.1.A, 3.2)**.** *Let (M, g) be a compact real-analytic pseudo-Riemannian manifold.*

1. *For all $x \in M$, the orbit $\mathcal{O}(x)$ is a semi-analytic subset of M. It is locally closed and has finitely-many components. The same holds for $\mathcal{O}_X(x)$.*
2. *For all $x \in M$, the closure $\overline{\mathcal{O}(x)}$ is locally connected and contains a closed $Conf^{loc}(M)$-orbit. The analogous properties hold for $\overline{\mathcal{O}_X(x)}$.*

Recall that a subset S of a topological space is *locally closed* if S is open in the closure $\overline{S}$. A set is *semianalytic* if it is locally cut out by finitely many analytic equalities and inequalities; see [3] for properties of these sets. The closure of a semianalytic set is again semianalytic. Local connectedness of semianalytic sets can be found in [3, Cor 2.7].

In a slight abuse of language, we will call the $\mathfrak{z}_X$*-orbit* of a point x the set of points reachable from x by flowing along finitely many local vector fields in $\mathfrak{z}_X$. A consequence of the proof of theorem 3.3 is that the $\mathfrak{z}_X$-orbit of x is the connected component of $\mathcal{O}_X(x)$ containing x. We will implicitly make this identification several times below.

3.4. **Recurrence produces isotropy.** In [20] and [19], the first author combined the Frobenius theorem 3.1 with Poincaré recurrence to produce nontrivial local isotropy for isometric actions. Recall that a *recurrent* point for an unbounded subgroup $H < \mathrm{Conf}(M,[g])$ is $x \in M$ with $h_k.x \to x$ for some unbounded sequence $\{h_k\} \subset H$.

Proposition 3.4. *(compare* [20, Prop. 5.1], [19, Prop. 3.3]*) Let (M,g) be a compact, real-analytic pseudo-Riemannian manifold, and let $\{\varphi_X^t\}$ be a conformal flow, which is not relatively compact. Then at each recurrent point x for $\{\varphi_X^t\}$, the local isotropy $Is_X^{loc}(x)$ has noncompact identity component.*

Proof: Let $x \in M$ be recurrent for $\{\varphi_X^t\}$, and consider $\hat{x} \in \pi^{-1}(x)$. There are $t_k \to \infty$ and $\{p_k\} \subset P$ such that $\varphi_X^{t_k}.\hat{x}.p_k^{-1} \to \hat{x}$. Because $\mathrm{Conf}(M)$ acts properly on $\hat{M}$, the sequence $\{p_k\}$ necessarily tends to infinity. Let $D^{i_0}\kappa_X : \hat{M} \to \mathbb{U}^{(i_0)}$ be the P-equivariant map given by theorem 3.1. This map is also $\mathrm{Conf}(M)$-invariant, so that $p_k.D^{(i_0)}\kappa_X(\hat{x}) \to D^{(i_0)}\kappa_X(\hat{x})$. Now P acts algebraically on $\mathbb{U}^{(i_0)}$ with locally closed orbits (see [42, Thm 3.1.1]), which implies $p_k.D^{(i_0)}\kappa_X(\hat{x}) = \epsilon_k.D^{(i_0)}\kappa_X(\hat{x})$ for some ϵ_k tending to the identity in P. This implies existence of a noncompact stabilizer of $D^{(i_0)}\kappa_X(\hat{x})$, coinciding with $(\hat{I}_X)_x$, again by theorem 3.1. Because this stabilizer is moreover algebraic in P, it has noncompact identity component. $\diamondsuit$

4. Conformal curvature and vanishing conditions

In this section, we gather several sufficient conditions for conformal flatness, which will be applied throughout our proof. Although some definitions and results later in this section will be valid in higher dimensions, we assume for now that (M,g) is a 3-dimensional, smooth Lorentzian manifold.

4.1. **Cotton-York tensor.** Recall that in dimension 3, the Weyl curvature vanishes, and the obstruction to conformal flatness is the Cotton-York tensor $C \in \Gamma(\wedge^2 T^*M \otimes T^*M)$ (see [11, II.28]):

$$C_x(u,v,w) = (\nabla_v P)(u,w) - (\nabla_u P)(v,w)$$

for $u,v,w \in T_xM$, where P is the Schouten tensor

$$P_x(v,w) = \mathrm{Ric}_x(v,w) - \frac{1}{4}\mathrm{Sc}(x)g_x(v,w)$$

and Ric and Sc denote the Ricci and scalar curvatures of g, respectively. The Cotton-York tensor is conformally invariant, meaning it is independent of a choice of metric in the conformal class $[g]$. For $f \in \mathrm{Conf}(M)$, in particular,

$$C_{f(x)}(f_{*x}u, f_{*x}v, f_{*x}w) = C_x(u,v,w) \qquad \forall x \in M,\ u,v,w \in T_xM$$

This tensor moreover satisfies the Bianchi identity, meaning it is in the kernel of the map to $\wedge^3 T^*M$, and is totally trace-free.

The Cotton-York tensor is a section of the vector bundle associated to the following module $\mathbb{U}$. Choose

$$\mathbb{I} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

and write $\mathrm{SO}(1,2)$ for $\mathrm{SO}(\mathbb{I})$. Write E_1, E_2, E_3 for the standard basis of $\mathbf{R}^3$, and take $\xi_i = E_i^t \mathbb{I}$ as basis for $\mathbf{R}^{3*}$, $i = 1, 2, 3$. Denote

$$\mathbb{U} = (\wedge^2 \mathbf{R}^{3*} \otimes \mathbf{R}^{3*})^C$$

the 5-dimensional $\mathrm{SO}(1,2)$-module of trace-free tensors satisfying the Bianchi symmetry. It is the sum of five 1-dimensional weight spaces, for weights $w = -2, -1, 0, 1, 2$. The positive weight spaces are

$$\mathbb{U}^{+2} = \mathbf{R}(\xi^1 \wedge \xi^2 \otimes \xi^1) \qquad \mathbb{U}^{+1} = \mathbf{R}(\xi^1 \wedge \xi^3 \otimes \xi^1 - \xi^1 \wedge \xi^2 \otimes \xi^2)$$

Denote $\mathbb{U}^+ = \mathbb{U}^{+1} + \mathbb{U}^{+2}$. Denote $G_0 = \mathbf{R}^* \times \mathrm{SO}(1,2) = \mathrm{CO}(1,2)$. Letting $d \in \mathbf{R}^*$ act by the scalar d^{-3} extends the representation on $\mathbb{U}$ to G_0.

4.2. **Cartan curvature.** Let $(M, \hat{M}, \omega)$ be the canonical Cartan geometry modeled on $\mathrm{Ein}^{1,2}$ associated to $(M, [g])$, from section 3. Recall the notation $G = \mathrm{PO}(2,3)$.

The *Cartan curvature*

$$\Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]$$

is a semi-basic 2-form on $\hat{M}$. It is the obstruction to $(M, \hat{M}, \omega)$ being locally isomorphic to the model Cartan geometry, which in our case is $(\mathrm{Ein}^{1,2}, G, \omega_G)$, where ω_G is the Maurer-Cartan form of G. The values of Ω lie in the nilpotent radical $\mathfrak{p}^+$ of $\mathfrak{p}$, which is identified as a G_0-representation with $\mathbf{R}^{1,2*}$.

Via the Cartan connection ω, the curvature Ω can be identified with a P-equivariant function

$$\kappa : \hat{M} \to \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}^+$$

where the P-representation on the target is built from the adjoint representation of G restricted to P. It factors through $P/P^+ \cong G_0$. Correspondingly, κ factors through the quotient $\hat{M}/P^+$. As G_0-modules,

$$\wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}^+ \cong \wedge^2 \mathbf{R}^{1,2*} \otimes \mathbf{R}^{1,2*}$$

The Cartan connection identifies T^*M with $\hat{M} \times_P \mathfrak{p}^+$. The factorization of κ, with these identifications of G_0-modules, is the same as C (see [6, Cor 1.6.8]); here we are identifying G_0-equivariant maps $\hat{M}/P^+ \to \mathbb{U}$ with sections of the associated vector bundle, and identifying $\hat{M}/P^+$ with the first-order conformal frame bundle. In particular, κ factors through a G_0-equivariant map to $\mathbb{U}$.

4.3. **Holonomy sequences.** Because M is compact, while local conformal transformations act freely and properly on $\hat{M}$, we can associate to an unbounded sequence in $\mathrm{Conf}^{loc}(M)$ an unbounded sequence in the principal group P.

Definition 4.1. *Let $\hat{x} \in \hat{M}$ and $\{f_k\} \subset Conf^{loc}(M)$, with f_k defined on a neighborhood of $x = \pi(\hat{x})$ for all k. A* holonomy sequence for $\{f_k\}$ at x with respect to $\hat{x}_k \to \hat{x}$, *is $\{p_k\} \subset P$ such that*

$$f_k.\hat{x}_k.p_k^{-1} \to \hat{y} \qquad \text{for some } \hat{y} \in \hat{M}$$

A pointwise holonomy sequence *is $\{p_k\}$ as above for which $\hat{x}_k \in \pi^{-1}(x)$ $\forall k$.*

Note that given a holonomy sequence as above, $f_k.x_k \to y$, where $x_k = \pi(\hat{x}_k), y = \pi(\hat{y})$. Given any sequence $\{f_k\} \subset \mathrm{Conf}^{loc}(M)$ and any $x_k \to x \in M$, we may assume, after passing to a subsequence, that $f_k.x_k$ converges in M, since M is compact. Then there are holonomy sequences, including pointwise holonomy sequences, for $\{f_k\}$ at x.

A conformal transformation is, in dimension at least 3, determined by its 2-jet at a point (see, eg, [25, Sec IV.6]). A holonomy sequence captures the 2-jets of $\{f_k\}$ along the sequence $\{x_k\}$ and thus turns out to be a useful tool to understand the local behavior of $\{f_k\}$ around $\{x_k\}$.

Since the 2-jet of a map can be read in different 2-frames, holonomy sequences are far from unique. Some of the choices involved in their construction are accounted for by vertical equivalence: two holonomy sequences $\{p_k\}$, $\{q_k\}$ are *vertically equivalent* if $q_k = a_k p_k b_k$ for $\{a_k\}, \{b_k\} \subset P$ bounded.

4.3.1. *Taxonomy of holonomy sequences.* The reductive group $G_0 \cong \mathrm{CO}(1,2)$ has a KAK decomposition, in which the $\mathbf{R}$-split Cartan subgroup A is two-dimensional. Under the embedding of G_0 in $G = \mathrm{PO}(2,3)$ as the stabilizer of a pair of nonorthogonal isotropic lines in $\mathbf{R}^{2,3}$, the torus A equals the $\mathbf{R}$-split Cartan subgroup of G. Up to vertical equivalence, a holonomy sequence in P can be written $p_k = d_k \tau_k$, with $d_k \in A$ and $\tau_k \in P^+$.

Denote by $\mathfrak{a}$ the subalgebra corresponding to A. The standard choice of simple roots spanning $\mathfrak{a}^*$ comprises a long root, γ, and a short root, β. The latter vanishes on the $\mathbf{R}$-factor in the decomposition $\mathfrak{g}_0 = \mathbf{R} \oplus \mathfrak{o}(1,2)$, and it can be thought of as the generator of the root space of $\mathfrak{o}(1,2)$. We take $\alpha = \gamma - \beta$; it corresponds to the negative log conformal dilation in the standard representation on $\mathbf{R}^{1,2}$. Explicitly, for the quadratic form $2x_0x_4 + 2x_1x_3 + x_2^2$ on $\mathbf{R}^{2,3}$,

$$\mathfrak{a} = \left\{ \begin{pmatrix} a & & & & \\ & b & & & \\ & & 0 & & \\ & & & -b & \\ & & & & -a \end{pmatrix},\ a, b \in \mathbf{R} \right\}$$

with α and β dual to the a and b parameters, respectively.

The symmetry $(\alpha, \beta) \mapsto (\alpha, -\beta)$ can be realized by conjugation in P, by a compact element. Denote A' the semigroup comprising all $d \in A$ with $\beta(\ln d) \leq 0$. Up to vertical equivalence, a holonomy sequence in P may be assumed to be in $A'P^+$.

Definition 4.2. *Let $\{d_k \tau_k\}$ be an unbounded sequence in $A'P^+$, and let $D_k = \ln d_k$. The sequence is said to be*

- *of* bounded distortion *if $\alpha(D_k)$ is bounded while $\beta(D_k) \to -\infty$*
- contracting *if $(\alpha + \beta)(D_k) \to \infty$.*
- balanced *if $\alpha(D_k) + \beta(D_k)$ is bounded, but each term is unbounded.*
- mixed *if $\beta(D_k) \to -\infty$ and $\alpha(D_k) \to \infty$, while $(\alpha + \beta)(D_k) \to -\infty$.*

It is called linear *if $\tau_k \equiv 1$.*

4.3.2. *Stability and propagation of holonomy.* The following definition is inspired by [41] (see also [9, Sec 7.4], [5, Def 2.10] for a non-approximate version, and [16, Sec 4.4] for a related notion of stability and stable foliations).

Definition 4.3. *Let $\mathbb{V}$ be a P-module, and let $\{p_k\}$ be a sequence in P. The* approximately stable set *for $\{p_k\}$ in $\mathbb{V}$ is*

$$\mathbb{V}^{AS}(p_k) = \{v = \lim v_k \in \mathbb{V} \ :\ p_k.v_k \text{ is bounded}\}$$

The following proposition is a version for sequences of [5, Prop 2.9] (see [31, Prop 3.13] for the one-line proof):

Proposition 4.4. *Given a P-module $\mathbb{V}$, represent a continuous, $\{f_k\}$-invariant section of the associated bundle $\hat{M} \times_P \mathbb{V}$ by a continuous, P-equivariant, $\{f_k\}$-invariant map $\sigma : \hat{M} \to \mathbb{V}$. Given any holonomy sequence $\{p_k\}$ for $\{f_k\}$ with respect to $\hat{x}_k \to \hat{x}$, the value $\sigma(\hat{x}) \in \mathbb{V}^{AS}(p_k)$.*

An unbounded sequence $\{d_k \tau_k\}$ in $A'P^+$ is called *stable* if it is linear and contracting or balanced. In general, holonomy sequences for a given $\{f_k\} \subset \mathrm{Conf}^{loc}(M)$ can be of different types at different nearby points; however, stable sequences enjoy the property of local *propagation of holonomy.*

The exponential map of the Cartan connection is the vehicle for propagation of holonomy. Any $X \in \mathfrak{g}$ defines a vector field $\hat{X}$ on $\hat{M}$ by $\omega(\hat{X}) \equiv X$.

Denote the flow along $\hat{X}$ by $\{\varphi^t_{\hat{X}}\}$. The *exponential map* at $\hat{x} \in \hat{M}$ is

$$\exp_{\hat{x}}(X) = \varphi^1_{\hat{X}}(\hat{x}) \in \hat{M}$$

for X in a sufficiently small neighborhood of 0 in $\mathfrak{g}$. The alternative notation $\exp(\hat{x}, X)$ will also be used below. A holonomy sequence for $\{f_k\}$ at x turns out to be also valid along certain exponential curves from x (see [17, Prop 6.3] or [31, Prop 3.9]):

Proposition 4.5. *Let $\{p_k\}$ be a holonomy sequence for f_k at x, with respect to $\hat{x}_k \in \pi^{-1}(x_k)$. Suppose given $Y_k \to Y \in \mathfrak{g}\backslash\mathfrak{p}$ for which $\operatorname{Ad} p_k(Y_k)$ converges. Then, provided Y is in the domain of $\exp_{\hat{x}}$, $\{p_k\}$ is also a holonomy sequence for $\{f_k\}$ at $x' = \pi \circ \exp_{\hat{x}}(Y)$ with respect to $\hat{x}'_k = \exp_{\hat{x}_k}(Y_k)$.*

One derives easily from proposition 4.5 the following corollary.

Corollary 4.6. [16, Lem 4.3, 4.6] *If $\{f_k\}$ has a stable holonomy sequence $\{p_k\}$ at x, then $\{p_k\}$ is also a holonomy sequence for $\{f_k\}$ on a neighborhood of x.*

4.4. **Stability implies conformal flatness.** We record some immediate consequences of the properties outlined above.

Proposition 4.7. *If there is a balanced or a contracting holonomy sequence at $x \in M$, then the Cotton-York tensor vanishes at x.*

Proof. Let $\{p_k = d_k\tau_k\}$ be a holonomy sequence at x, with $\{d_k\} \subset A'$ satisfying the balanced or contracting condition in definition 4.2. The P-representation on the Cotton module $\mathbb{U}$ factors through the projection to G_0, which has weights $3\alpha + w\beta$, $w = -2, -1, 0, 1, 2$. Now p_k acts by d_k, for which $(3\alpha + w\beta)(\ln d_k) \to \infty$, for all possible w. Thus $\mathbb{U}^{AS}(p_k) = 0$. The conclusion follows from proposition 4.4. □

Assuming (M, g) is not conformally flat, we may thus assume that the set of points admitting a balanced or a contracting holonomy sequence is nowhere dense.

Proposition 4.8. *Assume that (M, g) is real-analytic. If there exists an unbounded sequence $\{f_k\} \subset Conf^{loc}(M)$, all defined on a neighborhood U of $x \in M$, admitting a stable holonomy sequence at x, then (M, g) is conformally flat.*

Proof: A stable, unbounded holonomy sequence $\{p_k = d_k\tau_k\}$ is balanced or contracting. As remarked above, proposition 4.5 implies that all points in a neighborhood $V \subseteq U$ of x admit the same stable holonomy sequence. By proposition 4.7, the Cotton-York tensor vanishes on V (see also [14, Prop

5]). By the analyticity assumption, (M, g) is conformally flat everywhere. $\diamondsuit$

4.5. **Big isotropy implies conformal flatness.** Next we recall a key linearization theorem for conformal vector fields:

Theorem 4.9 (Frances–Melnick [22] Thm 1.2)**.** *Let (M, g) be a real-analytic Lorentzian manifold of dimension $n \geq 3$. Let $x \in M$ and let $X \in \mathcal{X}^{conf}(M)$ vanish at x, with local flow $\{\varphi_X^t\} < Is^{loc}(x)$. If for some $\hat{x} \in \pi^{-1}(x)$, the image $\iota_{\hat{x}}(\{\varphi_X^t\}) < G_0 \cong CO(1, n-1)$, then $\{\varphi_X^t\}$ is linearizable in a neighborhood of x. Otherwise, (M, g) is conformally flat.*

Combining the theorem above with the previous propositions yields two useful corollaries.

Corollary 4.10. *Assume that (M^3, g) is real-analytic. If for some $x \in M$, the isotropy algebra $\mathfrak{Is}^{loc}(x)$ contains a 2-dimensional abelian subalgebra, then (M, g) is conformally flat.*

Proof: Consider the linear part of the isotropy: given $\hat{x} \in \pi^{-1}(x)$, compose the isotropy homomorphism $(\iota_{\hat{x}})_{*e} : \mathfrak{Is}^{loc}(x) \to \mathfrak{p}$ with the projection to $\mathfrak{g}_0$, to obtain $\lambda_{\hat{x}} : \mathfrak{Is}^{loc}(x) \to \mathfrak{g}_0$. Because (M, g) is real analytic, if $\lambda_{\hat{x}}$ is not injective, then (M, g) is conformally flat by theorem 4.9. Assuming $\lambda_{\hat{x}}$ is injective, it has 2-dimensional, abelian image in $\mathfrak{g}_0 \cong \mathbf{R} \oplus \mathfrak{o}(1, 2)$, necessarily containing the center. There is thus a linear, contracting 1-parameter subgroup of $\mathrm{Is}^{loc}(x)$. Proposition 4.8 implies in this case that (M, g) is conformally flat. $\diamondsuit$

Now let $X \in \mathcal{X}^{conf}(M)$ and $x \in M$. Let $Z_X^{loc} \subset \mathrm{Conf}^{loc}(M)$, $\mathfrak{z}_X$, and $\mathrm{Is}_X^{loc}(x)$ be as in section 3.2. The Lie algebra of $\mathrm{Is}_X^{loc}(x)$ is denoted $\mathfrak{Is}_X^{loc}(x)$.

Corollary 4.11. *Assume (M^3, g) is real-analytic. If the dimension of $\mathfrak{Is}_X^{loc}(x)$ is at least 2 for some $x \in M$, then (M, g) is conformally flat.*

Proof: If $X(x) = 0$, namely $X \in \mathfrak{Is}_X^{loc}(x)$, then $\mathfrak{Is}_X^{loc}(x)$ contains a two-dimensional abelian subalgebra, and we conclude by Corollary 4.10.

Otherwise, $\mathfrak{Is}_X^{loc}(x)$ annihilates $X(x) \neq 0$. Again by theorem 4.9, we can assume that the image $(\iota_{\hat{x}})_{*e}(\mathfrak{Is}^{loc}(x))$ is in $\mathfrak{g}_0$ for some $\hat{x} \in \pi^{-1}(x)$. A 0-eigenvector in $\mathbf{R}^{1,2}$ of a 2-dimensional subalgebra of $\mathfrak{g}_0$ is necessarily lightlike, and the annihilator of a lightlike vector has dimension exactly 2. It includes a diagonal subgroup $\{d^t = e^{tD}\}$ with $\alpha(D) = -\beta(D)$; any unbounded sequence $\{d^{t_k}\}$ has balanced, linear holonomy with respect to $\hat{x}$. Proposition 4.8 thus applies and ensures that (M, g) is conformally flat. $\diamondsuit$

4.6. **A new curvature vanishing result.** We will need the following strengthening of [22, Thm 1.4]. The proof is somewhat shorter for 3-dimensional manifolds, so we restrict to that case here.

Theorem 4.12. *Let (M, g) be a real-analytic, compact, 3-dimensional Lorentzian manifold. Let $\{f_k\} \subset Conf(M, [g])$ be an unbounded sequence. If $\{f_k\}$ admits a holonomy sequence at $x \in M$ contained in P^+, then (M, g) is conformally flat.*

Notice that the hypothesis that $\{f_k\}$ admits a holonomy sequence at $x \in M$ contained in P^+ is equivalent to both sequences of differentials $\{D_x f_k\}$ and $\{(D_x f_k)^{-1}\}$ being bounded. The proof of the theorem is somewhat technical, and is deferred to section 9, at the end of the paper.

5. Case $\mathfrak{z}_X$ is 4-dimensional

We begin the proof of the main theorem 1.2. Recall that (M, g) is 3-dimensional, compact, real-analytic, and Lorentzian. The group $\mathrm{Conf}^0(M, [g])$ is assumed to be essential. By proposition 2.1, it admits an essential conformal vector field $X \in \mathcal{X}^{conf}(M)$. Denote by $\mathfrak{z}_X$ the algebra of local conformal vector fields on M commuting with X (see section 3.2). By corollary 4.11, the dimension of $\mathfrak{z}_X$ is at most four, if M is not conformally flat. Of course, $X \in \mathfrak{z}_X$, so it has dimension at least one. We will prove theorem 1.2 by analyzing each possible value of this dimension.

Suppose $\mathfrak{z}_X$ has dimension four and M is not conformally flat. Corollary 4.11 implies that all $\mathfrak{z}_X$-orbits have dimension 3, hence there is only one such orbit. In particular, (M, g) is locally conformally homogeneous and X is nowhere vanishing. Given $x \in M$, the isotropy $\mathrm{Is}_X^{loc}(x)$ fixes the nonzero vector $X(x)$. By theorem 4.9, (M, g) is conformally flat, or there exists $\hat{x} \in \pi^{-1}(x)$ with $(\hat{I}_X)_{\hat{x}} < G_0$. This subgroup is 1-dimensional, and it is algebraic by theorem 3.2; two cases may occur.

If $(\hat{I}_X)_{\hat{x}} < G_0$ does not act by unimodular transformations on $\mathfrak{g}/\mathfrak{p}$, then it is conjugate in G_0 to a diagonal subgroup $\{\mathrm{diag}(1, \lambda, \lambda^2) \mid \lambda \in \mathbf{R}^*\}$. Any unbounded sequence of this group is balanced and linear. Then proposition 4.8 ensures that (M, g) is conformally flat.

Next suppose $(\hat{I}_X)_{\hat{x}}$ acts by unimodular transformations on $\mathfrak{g}/\mathfrak{p}$. Then there is $\lambda_x \in [g]_x$ on $T_x M$ which is preserved by $D_x f$ for every $f \in \mathrm{Is}_X^{loc}(x)$. Given $y \in M$, choose $f \in Z_X^{loc}$ sending x to y, and define $\lambda_y = f_* \lambda_x$. This does not depend on the choice of f, because for another choice, say $h \in Z_X^{loc}$, the difference $h^{-1} \circ f \in \mathrm{Is}_X^{loc}(x)$, which preserves λ_x. The result is a metric $\lambda \in [g]$ which is Z_X^{loc}-invariant. Note that λ is analytic; indeed, given $y \in M$, there are $Y, Z, T \in \mathfrak{z}_X$ defined on a neighborhood of y, with values at y spanning $T_y M$. The map $(u, v, w) \mapsto \varphi_Y^u \circ \varphi_Z^v \circ \varphi_T^w(y)$ provides a local analytic chart

around y, in which λ is analytic. We conclude that $\{\varphi_X^t\}$ is inessential, a contradiction.

6. Case $\mathfrak{z}_X$ is 3-dimensional

The center of $\mathfrak{z}_X$ is nontrivial because it contains X. Thus if $\mathfrak{z}_X$ is 3-dimensional, it could be isomorphic to $\mathbf{R}^3$, $\mathfrak{heis}(3)$, or $\mathfrak{aff}(\mathbf{R}) \oplus \mathbf{R}$. For the sake of efficiency, we will assume for the rest of this section that (M, g) is *not* conformally flat, in order to arrive at a contradiction with the fact that $\{\varphi_X^t\}$ is essential. The results collected thus far lead to:

Proposition 6.1. (1) *The flow $\{\varphi_X^t\}$ has no singularities.*
(2) *All $\mathfrak{z}_X$-orbits have dimension at least two.*
(3) *There is a closed $\mathfrak{z}_X$-orbit Σ, which is a torus or a Klein bottle, on which X is lightlike.*

Proof: If there were a singularity x, the differential $\{D_x\varphi_X^t\}$ at x would fix two linearly independent vectors, the values at x of two elements of $\mathfrak{z}_X$ linearly independent modulo X. From the fact that the differential preserves g_x up to scale, simple linear algebra leads to the conclusion that $D_x\varphi_X^t = \mathrm{Id}_{T_xM}$ for all t, a contradiction with theorem 4.9.

Point (2) follows from corollary 4.11.

If $g(X, X)$ were nonvanishing, then $\{\varphi_X^t\}$ woud preserve $g/g(X, X)$ and be inessential. Thus the zero set Λ of $g(X, X)$ is nonempty and closed. By theorem 3.3 (2), there exists $x \in \Lambda$ such that the $\mathfrak{z}_X$-orbit Σ of x is closed. If Σ is 3-dimensional then it equals M, and the identity component of $\mathrm{Is}_X^{loc}(y)$ is trivial at each $y \in M$. But since M is compact, the flow $\{\varphi_X^t\}$ has recurrent points, which leads to a contradiction with proposition 3.4. Thus Σ must be a closed surface. It is a torus or a Klein bottle because X is nonvanishing. By construction, X is lightlike on Σ. Point (3) is proved. $\diamondsuit$

Arguments follow for each possible isomorphism type of $\mathfrak{z}_X$.

6.1. **Case $\mathfrak{z}_X$ is isomorphic to $\mathbf{R}^3$.** Let Σ be a 2-dimensional orbit, as guaranteed by proposition 6.1 (3), and let $x \in \Sigma$. Because $\mathfrak{z}_X$ is abelian, the isotropy at x fixes two linearly independent vectors, spanning $T_x\Sigma$. As in the proof of proposition 6.1 (1), we have a contradiction with theorem 4.9.

6.2. **Case $\mathfrak{z}_X$ is isomorphic to $\mathfrak{heis}(3)$.** In this case, X generates the center of $\mathfrak{heis}(3)$.

First suppose there exists an open $\mathfrak{z}_X$-orbit Ω. On a sufficiently small open subset $U \subset \Omega$, there is $g_0 \in [g]|_U$ such that (U, g_0) is isometric to an open subset of Heis(3) endowed with a left-invariant Lorentzian metric. Left-invariant Lorentzian metrics on Heis(3) were classified in [36]; there are three isometry types, according to the sign of $\langle X, X\rangle$. If $\langle X, X\rangle = 0$, the

metric is flat. By the analyticity assumption, (M, g) is conformally flat, contradicting our current hypothesis. If $\langle X, X\rangle \neq 0$, the isometry group of the metric on Heis(3) is 4-dimensional and centralizes the center of Heis(3). Then $\mathfrak{z}_X$ has dimension at least 4, contradicting our current assumption that it is 3.

We conclude that the $\mathfrak{z}_X$-orbits are all 2-dimensional. The following proposition, when combined with our analyticity assumption, concludes this case.

Proposition 6.2. *Let (M, g) be a smooth, 3-dimensional, Lorentzian manifold. Suppose there is a nonempty open subset $\Omega \subseteq M$ with a local conformal action of $\mathfrak{heis}(3)$, such that all pseudo-orbits are 2-dimensional. Then (M, g) is conformally flat.*

Proof: Let Y and Z be the further generators of $\mathfrak{heis}(3)$, such that $[X, Y] = [X, Z] = 0$, and $[Y, Z] = X$. Since X and Y commute and span a 2-dimensional space at each point of Ω, there exist local coordinates (x, y, z) in which $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$. Because the $\mathfrak{z}_X$-orbits are 2-dimensional, Z is of the form $\lambda\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial y}$ for some functions λ and μ. The bracket relations lead to

$$0 = \frac{\partial\lambda}{\partial x} = \frac{\partial\mu}{\partial x} = \frac{\partial\mu}{\partial y} \qquad \text{and} \qquad \frac{\partial\lambda}{\partial y} = 1$$

Hence we can write

$$Z = (y + a(z))\frac{\partial}{\partial x} + b(z)\frac{\partial}{\partial y}.$$

Observe that replacing Z by $Z - a(0)X - b(0)Y$ will not affect the bracket relations between Z, Y and X, so we may assume that $a(0) = b(0) = 0$.

Given a point $p = (p_1, p_2, p_3)$ in the domain of such a coordinate chart, the vector field $U = Z - (p_2 + a(p_3))X - b(p_3)Y$ is nonzero and vanishes at p. At p,

$$\left[U, \frac{\partial}{\partial x}\right] = 0, \ \left[U, \frac{\partial}{\partial y}\right] = -\frac{\partial}{\partial x}, \ \left[U, \frac{\partial}{\partial z}\right] = -a'(p_3)\frac{\partial}{\partial x} - b'(p_3)\frac{\partial}{\partial y}.$$

Since U belongs to $\mathfrak{heis}(3)$, hence is conformal for g, the matrix

$$A = \begin{pmatrix} 0 & -1 & a'(p_3) \\ 0 & 0 & b'(p_3) \\ 0 & 0 & 0 \end{pmatrix}$$

which is the matrix of $\nabla U(p)$ in the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$, must satisfy the identity

$$g_p(A\cdot, \cdot) + g_p(\cdot, A\cdot) = \alpha g_p, \ \alpha \in \mathbf{R}$$

The matrix A generates a 1-parameter group $\{e^{tA}\}$ in $\mathbf{R} \times \mathrm{O}(1, 2)$, which is nontrivial because the rank of A is at least 1. If the rank of A were 1, then

$\{e^{tA}\}$ would fix two linearly independent vectors. But no nontrivial flow in $\mathbf{R} \times \mathrm{O}(1,2)$ has this property, so that we infer $b'(p_3) \neq 0$.

As p was arbitrary, we may assume the derivative b' does not vanish at any point of such a coordinate chart. Now let ψ be a smooth diffeomorphism on an interval around 0 such that $\psi(0) = 0$ and $b(\psi(z)) = z$. The transformation

$$\varphi : (x, y, z) \mapsto (x, y - a(\psi(z)), \psi(z)).$$

then yields a local diffeomorphism fixing the origin. Applying $(\varphi^{-1})_*$ to the generators yields

$$Z' = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y},\ Y' = \frac{\partial}{\partial y},\ \text{and}\ X' = \frac{\partial}{\partial x}$$

which are conformal for the metric $g' = \varphi^* g$.

Again, let $p = (p_1, p_2, p_3)$ be a point in our coordinate chart. The vector field $U' = Z' - p_2 X' - p_3 Y'$ vanishes at p and is conformal for g'. A straigthforward computation yields

$$\left[U', \frac{\partial}{\partial x}\right] = 0,\ \left[U', \frac{\partial}{\partial y}\right] = -\frac{\partial}{\partial x},\ \text{and}\ \left[U', \frac{\partial}{\partial z}\right] = -\frac{\partial}{\partial y}$$

everywhere. Now the matrix

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

satisfies

$$g'_p(B\cdot, \cdot) + g'_p(\cdot, B\cdot) = \alpha g'_p,\ \alpha \in \mathbf{R}$$

The matrix of g'_p in the basis $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$ is thus of the form

$$\begin{pmatrix} 0 & 0 & -\beta(p) \\ 0 & \beta(p) & 0 \\ -\beta(p) & 0 & \gamma(p) \end{pmatrix},\ \beta(p) > 0$$

Replace g' by $\frac{1}{\beta} g'$, which amounts to assuming $\beta = 1$.

Now, X' and Y' are conformal Killing fields for g'. But $g'(Y', Y') = 1$, and Y' commutes with X' and Y'. It follows that X' and Y' are actually isometric Killing fields for g'. In particular, the function γ only depends on the variable z, and the metric g' is:

$$-2dxdz + dy^2 + \gamma(z)dz^2.$$

Now, if $\zeta(z)$ is an antiderivative of $\gamma(z)/2$, then the change of coordinates

$$(x, y, z) \mapsto (x + \zeta(z), y, z)$$

converts g' to $-2dxdz + dt^2$. We conclude that the conformal class $[g]$, restricted to a nonempty open subset of Ω, contains a flat metric. $\diamondsuit$

6.3. **Case $\mathfrak{z}_X$ is isomorphic to $\mathfrak{aff}(\mathbf{R}) \oplus \mathbf{R}$.** Let Σ be as in proposition 6.1 (3) and $x_0 \in \Sigma$. Let Y and Z be further generators of $\mathfrak{z}_X$ on a neighborhood of x_0 such that $[Y, Z] = Z$ and all other brackets are zero. The isotropy at x_0 can be of three types:

- isotropy generated by $U \in \mathfrak{z}_X$ transverse to $\mathrm{Span}(Z, X)$: The tangent vector $X(x_0)$ is lightlike and fixed by the isotropy. Rescaling U if necessary gives
$$D_{x_0}\varphi_U^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}.$$
By theorem 4.9 and proposition 4.8, (M, g) is conformally flat, a contradiction.
- isotropy generated by Z: Then $D_{x_0}\varphi_Z^t$ is trivial, which implies conformal flatness by theorem 4.9. We thus discard this case too.
- isotropy generated by $Z + cX$ with $c \neq 0$: We handle this case below.

First we construct a model for the geometry on Σ. Consider $\mathbf{R}^2$ with coordinates (x, y), and denote by $\mathcal{H}^+$ the upper half-space defined by $y > 0$. On $\mathcal{H}^+$, let $A = y\frac{\partial}{\partial y}$, $B = y\frac{\partial}{\partial x}$, and $C = \frac{\partial}{\partial x}$. The only nontrivial bracket relation is $[A, B] = B$, so that the Lie algebra $\mathfrak{h}$ generated by A, B, and C is isomorphic to $\mathfrak{aff}(\mathbf{R}) \oplus \mathbf{R}$. These vector fields are complete and integrate to a genuine action of $H \simeq \mathrm{Aff}(\mathbf{R}) \times \mathbf{R}$ on $\mathcal{H}^+$ given by the affine transformations:
$$\begin{pmatrix} 1 & b \\ 0 & e^a \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix}, \ a, b, c \in \mathbf{R}.$$

On $\mathcal{H}^+$, the isotropy for the local action of $\mathfrak{h}$ is always generated by an element of $\mathrm{Span}(B, C)$ transverse to $\mathbf{R}C$. Thus the local action of $\mathfrak{z}_X$ on Σ is locally modeled on that of $\mathfrak{h}$ on $\mathcal{H}^+$.

Let G be the 4-dimensional Lie group given by the following transformations of $\mathcal{H}^+$:
$$(x, y) \mapsto (x + \beta y + \gamma \ln(y) + \tau, e^\alpha y) \qquad \alpha, \beta, \gamma, \tau \in \mathbf{R}$$

In coordinates $(\alpha, \beta, \gamma, \tau)$, the subgroup comprising elements of the form $(0, \beta, \gamma, \tau)$ is normal and abelian. The action of $(\alpha, 0, 0, 0)$ on it is given by the matrix:
$$\begin{pmatrix} e^{-\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha & 1 \end{pmatrix}.$$

The group G is thus isomorphic to a semi-direct product $\mathbf{R} \ltimes \mathbf{R}^3$.

Observe that H is a subgroup of G, corresponding to $\gamma = 0$. We have the following Liouville Theorem for $(G, \mathcal{H}^+)$:

Lemma 6.3. *Let U and V be two connected open subsets of $\mathcal{H}^+$, and $f : U \to V$ a diffeomorphism such that $f_*C = C$ and $f_*(\mathfrak{h}) = \mathfrak{h}$. Then f is the restriction of a unique element of G.*

Proof: Because f preserves C, it is of the form $f(x, y) = (x + \eta(y), \psi(y))$. Now f also preserves $\mathfrak{h}$, hence f_* acts as an automorphism of $\mathfrak{h}$. In particular $f_*B = bB$, with $b \neq 0$. We get $\psi(y) = \frac{1}{b}y$, showing that $b > 0$. Set $b = e^\alpha$. Next, $f_*(A)$ has the form $A + cB + dC$, from which we deduce

$$\eta(y) = \beta y + \gamma \ln(y) + \tau, \qquad \beta, \gamma, \tau \in \mathbf{R}$$

$\diamondsuit$

Corollary 6.4. *The surface Σ is endowed with a $(G, \mathcal{H}^+)$-structure.*

Proof: Given an open subset $U \subset \Sigma$, denote $\mathfrak{z}_X(U)$ the Lie algebra of all local conformal vector fields defined on U commuting with X. For each $x_0 \in \Sigma$, there exists a small neighborhood U containing x_0, an open subset $V \subset \mathcal{H}^+$, and a diffeomorphism $\psi : U \to V$ such that $\psi_*(\mathfrak{z}_X(U)) = \mathfrak{h}|_V$ and $\psi_*(X) = C$. The corollary then follows from Lemma 6.3. $\diamondsuit$

Lemma 6.5. *The $(G, \mathcal{H}^+)$-structure on Σ is complete.*

Proof: Let $\tilde{\Sigma}$ be the universal cover of Σ and $\delta : \tilde{\Sigma} \to \mathcal{H}^+$ a developing map, with associated holonomy morphism $\rho : \pi_1(\Sigma) \to G$. By construction, $\delta_*(\tilde{X}) = C$, where $\tilde{X}$ is the lift of X to $\tilde{\Sigma}$. In particular, the relation $\delta \circ \varphi^t_{\tilde{X}} = \varphi^t_C \circ \delta$ shows that $\delta(\tilde{\Sigma})$ is a union of lines $y \equiv c$, for $c \in \mathbf{R}$. Thus $\delta(\tilde{\Sigma})$ is a horizontal strip in $\mathcal{H}^+$.

Next observe that G preserves the degenerate metric $h_0 = dy^2/y^2$ on $\mathcal{H}^+$, from which Σ inherits a degenerate metric h. On a 2-fold cover of Σ, there is $W \in \mathcal{X}(\Sigma)$ transverse to X and satisfying $h(W, W) = 1$. For $\tilde{x} \in \tilde{\Sigma}$, the trajectory $\delta(\varphi^t_{\tilde{W}}.\tilde{x})$ is a curve in $\mathcal{H}^+$ with velocity of constant h_0-length 1, defined on $\mathbf{R}$ since W is complete on Σ. This curve must cross every horizontal line in $\mathcal{H}^+$; we conclude that $\delta(\tilde{\Sigma}) = \mathcal{H}^+$. In fact, the open set $\Omega = \{\varphi^t_{\tilde{X}}\varphi^s_{\tilde{W}}.\tilde{x} \mid s, t \in \mathbf{R}\} \subset \tilde{\Sigma}$ is mapped diffeomorphically by δ onto $\mathcal{H}^+$. The boundary $\partial\Omega$ is empty. Indeed, if $x \in \partial\Omega$, then there is $x' \in \Omega$ satisfying $\delta(x') = \delta(x) = y$. Disjoint neighborhoods of both x and x' in $\tilde{\Sigma}$ map diffeomorphically under δ to a neighborhood of y. On the other hand, both neighborhoods in $\tilde{\Sigma}$ intersect Ω, which contradicts injectivity of δ on Ω. We conclude that $\Omega = \tilde{\Sigma}$, and completeness follows. $\diamondsuit$

The developing map δ identifies $\pi_1(\Sigma)$ with a discrete subgroup of G, and by proposition 6.1 (3), this discrete group contains an index 2 subgroup Λ isomorphic to $\mathbf{Z}^2$. There must be in Λ an element $\gamma_0 = (\alpha_0, \beta_0, \gamma_0, \tau_0)$ with $\alpha_0 \neq 0$, otherwise Λ would preserve the lines $y \equiv c$ in $\mathcal{H}^+$, and could not

act cocompactly. Now it is readily checked that the centralizer L of γ_0 in G comprises elements of G of the form $\left(\alpha, \frac{\beta_0}{e^{\alpha_0}-1}(e^\alpha - 1), \frac{\gamma_0}{\alpha_0}\alpha, \tau\right), \alpha, \tau \in \mathbf{R}$. Hence L is isomorphic to $\mathbf{R}^2$, and $\{\varphi_C^t\}$ acts by translations on the torus L/Λ. Because $\overline{\{\varphi_C^t\}}$ is compact in L/Λ, this would make $\{\varphi_X^t\}$ relatively compact in $\mathrm{Diff}(\Sigma)$. Then there are $t_k \to \infty$ and $x \in \Sigma$ such that $\left\{D_x(\varphi_X^{t_k}\big|_\Sigma)\right\}$ and $\left\{\left(D_x(\varphi_X^{t_k}\big|_\Sigma)\right)^{-1}\right\}$ are bounded. Since Σ has codimension one, this implies boundedness of $D_x\varphi_X^{t_k}$ and $(D_x\varphi_X^{t_k})^{-1}$. Then there is a holonomy sequence for $\{\varphi_X^{t_k}\}$ at x contained in P^+, which implies conformal flatness by theorem 4.12. We have reached the desired contradiction.

7. Case $\mathfrak{z}_X$ is 2-dimensional

The case in which $\mathfrak{z}_X$ is 2-dimensional, necessarily isomorphic to $\mathbf{R}^2$, is the most involved. What we will actually prove in this section is the following:

Theorem 7.1. *Let (M, g) be a 3-dimensional, compact, real-analytic, Lorentzian manifold. Assume that $\{\varphi_X^t\}$ is 1-parameter group of conformal transformations, which does not have compact closure in $Conf(M, [g])$. If the Lie algebra $\mathfrak{z}_X$ is 2-dimensional, then (M, g) is conformally flat.*

7.1. **Ideas of the proof.** For the reader's convenience, we outline below the general strategy to prove theorem 7.1. The proof is by contradiction, and our standing assumption throughout section 7 will thus be that (M, g) is not conformally flat. There are basically four steps, the last three ones heavily relying on section 6 of [31]—also the most difficult part of the proof in that paper—to arrive at a contradiction.

Step one. We first show in subsection 7.2 that the infinitesimal local action of $\mathfrak{z}_X$ integrates into a global conformal action of a cylinder $Z_X \simeq S^1 \times \mathbf{R}$ on M.

Step two. In Section 7.3, we carefully analyze the orbits of the action of Z_X on M. We show in Proposition 7.6 that orbits are of exactly two kinds. First, there are 2-dimensional lightlike orbits, which fill a dense open subset $\Omega \subset M$. Each such 2-dimensional orbit accumulates on one or two closed, lightlike, 1-dimensional orbits. The union of those 1-dimensional orbits is a closed, analytic, nowhere-dense subset of M, which we denote by Λ. Our assumption that M is not conformally flat moreover provides precise information about the isotropy on 1-dimensional orbits: It is linearizable and unipotent.

Step three. The main issue is now to understand how 2-dimensional orbits accumulate on 1-dimensional ones, which is the purpose of sections 7.4 and 7.5. A fundamental tool is to show in lemma 7.11 that on a dense, open subset Ω (the subset where the Cotton tensor does not vanish), 2-dimensional orbits enjoy a nice geometric structure, making them akin to totally geodesic hypersurfaces: They are projections of leaves of an integrable distribution in a reduction of the Cartan bundle over Ω.

In the same way, any point in a 1-dimensional orbit is also contained in a distinguished degenerate surface, which we call *distinguished plaques*. Actually, these distinguished plaques turn out to be analytic continuations of the 2-dimensional orbits in the following sense. Any 2-dimensional orbit accumulating on a point in some 1-dimensional orbit must intersect the distinguished plaque of this point in an open subset. A crucial consequence is that each 1-dimensional orbit can attract at most countably-many 2-dimensional ones (corollary 7.17).

Step four. In this last step, we investigate in more details the analytic set Λ comprising all 1-dimensional orbits. If this set is 1-dimensional, it is a finite union of 1-dimensional orbits. When Λ has dimension 2, then a careful analysis of the smooth 2-dimensional part of Λ shows that this smooth part is a union of distinguished plaques. By step three, no 1-dimensional orbit staying in this 2-dimensional smooth part can be accumulated by a 2-dimensional orbit. It follows that only finitely many 1-dimensional orbits can be accumulated, yielding that there exists at most countably many 2-dimensional orbits by step three. This contradicts the fact that such orbits fill an open subset.

7.2. **Global conformal action on M.** The result of this subsection is that, possibly after passing to a finite cover of M, the vector fields in $\mathfrak{z}_X$ are globally defined, necessarily complete, generating a conformal action of a cylinder Z_X.

Recall that we are assuming that $\{\varphi_X^t\}$ is not relatively compact, and (M, g) is not conformally flat. From these assumptions we can immediately deduce the following facts about $\mathfrak{z}_X$-orbits:

Lemma 7.2. (1) *There is no $\mathfrak{z}_X$-orbit of dimension* 0.
(2) *There exists a closed $\mathfrak{z}_X$-orbit Σ of dimension* 1

Proof: Corollary 4.11 rules out any $\mathfrak{z}_X$-orbit of dimension 0. Because M is compact, there are recurrent points for $\{\varphi_X^t\}$. At such points, the isotropy is nontrivial by proposition 3.4, and the $\mathfrak{z}_X$-orbit is of dimension 1. It is closed because of theorem 3.3 (2) and point (1). $\diamondsuit$

Now we prove:

Proposition 7.3. *After possibly replacing M by a finite cover, every local conformal vector field on M commuting with X extends to a conformal vector field defined on all of M. The resulting subalgebra of $\mathcal{X}^{conf}(M)$, isomorphic to $\mathfrak{z}_X$, integrates to a subgroup group $Z_X \leq Conf(M)$ isomorphic to $S^1 \times \mathbf{R}$, acting locally freely on an open, dense subset of M.*

Proof: Let $\tilde{M}$ be the universal cover of M, with group of deck transformations $\Gamma < \mathrm{Conf}(\tilde{M})$. Denote the lifts of X and $\mathfrak{z}_X$ to $\tilde{M}$ also by X and $\mathfrak{z}_X$. By Amores' theorem [1] (see section 3.2), the lifts of $\mathfrak{z}_X$ form a globally defined subalgebra of $\mathcal{X}^{conf}(\tilde{M})$. Let Γ_0 be the kernel of the holonomy representation of Γ on $\mathfrak{z}_X$, and set $\tilde{M}' = \tilde{M}/\Gamma_0$. We will again denote by $\mathfrak{z}_X$ the corresponding subalgebra of $\mathcal{X}^{conf}(\tilde{M}')$. The manifold M is a quotient of $\tilde{M}'$ by a group $\Gamma' \cong \Gamma/\Gamma_0$, and the holonomy representation of Γ' on $\mathfrak{z}_X$ is faithful.

Lift the closed orbit Σ given by lemma 7.2 to $\tilde{M}'$, and let $\tilde{\Sigma}$ be a connected component of this lift. It is the $\mathfrak{z}_X$-orbit of a point $\tilde{x}_0$ in $\tilde{M}'$.

Lemma 7.4. *The manifold $\tilde{\Sigma} \subset \tilde{M}'$ is a circle.*

Proof: Assume, for the sake of contradiction, that $\tilde{\Sigma}$ is diffeomorphic to $\mathbf{R}$. In this case, the stabilizer of $\tilde{\Sigma}$ in Γ' is a cyclic subgroup $\Gamma'_{\tilde{\Sigma}} = \langle\gamma\rangle \cong \mathbf{Z}$. First suppose $X(\tilde{x}_0) \neq 0$, which implies X is nonvanishing on $\tilde{\Sigma}$. Then there is $T_0 \neq 0$ such that $\gamma\varphi_X^{T_0} \in \mathrm{Is}_X^{loc}(\tilde{x}_0)$. This group has finitely-many components by theorem 3.2, so a power

$$(\gamma\varphi_X^{T_0})^\ell = \gamma^\ell \varphi_X^{\ell T_0} = \varphi_Z^1$$

on a neighborhood U of $\tilde{x}_0$, where Z is a generator of $\mathfrak{Is}_X(\tilde{x}_0)$, and we assume φ_Z^t is defined on U for all $t \in [0,1]$. Let $Y = Z - \ell T_0 X$. For $t \in [0,1]$ and $x \in U$, the composition $\varphi_X^{-tT_0} \circ \varphi_Z^t(x)$ is well-defined. Because X and Z commute, $\varphi_Y^t(x) = \varphi_X^{-tT_0} \circ \varphi_Z^t(x)$, and now

$$\varphi_Y^1\big|_U = \gamma^k\Big|_U .$$

Then $(\gamma^k)_* Y = Y$ on U, which implies by analyticity that $(\gamma^k)_* Y = Y$ on $\tilde{M}'$. Because Γ' fixes X and Y is independent of X, the element $\gamma^k \neq 1$ would centralize $\mathfrak{z}_X$, contradicting faithfulness of Γ' on $\mathfrak{z}_X$.

Next suppose $X(\tilde{x}_0) = 0$, which implies $X \equiv 0$ on $\tilde{\Sigma}$, and let $Z \in \mathfrak{z}_X \backslash \mathbf{R}X$. Such Z is nonvanishing by corollary 4.11. There is $T_0 \neq 0$ such that $\varphi_Z^{T_0}(\tilde{x}_0) = \gamma(\tilde{x}_0)$, and φ_Z^t is defined on a neighborhood U of $\tilde{x}_0$ for all $t \in [0, T_0]$. Then $\gamma^{-1}\varphi_Z^{T_0} \in \mathrm{Is}_X^{loc}(\tilde{x}_0)$ on U. Again by theorem 3.2, the latter group contains $\{\varphi_X^s\}$ as a finite-index subgroup. Thus, for some $k \in \mathbf{Z}$ and $S_0 \in \mathbf{R}$,

$$(\gamma^{-1} \circ \varphi_Z^{T_0})^k = \varphi_X^{S_0}$$

on a neighborhood of $\tilde{x}_0$.

Because $[X,Z]=0$, the differential $D_{\tilde{x}_0}\varphi_X^{S_0}(Z(\tilde{x}_0)) = Z(\tilde{x}_0)$. Next

$$D_{\tilde{x}_0}(\gamma^{-1}\circ\varphi_Z^{T_0})(Z(\tilde{x}_0)) = \pm Z(\tilde{x}_0) = (\gamma_*^{-1}Z)(\tilde{x}_0)$$

Then $\gamma_*(Z) = \pm Z + \beta X$ for some $\beta\in\mathbf{R}$. If $\gamma_*(Z)$ were congruent to $-Z$ modulo $\mathbf{R}X$, then, given that $\varphi_Z^{-T_0}\gamma(\tilde{x}_0)=\tilde{x}_0$, we would have

$$\gamma^{-1}(\tilde{x}_0) = \gamma^{-1}\varphi_Z^{-T_0}\gamma(\tilde{x}_0) = \varphi_Z^{T_0}(\tilde{x}_0) = \gamma(\tilde{x}_0)$$

which is absurd, since Γ acts freely and γ has infinite order. Thus $\gamma_*(Z) = Z+\beta X$. Now there is S_1 such that

$$(\gamma^{-1}\circ\varphi_Z^{T_0})^k = \gamma^{-k}\varphi_X^{S_1}\varphi_Z^{T_0} \in \mathrm{Is}_X^{loc}(\tilde{x}_0)$$

Thus, for some S_2,

$$\gamma^k = \varphi_X^{S_2}\varphi_Z^{T_0}$$

on U. Let $Y = S_2X + T_0Z$. As above, the flow along Y is well-defined on U for $t\in[0,1]$, and

$$\varphi_Y^1\big|_U = \gamma^k\Big|_U.$$

This leads again to $(\gamma^k)_*Y = Y$ on $\tilde{M}'$, yielding a contradiction with the faithfulness of the action of Γ' on $\mathfrak{z}_X$. $\diamondsuit$

Next we show that $\mathfrak{z}_X$ globalizes on a neighborhood of the closed orbit $\tilde{\Sigma}$.

Lemma 7.5. *There exists $Y\in\mathfrak{z}_X$, and $\tilde{N}$ an open neighborhood of $\tilde{\Sigma}$, such that for all $y\in\tilde{N}$, the orbit $\{\varphi_Y^t.y\}$ is defined on $\mathbf{R}$, included in $\tilde{N}$, and 1-periodic.*

Proof: If $X(\tilde{x}_0)\neq 0$, then $\{\varphi_X^t.\tilde{x}_0\}$ is periodic. Because $\mathrm{Is}_X^{loc}(\tilde{x}_0)$ has finitely-many components, there are $T_0>0$, $S_0\in\mathbf{R}$ such that $\varphi_X^{T_0}\circ\varphi_Z^{S_0}$ is trivial on a neighborhood of $\tilde{x}_0$, where Z generates $\mathfrak{Is}_X(\tilde{x}_0)$. Note that, for any S, the flow $\{\varphi_Z^s\}$ is defined for $s\in[0,S]$ in some neighborhood of $\tilde{x}_0$, and it commutes with $\{\varphi_X^t\}$. Now $Y = T_0X + S_0Z\in\mathfrak{z}_X$ generates a periodic flow $\{\varphi_X^{rT_0}\circ\varphi_Z^{rS_0}\}$, defined for $r\in\mathbf{R}$ in a neighborhood U of $\tilde{x}_0$. We can then put $\tilde{N} := \bigcup_{t\in\mathbf{R}}\varphi_Y^t(U)$, which is indeed an open neighborhood of $\tilde{\Sigma}$ satisfying the conclusions of the lemma.

If X vanishes on $\tilde{\Sigma}$, then by corollary 4.11, any $Z\in\mathfrak{z}_X\backslash\mathbf{R}X$ does not vanish on $\tilde{\Sigma}$. Then $\bar{Z} = Z|_{\tilde{\Sigma}}$ is nonvanishing and complete because $\tilde{\Sigma}$ is compact by lemma 7.4. Let $T_0>0$ be such that $\varphi_Z^{T_0}.\tilde{x}_0 = \tilde{x}_0$. The flow along Z is defined on $[0,T_0]$ in a neighborhood $\tilde{N}$ of $\tilde{\Sigma}$. By theorem 3.2, $\{\varphi_X^s\}$ has finite index in $\mathrm{Is}_X^{loc}(\tilde{x}_0)$ and is noncompact by assumption. After replacing T_0 by a finite integer multiple, and shrinking $\tilde{N}$ if necessary, $\varphi_Z^{T_0}\varphi_X^{S_0}$ will be trivial in restriction to $\tilde{N}$, for some $S_0\in\mathbf{R}$. Because X is complete and $[Z,X]=0$, the flow along $Y = T_0Z+S_0X$ restricted to $\tilde{N}$ is periodic with

period 1 and complete. Again, $\tilde{N} := \bigcup_{t\in\mathbf{R}} \varphi_Y^t(U)$ is an open neighborhood of $\tilde{\Sigma}$ satisfying the conclusions of the lemma. $\diamondsuit$

Let $\Omega = \bigcup_{t\in\mathbf{R}} \varphi_X^t.\tilde{N}$, an open set; let Y be as in lemma 7.5. For every $y \in \Omega$, the pseudo-orbit $\{\varphi_Y^t.y\}$ is defined for every $t \in \mathbf{R}$, included in Ω, and 1-periodic. Any $Z \in \mathfrak{z}_X$ equals $r_0X + s_0Y$, for some r_0 and s_0, so for every $y \in \tilde{M}'$, we have $\varphi_Z^t.y = \varphi_X^{tr_0} \circ \varphi_Y^{ts_0}.y$, for all t such that the expression is defined. For $y \in \Omega$, in particular, $\varphi_Z^t.y$ is defined for every $t \in \mathbf{R}$ and lies in Ω. All vector fields of $\mathfrak{z}_X|_\Omega$ are thus complete, defining an action of a 2-dimensional abelian Lie group Z_X on Ω. For the time being, we consider Z_X as a subgroup of $\mathrm{Conf}(\Omega)$. Since the flow $\{\varphi_Y^t\}$ is cyclic on Ω, we infer that Z_X is isomorphic to a cylinder $S^1 \times \mathbf{R}$ or a torus $\mathbf{T}^2$. The latter possibility would mean that $\overline{\{\varphi_X^t\}}$ is compact in $\mathrm{Conf}(\Omega)$, hence in $\mathrm{Conf}(\tilde{M}')$, and finally in $\mathrm{Conf}(M)$, contradicting our standing hypotheses.

Now assume Z_X is a cylinder. There exists a homomorphism $\rho : \Gamma' \to \mathrm{Aut}(Z_X)$ integrating the representation $\Gamma' \to \mathrm{Aut}(\mathfrak{z}_X)$. Indeed, for $\gamma \in \Gamma'$ and $Z \in \mathfrak{z}_X$ the local flows $\{\gamma\varphi_Z^s\gamma^{-1}\}$ and $\{\varphi_{\gamma_*Z}^s\}$ coincide on Ω; in particular, $\gamma\varphi_Z^1\gamma^{-1}$ is well-defined on Ω and belongs to Z_X.

Because $\rho(\Gamma')$ fixes φ_X^t for all t, it follows that $|\rho(\Gamma')| \leq 2$. Because ρ is faithful, $|\Gamma'| \leq 2$. Then $\tilde{M}'$ is compact, and all vector fields of $\mathfrak{z}_X$ are complete. Thus $\mathfrak{z}_X$ integrates to the action of a group $Z_X \cong S^1 \times \mathbf{R}$, as claimed in the proposition.

We replace M by $\tilde{M}'$ in the sequel. To complete the proof of the proposition, it remains to check that the action of Z_X is locally free on a dense open subset of M. If not, there would be a nonempty open subset U on which all Z_X-orbits have dimension 1. The identity component of the isotropy group $\mathrm{Is}_X^{loc}(x_0)$ at a point $x_0 \in U$ would fix a nonzero vector v tangent to the orbit and act trivially on the quotient $T_{x_0}M/\mathbf{R}v$. Basic linear algebra shows that the differential of the isotropy at x_0, identified with a subgroup of $\mathrm{CO}(1,2)$, must be trivial in this case. On the other hand, theorem 4.9 says, given our assumption on (M,g), that the isotropy must be linearizable, yielding a contradiction. $\diamondsuit$

Now we replace M by the finite covering given by the above proposition.

7.3. **Description of the Z_X-orbits.**

Proposition 7.6. *There are two types of Z_X-orbits in M, both of which occur:*

1. *Circular lightlike orbits, with linear, unipotent isotropy. These form an analytic subset Λ of M.*
2. *Cylindrical orbits on which the metric is degenerate. Each of these contains an orbit of type (1) in its closure. These fill an open, dense subset Ω_f of M.*

Proof: By corollary 4.10, there are no Z_X-fixed points; moreover, all 1-dimensional orbits are closed—otherwise, there would be a fixed point in the closure by theorem 3.3 (2). The closed, lightlike orbits are precisely the zero-set of the analytic function, given by a choice of metric g in the conformal class and $Y \in \mathfrak{z}_X \backslash \mathbf{R}X$ by

$$\varphi(x) = g_x(X,X)^2 + g_x(Y,Y)^2 + g_x(X,Y)^2$$

Since we assume that (M,g) is not conformally flat, the elements of the isotropy algebra can be assumed linearizable by theorem 4.9. In a 1-dimensional, lightlike orbit, the Z_X-isotropy fixes a lightlike tangent vector. Such isotropy is easily seen to be balanced or unipotent. In the first case, the isotropy is stable and leads to conformal flatness by proposition 4.8. We have proved all the claimed properties of orbits of type (1).

Let Ω_f be the set on which Z_X acts locally freely; it is open and dense by proposition 7.3. If there were a closed, 2-dimensional orbit, then $\{\varphi_X^t\}$ would have a recurrent point on this orbit, contradicting proposition 3.4. Thus every 2-dimensional orbit is not closed, and, by theorem 3.3 (2), contains a closed 1-dimensional orbit in its closure.

Now we focus on the linear part of a holonomy sequence $\{p_k\}$ for an unbounded sequence $\{h_k\} \subset Z_X$ at $x \in \Omega_f$. By proposition 4.4, the subspace $\mathfrak{z}_X(x) \subset T_xM$ is approximately stable for $\{D_x h_k\}$, because Z_X centralizes $\mathfrak{z}_X$. This means that $\omega_{\hat{x}}(\mathfrak{z}_X)$ belongs modulo $\mathfrak{p}$ to $(\mathfrak{g}/\mathfrak{p})^{AS}(p_k)$, for any $\hat{x} \in \pi^{-1}(x)$. Assuming p_k is in $A'P^+$-form, the presence of a 2-dimensional approximately stable subspace in $\mathfrak{g}/\mathfrak{p}$ makes it contracting, balanced, mixed, or of bounded distortion, as in definition 4.2.

Let $\Omega \subseteq \Omega_f$ be the open subset where the Cotton tensor is nonzero; by our standing assumption, it is also dense. For $x \in \Omega$, a holonomy sequence $\{p_k\}$ as above is of bounded distortion or mixed type by proposition 4.7. For these types, $(\mathfrak{g}/\mathfrak{p})^{AS}(p_k)$ is a degenerate plane. Thus all orbits in Ω are degenerate. On the other hand, if a point $x \in \Omega_f$ has Riemannian or Lorentzian orbit, then so do the points in a neighborhood of x. Since Ω is open and dense, we conclude that all orbits of Ω_f—that is, all 2-dimensional orbits—are degenerate.

We rule out nondegenerate orbits of dimension 1 in lemma 7.7 directly below, which completes the proof. $\diamondsuit$

Lemma 7.7. *There are no 1-dimensional spacelike or timelike orbits.*

The proof of this lemma makes use of the exponential map of the Cartan connection, defined in subsection 4.3.2. The exponential map will appear frequently in the remainder of this section.

Proof. If a point x has a 1-dimensional timelike orbit, then some $Y \in \mathfrak{z}_X\ X$ is timelike in a neighborhood of x. This neighborhood intersects the open, dense set from above in which all orbits are degenerate, a contradiction.

Now suppose that x has a 1-dimensional spacelike orbit, and let $\hat{x} \in \pi^{-1}(x)$ be such that the isotropy image $(\hat{I}_X)_{\hat{x}}$ is linear, contained in $G_0 = \mathrm{CO}(1,2)$. It fixes a spacelike vector, which makes it conjugate in G_0 to a 1-parameter diagonal group

$$\{\hat{h}^t : (x^1, x^2, x^3) \mapsto (e^t x^1, x^2, e^{-t} x^3)\}$$

Let $\hat{Z} \in \mathfrak{co}(1,2)$ be the generator of $\{\hat{h}^t\}$, corresponding under $\iota_{\hat{x}}$ to a generator $Z \in \mathfrak{Is}_X(x)$. Note that the corresponding linear vector field $\hat{Z}$ on $\mathbf{R}^{1,2}$ is timelike along the line $\mathbf{R}(E_1 + E_3)$ (with the metric $\mathbb{I}$ of section 4.1).

Let $\hat{\gamma}(t) = \exp_{\hat{x}}(t(E_1 + E_3))$ and $\gamma = \pi \circ \hat{\gamma}$. The lift $\hat{\gamma}$ determines a metric along γ in the conformal class $[g]$ by

$$\langle u, v \rangle_{\gamma(t)} = \mathbb{I}\left(\omega^{(-1)}_{\hat{\gamma}(t)} \hat{u}, \omega^{(-1)}_{\hat{\gamma}(t)} \hat{v}\right)$$

where $\omega^{(-1)}$ is the component of ω on $\mathfrak{g}_{-1} \cong \mathbf{R}^{1,2}$, and $\hat{u}, \hat{v}$ are any lifts of u, v to $T_{\hat{\gamma}(t)}\hat{M}$. We will approximate $\langle Z, Z \rangle_{\gamma(t)}$ by computing

$$\left.\frac{d}{dt}\right|_0 \omega^{(-1)}_{\hat{\gamma}(t)}(Z) = (\hat{E}_1 + \hat{E}_3)_{\hat{x}}.\omega^{(-1)}(Z)$$

where $\hat{E}_1 + \hat{E}_3$ is the vector field on $\widehat{M}$ satisfying $\omega(\hat{E}_1 + \hat{E}_3) \equiv E_1 + E_3$. Using that $L_Z \omega(\hat{E}_1 + \hat{E}_3) \equiv 0$, the Cartan curvature gives

$$0 = \Omega_{\hat{x}}(\hat{E}_1 + \hat{E}_3, Z) = (\hat{E}_1 + \hat{E}_3)_{\hat{x}}.\omega(Z) + \left[E_1 + E_3, \hat{Z}\right]$$

Thus

$$(\hat{E}_1 + \hat{E}_3)_{\hat{x}}.\omega^{(-1)}(Z) = \left[\hat{Z}, E_1 + E_3\right]^{(-1)} = E_1 - E_3$$

Now

$$\omega^{(-1)}_{\hat{\gamma}(t)}(Z) = t(E_1 - E_3 + R(t)), \qquad \lim_{t \to 0} R(t) = 0$$

and

$$\begin{aligned} \langle Z, Z \rangle_{\gamma(t)} &= \mathbb{I}\left(t(E_1 - E_3 + R(t)), t(E_1 - E_3 + R(t))\right) \\ &= t^2(-2 + Q(t)) \qquad \lim_{t \to 0} Q(t) = 0 \end{aligned}$$

For sufficiently small t, this inner product is negative along γ. This would mean that $Z \in \mathfrak{z}_X$ is timelike on a nonempty open set, contradicting that all Z_X-orbits in the open, dense set Ω_f are degenerate. □

Corollary 7.8. *Let $\{h_k\} \subset Z_X$ be an unbounded sequence, and suppose that $h_k.x \to y$ for $x \in \Omega$ and $y \in \Lambda$. Then any holonomy sequence for $\{h_k\}$ at x is of mixed type.*

Proof. We established during the proof of proposition 7.6 above that holonomy sequences at points of Ω are of bounded distortion or mixed type.

By proposition 7.6, there is $Z \in \mathfrak{z}_X$ with $Z(x)$ spacelike. It is also approximately stable for $\{D_x h_k\}$. If $\{h_k\}$ has bounded distortion at x, then $\lim_k D_x h_k(Z)$ is a nonzero spacelike vector tangent to the orbit of y. This would contradict the result from proposition 7.6 above that the orbit of y is one-dimensional and lightlike. □

7.4. **Z_X-orbits in the Cartan bundle.** In this section we construct a Z_X-invariant reduction of $\hat{M}\Big|_\Omega$ and show that Z_X-orbits there are tangent to a special distribution defined by the Cartan connection. These properties will be key to controlling the accumulation of 2-dimensional orbits in Ω on 1-dimensional orbits in $\partial\Omega$.

7.4.1. *Adjoint approximately stable spaces.* As in [31], we will call $\{p_k\}$ an *ACL holonomy sequence* at $x \in \Omega_f$ if it is in $A'P^+$-form; it corresponds to $h_k.x \to y$ for $y \in \Lambda$; and the isotropy in Z_X with respect to $\hat{y} = \lim h_k.\hat{x}_k.p_k^{-1}$ is linear. Every ACL holonomy sequence at $x \in \Omega$ has $(\mathfrak{g}/\mathfrak{p})^{AS}(p_k) = E_1^\perp$, because its linear component is of mixed type by corollary 7.8.

We will next describe $\mathfrak{g}^{AS}(p_k)$, which also reflects the nonlinear part of $\{p_k\}$. We use the G_0-invariant decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{g}_{-1} \cong \mathbf{R}^{1,2}$ to identify $E_1^\perp$ with a subspace of $\mathfrak{g}$.

Proposition 7.9 ([31] Prop 6.5)**.** *For $\{p_k\}$ an ACL holonomy sequence at $x \in \Omega$,*

$$E_1^\perp \subset \mathfrak{g}^{AS}(p_k)$$

Proof: Write $\{p_k = d_k\tau_k\}$. By [31, Lem 6.4], $\{\xi_k = \ln \tau_k\}$ is contained in the line $\mathbf{R}E_1^t\mathbb{I} \subset \mathfrak{p}^+ \cong \mathbf{R}^{1,2*}$. Moreover, for $D_k = \ln d_k$, the sequence $\{e^{\beta(D_k)}\xi_k\}$ is bounded. Now, it follows that for $\{p_k\}$ as above and $v \in E_1^\perp$,

$$\mathrm{Ad}(p_k).v = \mathrm{Ad}(d_k)(v + [\xi_k, v]) = \mathrm{Ad}(d_k).v + e^{\beta(D_k)}[\xi_k, v],$$

noting that $[\xi_k, v]$ is in the root space $\mathfrak{g}_\beta$ and $[\xi_k, [\xi_k, v]] = 0$. This expression is bounded, so the desired inclusion follows. ♢

7.4.2. *Reduction of $\hat{M}$ over Ω.* Write $\hat{\Omega}_f = \hat{M}\Big|_{\Omega_f}$. Recall that Z_X acts locally freely in Ω_f with degenerate orbits, by proposition 7.6. Given $x \in \Omega_f$, the orthogonal $n(x)$ to the orbit of x is a lightlike line, tangent to the orbit. We define a reduction of $\hat{\Omega}_f$ given by the conformal frames in which

this orthogonal is the line $[E_1]$, as follows. Denote $\mathcal{N}$ the null cone of the Lorentzian inner product $\mathbb{I}$ in $\mathbf{R}^{1,2}$, and by $\mathbf{P}(\mathcal{N})$ its projectivization. Let

$$\begin{aligned} \eta : \hat{\Omega}_f &\to \mathbf{P}(\mathcal{N}) \\ \hat{x} &\mapsto \omega_{\hat{x}}^{(-1)}(\widehat{n(x)}) \end{aligned}$$

where $\widehat{n(x)}$ is any lift of $n(x)$ to $T_{\hat{x}}\hat{M}$, and $\omega^{(-1)}$ denotes the component on $\mathfrak{g}_{-1} \cong \mathbf{R}^{1,2}$. This map is well-defined, and in fact analytic, in Ω_f. The level set of $[E_1]$ is a reduction $\mathcal{R}' \subset \hat{\Omega}_f$ to $Q_0 \ltimes P^+$, where $Q_0 < G_0$ is the stabilizer of $[E_1]$. The Z_X-action preserves orbits, so it preserves the orthogonals, and it leaves ω invariant; thus Z_X preserves $\mathcal{R}'$.

Now we restrict to $\hat{\Omega} = \hat{M}\Big|_{\Omega}$. Let $\mathfrak{q}_1$ be the annihilator of E_1 in $\mathfrak{p}^+ \cong \mathbf{R}^{1,2*}$, with corresponding connected subgroup $Q_1 < P^+$. Let $Q = Q_0 \ltimes Q_1 < P$. Define

$$\mathcal{R} = \{\hat{x} \in \hat{\Omega} : \omega_{\hat{x}}^{-1}(E_1^{\perp}) \subset T_{\hat{x}}\mathcal{R}'\}$$

This construction and the following proposition are very similar to [31, Sec 6.2].

Proposition 7.10. *The set $\mathcal{R}$ is a Z_X-invariant reduction of $\hat{\Omega}$ to Q.*

Proof: Let $x \in \Omega$. By proposition 7.6, there is $h_k \to \infty$ in Z_X such that $h_k.x \to y \in \Lambda$, and the isotropy at y is linear and unipotent. There is thus an ACL holonomy sequence $\{p_k\}$ for $\{h_k\}$ with respect to $\hat{x} \in \pi^{-1}(x)$. By corollary 7.8 it is of mixed type. As observed in the proof of proposition 7.6, $\omega_{\hat{x}}(\mathfrak{z}_X)$ belongs modulo $\mathfrak{p}$ to $(\mathfrak{g}/\mathfrak{p})^{AS}(p_k)$, which equals $E_1^{\perp}$. For $x = \pi(\hat{x})$, the projection $\mathfrak{z}_X(x)$ is the tangent space to the orbit of x. Thus the orthogonal $n(x)$ corresponds under ω to $[E_1]$, and $\hat{x} \in \mathcal{R}'$. Whenever there is an ACL holonomy sequence of mixed type with respect to $\hat{x} \in \hat{\Omega}$, then $\hat{x} \in \mathcal{R}'$.

By proposition 7.9, $E_1^{\perp} \subset \mathfrak{g}^{AS}(p_k)$. By proposition 4.5, for all $X \in E_1^{\perp}$, for s sufficiently small, $\{p_k\}$ is also a holonomy sequence at $\hat{\gamma}(s) = \exp_{\hat{x}}(sX)$, and this point is in $\hat{\Omega}$. Thus $\hat{\gamma}(x)$ is in $\mathcal{R}'$, for s sufficiently small, which implies $\omega_{\hat{x}}^{-1}(X) \in T_{\hat{x}}\mathcal{R}'$. Then $\hat{x} \in \mathcal{R}$; moreover, every π-fiber of $\hat{\Omega}$ intersects $\mathcal{R}$.

To verify that $\mathcal{R}$ is a reduction of $\mathcal{R}'$ to Q, we will express it as the level set of a smooth—actually, analytic—map on $\mathcal{R}'$. As $\mathcal{R}'$ is a reduction of $\hat{\Omega}$, there is, at each $\hat{x} \in \mathcal{R}'$, a two-dimensional subspace of $\omega(T_{\hat{x}}\mathcal{R}')$ projecting modulo $\mathfrak{p}$ to $E_1^{\perp}$, varying smoothly with $\hat{x}$. This subspace can be expressed as the graph of a linear homomorphism $E_1^{\perp} \to \mathfrak{p}$, unique up to addition of a homomorphism $E_1^{\perp} \to \mathfrak{q}_0 \ltimes \mathfrak{p}^+$, corresponding to addition of vertical vectors tangent to $\mathcal{R}'$. Then we compose with the projection to $\mathfrak{p}/\mathfrak{p}^+ \cong \mathfrak{g}_0$ to define

$$\Phi : \mathcal{R}' \to \mathrm{Hom}(E_1^{\perp}, \mathfrak{g}_0/\mathfrak{q}_0)$$

Observe that $\hat{x} \in \mathcal{R}'$ belongs to $\mathcal{R}$ if and only if $\Phi(\hat{x}) = 0$.

The map Φ is $Q_0 \ltimes P^+$-equivariant, where, for $\varphi \in \mathrm{Hom}(E_1^\perp, \mathfrak{g}_0/\mathfrak{q}_0)$,

$$(g \cdot \tau)(\varphi) = (\mathrm{Ad}\, g|_{\mathfrak{g}_0/\mathfrak{q}_0}) \circ \varphi \circ (\mathrm{Ad}\, g^{-1}\big|_{E_1^\perp}) + (\mathrm{Ad}\, \tau - \mathrm{Id})|_{E_1^\perp}$$

For $\tau \in P^+$, the image $\mathrm{Ad}\, \tau(E_1^\perp + \mathfrak{q}_0) \equiv E_1^\perp + \mathfrak{q}_0 \mod \mathfrak{p}^+$ if and only if $\tau \in Q_1$. The affine $Q_0 \ltimes P^+$-action on $\mathrm{Hom}(E_1^\perp, \mathfrak{g}_0/\mathfrak{q}_0)$ factors through $\mathrm{Aff}(\mathbf{R})$. The orbit of 0 is 1-dimensional, with stabilizer Q.

Because every π-fiber of $\hat{\Omega}$ intersects $\mathcal{R}$, which is in turn contained in $\mathcal{R}'$, the image of the latter under Φ is contained in the orbit of 0. Now $\mathcal{R}$, the inverse image of 0, is a smooth Q-reduction of $\hat{M}$ over Ω. It is Z_X-invariant and analytic because $\mathcal{R}'$ and ω are. $\diamondsuit$

The geometric interpretation of $\mathcal{R}$ is as the conformal normalized 2-frames at points $x \in \Omega$ in which the orbits are totally geodesic (infinitesimally at x).

The fact from the proof of proposition 7.10 that $\{p_k\}$ is also a holonomy sequence at $\hat{\gamma}(s) = \exp_{\hat{x}}(sX)$ for all $X \in E_1^\perp$ gives that $\hat{\gamma}(s) \in \mathcal{R}$, for all s such that $\gamma(s) = \pi \circ \hat{\gamma}(s) \in \Omega$. It follows that $\omega_{\hat{x}}^{-1}(E_1^\perp) \subset T_{\hat{x}}\mathcal{R}$.

7.4.3. *Foliation of* $\mathcal{R}$. Let $\hat{\mathcal{D}} = \omega^{-1}(E_1^\perp + \mathfrak{q})$, an analytic distribution on $\hat{M}$. The restriction to $\mathcal{R}$ is tangent to $\mathcal{R}$ because, from the previous paragraph, $\omega^{-1}(E_1^\perp) \subset T\mathcal{R}$, and $\mathcal{R}$ is a principal Q-bundle. When M is 3-dimensional, we can prove integrability of $\hat{\mathcal{D}}$ in $\mathcal{R}$ without using the Cartan curvature.

Lemma 7.11. *The distribution* $\hat{\mathcal{D}}\Big|_{\mathcal{R}}$ *is integrable. The projection on* M *of the leaves of this distribution coincide with* Z_X*-orbits in* Ω.

Proof: The key fact here is that the Z_X-orbits in $\mathcal{R}$ are tangent to $\hat{\mathcal{D}}$. Indeed, as noted in the proofs of propositions 7.6 and 7.10, $\omega_{\hat{x}}(\mathfrak{z}_X)$ is congruent modulo $\mathfrak{p}$ to $E_1^\perp$ if there is a mixed ACL holonomy sequence with respect to $\hat{x}$, and more generally, if $\hat{x} \in \mathcal{R}$. Because $\mathcal{R}$ is Z_X-invariant, $\mathfrak{z}_X(\hat{x}) \subset T_{\hat{x}}\mathcal{R}$. Thus $\mathfrak{z}_X(\hat{x}) \subset \hat{\mathcal{D}}_{\hat{x}}$ for all $\hat{x} \in \mathcal{R}$.

The $(Z_X \times Q)$-orbits in $\mathcal{R}$ are integral leaves for $\hat{\mathcal{D}}$, projecting to the two-dimensional, degenerate Z_X-orbits in Ω. $\diamondsuit$

7.5. **Accumulation of 2-dimensional orbits on 1-dimensional orbits.** Before applying the results of the previous section, we focus on the geometry around 1-dimensional Z_X-orbits.

7.5.1. *Plaques at* 1*-dimensional orbits.* We define a distinguished degenerate surface around each 1-dimensional orbit. The following proposition is an aggregate of propositions 6.11 and 6.12 (see also remark 6.14) of [31].

Proposition 7.12. *Let* $y \in M$ *have closed, isotropic* Z_X*-orbit and suppose that* $Is_X(y)$ *is linear and unipotent with respect to* $\hat{y} \in \pi^{-1}(y)$. *Then:*

(1) *The point $\hat{y}$ belongs to the closure $\overline{\mathcal{R}}$.*
(2) *Let $\gamma : [0,1] \to M$ a continuous path, smooth on $[0,1)$, such that $\gamma([0,1)) \subset \Omega$ and $\gamma(1) = y$. Then there exists a continuous lift $\hat{\gamma} : [0,1] \to \hat{M}$, smooth on $[0,1)$, such that $\hat{\gamma}([0,1)) \subset \mathcal{R}$, and $\hat{\gamma}(1) = \hat{y}$.*
(3) *There is a neighborhood $\mathcal{U}$ of 0 in $E_1^\perp + \mathfrak{q}$ such that $\exp_{\hat{y}}(\mathcal{U})$ is an integral submanifold of $\hat{\mathcal{D}}$.*

The third point of the proposition follows rather easily from the first, because by analyticity, integrability of $\hat{\mathcal{D}}$ on $\mathcal{R}$ extends to the closure.

Definition 7.13. *Let $\exp_{\hat{y}}(\mathcal{U})$ be as in proposition 7.12. For $\mathcal{U}$ small enough, $\pi(\exp_{\hat{y}}(\mathcal{U}))$ is a degenerate 2-dimensional submanifold of M. It will be called a* plaque *at y, and denoted $\mathcal{P}_y$.*

Observe that if $\hat{y}$ and $\hat{y}'$ are two points of $\pi^{-1}(y)$ such that $\mathrm{Is}_X(y)$ is linear and unipotent with respect to $\hat{y}$ and $\hat{y}'$, then $\hat{y}' = \hat{y}.q$, where $q \in Q$ (actually q belongs to the subgroup $Q_0 \ltimes Q_1' \subset Q$, where Q_1' is the 1-dimensional subgroup of Q_1 normalized by Q_0). In particular $\mathrm{Ad}(q^{-1})(E_1^\perp + \mathfrak{q}) = E_1^\perp + \mathfrak{q}$. The relation $\exp_{\hat{y}'}(\mathrm{Ad}(q^{-1}).\mathcal{U}) = \exp_{\hat{y}}(\mathcal{U}).q$ is a consequence of the second axiom for ω part (2), and implies that the projections on M of $\exp_{\hat{y}'}(\mathrm{Ad}(q^{-1}).\mathcal{U})$ and $\exp_{\hat{y}}(\mathcal{U})$ are the same. Thus all plaques at y have the same germ, in the sense that if $\mathcal{P}_y$ and $\mathcal{P}_y'$ are two of them, then $\mathcal{P}_y \cap \mathcal{P}_y'$ is open in $\mathcal{P}_y$ and $\mathcal{P}_y'$.

Proposition 7.14 (see [31] Rem 6.17)**.** *Let Δ be a 1-dimensional Z_X-orbit. Let $\mathcal{P}_y$ be a plaque at $y \in \Delta$. If $\mathcal{O}$ is a 2-dimensional Z_X-orbit of Ω containing y in its closure, then $\mathcal{P}_y \cap \mathcal{O}$ has nonempty interior in $\mathcal{P}_y$.*

Proof: By theorem 3.3 (1), the orbit $\mathcal{O}$ is a semi-analytic set. We will use the following result, known as the "curve selecting lemma."

Lemma 7.15 (see [29] Sec 19, Prop 2)**.** *Let S be a semi-analytic subset of M, and $y \in \overline{S}$. Assuming y is not an isolated point of S, there exists an analytic arc $\gamma : [0,1) \to S$ extending continuously to 1, with $\gamma(1) = y$.*

This lemma applied to $S = \mathcal{O}$ provides a continuous path $\gamma : [0,1] \to M$, which is smooth on $[0,1)$, and satisfies $\gamma([0,1)) \subset \mathcal{O} \subset \Omega$, as well as $\gamma(1) = y$. Let $\hat{y} \in \pi^{-1}(y)$ with linear, unipotent holonomy. Pproposition 7.12 (2) yields a continuous lift $\hat{\gamma}$ of γ, such that $\hat{\gamma}$ is smooth on $[0,1)$, $\hat{\gamma}([0,1)) \subset \mathcal{R}$, and $\hat{\gamma}(1) = \hat{y}$.

Because $\gamma([0,1)) \subset \mathcal{O}$, the image $\hat{\gamma}((0,1))$ lies in an integral leaf of $\hat{\mathcal{D}}$ (this leaf is the preimage of $\mathcal{O}$ in $\mathcal{R}$). Let $\mathcal{U}$ be a neighborhood of 0 in $\mathfrak{g}$ sufficiently small that for every $t \in [0,1]$, the set $\mathcal{U}$ is mapped diffeomorphically by $\exp_{\hat{\gamma}(t)}$ onto its image. Set $\mathcal{V} = \mathcal{U} \cap (E_1^\perp + \mathfrak{q})$ and $\mathcal{L}_t = \exp_{\hat{\gamma}(t)}(\mathcal{V})$ for

$t \in [0,1]$. A suitable choice of $\mathcal{U}$ ensures that $\mathcal{V}$ is connected and that $\pi(\mathcal{L}_t)$ is a hypersurface for all t.

Lemma 7.16. *For t close enough to 1, $\mathcal{L}_t$ contains $\hat{y}$.*

Proof: For t_0 close enough to 1, the image $U = \exp_{\hat{\gamma}(t_0)}(\mathcal{U})$ contains $\hat{y}$. Let $(X_1, \dots, X_m)$ be a basis of $E_1^\perp + \mathfrak{q}$, and $\hat{X}_1, \dots, \hat{X}_m$ the corresponding vector fields on $\hat{M}$ with $\omega(\hat{X}_i) \equiv X_i \ \forall i$. Write

$$\hat{\gamma}'(t) = \sum_{i=1}^{m} a_i(t) \hat{X}_i(\hat{\gamma}(t)) \qquad \forall t \in (t_0 - \epsilon, 1)$$

where $a_i : (t_0 - \epsilon, 1) \to \mathbf{R}$ are some smooth functions. The ODE

$$\beta'(t) = \sum_{i=1}^{m} a_i(t) \hat{X}_i(\hat{\gamma}(t))$$

is defined on the open set $\exp_{\hat{\gamma}(t_0)}(\mathcal{U})$, but also on the submanifold $\mathcal{L}_{t_0}$. Uniqueness of solutions with the intial condition $\beta(t_0) = \hat{\gamma}(t_0)$ implies that $\hat{\gamma}([t_0, 1)) \subset \mathcal{L}_{t_0}$. Then $\hat{y} = \hat{\gamma}(1) \in \mathcal{L}_{t_0}$ because $\mathcal{L}_{t_0}$ is closed in $\exp_{\hat{\gamma}(t_0)}(\mathcal{U})$. $\diamondsuit$

A plaque $\mathcal{P}_y$ at y is, by definition, the projection of $\exp_{\hat{y}}(\mathcal{W})$, where $\mathcal{W}$ is a neighborhood of 0 in $E_1^\perp + \mathfrak{q}$. Now $\exp_{\hat{y}}(\mathcal{W})$ and $\mathcal{L}_{t_0}$ are two integral leaves of $\hat{\mathcal{D}}$ having a common point $\hat{y}$. It follows that the intersection $\exp_{\hat{y}}(\mathcal{W}) \cap \mathcal{L}_{t_0}$ is open in both $\exp_{\hat{y}}(\mathcal{W})$ and $\mathcal{L}_{t_0}$. Projecting to M gives an open subset of $\mathcal{P}_y \cap \mathcal{O}$. $\diamondsuit$

Corollary 7.17. *Let Δ be a 1-dimensional orbit of Z_X. The set of 2-dimensional Z_X-orbits in Ω containing Δ in their closure is at most countable. In particular, the set of 1-dimensional Z_X-orbits which are accumulated by 2-dimensional orbits is uncountable.*

Proof: Given $y \in \Delta$, the set of 2-dimensional Z_X-orbits $\mathcal{O} \subset \Omega$ with $\Delta \subset \overline{\mathcal{O}}$ coincides with the set of orbits $\mathcal{O}$ with $y \in \overline{\mathcal{O}}$. Denote this set of orbits $\mathcal{I}_y$. Let $\mathcal{P}_y$ be a plaque containing y. For every $\mathcal{O} \in \mathcal{I}_y$, the interior of $\mathcal{O} \cap \mathcal{P}_y$ is nonempty by proposition 7.14. Choose $U_\mathcal{O}$ a connected component of this interior. If $\mathcal{O} \neq \mathcal{O}'$ are two distinct orbits in $\mathcal{I}_y$, then $U_\mathcal{O} \cap U_{\mathcal{O}'} = \emptyset$. A collection of pairwise disjoint nonempty open subsets of $\mathcal{P}_y$ is at most countable, so $\mathcal{I}_y$ is also countable. $\diamondsuit$

7.6. **Stratification of the set of 1-dimensional orbits, conclusion.** Now we denote by Λ the complement of Ω. By proposition 7.6, Λ is a nowhere dense, analytic subset of M, comprising the points with 1-dimensional, closed, lightlike Z_X-orbits orbits. For $k \in \mathbf{N}$, we denote by $\Lambda^{(k)}$ the set of smooth points of dimension k in Λ—namely, the points contained in some

k-dimensional analytic submanifold of M contained in Λ. Observe that $\Lambda^{(k)} = \emptyset$ for every $k \geq 3$, since Λ is nowhere dense.

For a semi-analytic set Λ, the *dimension* of Λ equals the maximal integer k, denoted k_{max}, such that $\Lambda^{(k)} \neq \emptyset$. The set of singular points of Λ, denoted Λ_{sing}, is the complement $\Lambda \backslash \Lambda^{(k_{max})}$. By [29, Thm 4] (see also [3, Thm 7.2] and [3, Rem 7.3]), Λ_{sing} is a closed, semi-analytic subset of Λ, of dimension less than dim Λ.

Lemma 7.18. *The complement of* $\Lambda^{(2)}$ *in* Λ *is a finite union of* 1*-dimensional* Z_X*-orbits, possibly empty.*

Proof: If dim $\Lambda = 1$, then $\Lambda^{(2)} = \emptyset$, and Λ_{sing} is actually empty. Indeed, if $\Lambda_{sing} \neq \emptyset$, it is a closed semi-analytic set of dimension 0, namely, a finite number of points. These are all Z_X-fixed points because Λ_{sing} is Z_X-invariant. But corollary 4.11 rules out any Z_X-fixed points. Thus Λ is a compact, Z_X-invariant, analytic submanifold of M, which means it is a finite union of circular Z_X-orbits.

If dim $\Lambda = 2$, then Λ_{sing} is the complement of $\Lambda^{(2)}$ in Λ. If nonempty, it has dimension 1 or 0. Let Λ'_{sing} be the singular set of Λ_{sing}. If $\Lambda'_{sing} \neq \emptyset$, then dim $\Lambda'_{sing} = 0$, and, as above, it is a nonempty set of Z_X-fixed points, which leads to a contradiction. Therefore $\Lambda'_{sing} = \emptyset$, which means Λ_{sing} is a closed, 1-dimensional, Z_X-invariant, analytic submanifold of M, again a finite union of circular Z_X-orbits. $\diamondsuit$

Here is the key proposition leading to a contradiction.

Proposition 7.19 (see [31] Sec 6.4)**.** *Let* $y \in \Lambda^{(2)}$*. Then there exists a distinguished plaque* $\mathcal{P}_y$ *containing* y *and contained in* $\Lambda^{(2)}$*.*

Proof: We recall here the main points of the argument from [31] that Λ is covered by finitely-many plaques, with some simplifications for dimension 3. Denote by $\mathfrak{u}_+$ the unipotent subalgebra of $\mathfrak{g}_0$ annihilating E_1; it is 1-dimensional in our case. Let $\alpha : (-\epsilon, \epsilon) \to \Lambda$ be a differentiable path through y, which we can assume is contained in $\Lambda^{(2)}$. For each t, there is a point $\hat{\alpha}(t) \in \pi^{-1}(\alpha(t))$ with $\omega_{\hat{\alpha}(t)}(\mathfrak{z}_X) \subset \mathbf{R}E_1$ modulo $\mathfrak{p}$ and with $\mathfrak{u}_+ \subset \omega_{\hat{\alpha}(t)}(\mathfrak{z}_X)$, by proposition 7.6. In fact, there is a differentiable lift $\hat{\alpha}$ of α satisfying these conditions. Note that by proposition 7.12 (1), the image of $\hat{\alpha}$ is contained in $\overline{\mathcal{R}}$.

Because $\mathfrak{u}_+ \subset \omega_{\hat{\alpha}(t)}(\mathfrak{z}_X)$ for all t, the latter subspace, which is 2-dimensional, is abelian, a standard fact that can be deduced from part (2) of the second axiom for ω. Together with a generator U of $\mathfrak{u}_+$, it is spanned by an element of the form $E_1 + A + \xi$ with $A \in \mathfrak{a}$ and $\xi \in \mathfrak{q}'_1$, the centralizer of $\mathfrak{u}_+$ in $\mathfrak{p}^+$, which is contained in $\mathfrak{q}_1$. Note that centralizing $\mathfrak{u}_+$ means $A \in \ker \beta$. An argument similar to that in the proof of lemma 7.7 shows that if $\alpha(A) \neq 0$,

then an element of $\mathfrak{z}_X$ would be timelike somewhere (see also [31, Lem 6.13]), contradiciting proposition 7.6. Therefore $A = 0$, and $\omega_{\hat\alpha(t)}(\mathfrak{z}_X)$ is spanned by U and $E_1 + \xi$ for some $\xi \in \mathfrak{q}_1'$.

On the other hand, if $A_{t_0} = \omega(\hat\alpha'(t_0))$ were transverse to $E_1^\perp + \mathfrak{q}$, then calculations with the formula

$$\left.\frac{d}{dt}\right|_{t_0} \omega_{\hat\alpha(t)}(Z) = [A_{t_0}, \omega_{\hat\alpha(t_0)}(Z)] \qquad Z \in \mathfrak{z}_X$$

(see the end of [31, Sec. 6.4]) would give that $\omega_{\hat\alpha(t)}(\mathfrak{z}_X)$ is not contained in the subspace spanned by U and $E_1 + \xi$ for t close to t_0. From this contradiction, we conclude that $\hat\alpha'(t) \in \hat{\mathcal{D}}$ for all t. It follows that $\hat\alpha$ is contained in an integral leaf of $\hat{\mathcal{D}}$, and so α is contained in $\mathcal{P}_y$ for t near t_0. Varying α over paths through y in $\Lambda^{(2)}$ gives that a neighborhood of y in $\Lambda^{(2)}$ is contained in a plaque $\mathcal{P}_y$. Because both sets are smooth surfaces near y, we can shrink $\mathcal{P}_y$ so that it is contained in $\Lambda^{(2)}$. $\diamondsuit$

Proposition 7.19 together with proposition 7.14 shows that no 1-dimensional orbit of $\Lambda^{(2)}$ can be accumulated by a 2-dimensional orbit, because the latter is contained in Ω. By lemma 7.18, the complement of $\Lambda^{(2)}$ in Λ is a finite union of 1-dimensional orbits. We infer that only finitely many 1-dimensional orbits can be accumulated by 2-dimensional orbits. This is a contradiction to corollary 7.17.

8. Case $\mathfrak{z}_X$ is 1-dimensional

We are still considering, on a real-analytic Lorentzian manifold, a conformal Killing field X generating a 1-parameter group of conformal transformations $\{\varphi_X^t\}$. We assume here that there are no additional local Killing fields commuting with X, and we are going to prove:

Theorem 8.1. *Let (M, g) be a compact, real-analytic, Lorentzian manifold of dimension 3. Let $\{\varphi_X^t\}$ be a closed, noncompact 1-parameter group of conformal transformations of M. Suppose that $\mathfrak{z}_X = \mathbf{R}X$. Then (M, g) is conformally flat.*

Again, the proof is by contradiction, and we assume throughout this section that (M, g) is not conformally flat.

The 1-dimensional orbits of $Z_X = \{\varphi_X^t\}$ cannot be closed by theorem 3.2. Theorem 3.3 then ensures that the closure of every 1-dimensional orbit of Z_X contains some 0-dimensional orbits. Hence, the set of fixed points of Z_X is nonempty. Call it F_X.

Let x_0 be a point of F_X. By theorem 4.9, and because we are assuming M is not conformally flat, X is linearizable at x_0. Let $\hat x_0 \in \pi^{-1}(x_0)$ with $\omega_{\hat x_0}(X) = \hat X \in \mathfrak{g}_0$. Recall from theorem 3.2 that $(\hat I_X)_{\hat x_0}$ is algebraic, which means that it is closed under real Jordan decomposition (see, eg, [32, Thm

4.3.3]). Thus $\hat{X}$ is $\mathbf{R}$-semisimple, nilpotent, or elliptic in $\mathbf{R} \oplus \mathfrak{o}(1,2)$. There are no elliptic fixed points; indeed, this would contradict noncompactness of Z_X, since the isotropy monomorphism $\iota_{\hat{x}} : \mathrm{Is}^{loc}(x) \to P$ (see section 3.2) is proper for all $\hat{x}$.

Lemma 8.2. *The set of points for which the Z_X-orbit accumulates on F_X has Lebesgue measure zero.*

This immediately leads to a contradiction, because we already observed that points having a 1-dimensional Z_X-orbit do accumulate on F_X, and such points fill a dense open subset of M. Hence Theorem 8.1 follows directly from the lemma.

Proof: Consider a covering of F_X by finitely many open subsets $V_1, \ldots, V_s$, on which $\{\varphi_X^t\}$ is linearizable. More precisely, for each $j = 1, \ldots, s$, there is a diffeomorphism $h_j : V_j \to V$, with V the Euclidean unit ball in $\mathbf{R}^3$, conjugating the action of $\{\varphi_X^t\}\big|_{V_j}$, where defined, with that of a linear flow $\{\psi_j^t\} < \mathrm{CO}(1,2)$ on V. For each $j = 1, \ldots, s$, let

$$S_j^+ = \{y \in V_j : \ \exists\ T_j^+ > 0 \text{ such that } \varphi_X^t.y \in V_j\ \forall\, t > T_j^+\}$$

and

$$S_j^- = \{y \in V_j : \ \exists\ T_j^- < 0 \text{ such that } \varphi_X^t.y \in V_j\ \forall\, t < T_j^-\}.$$

The sets $h_j(S_j^\pm)$ will be denoted $\tilde{S}_j^\pm$.

If $\{\psi_j^t\}$ is a unipotent 1-parameter subgroup in $\mathrm{O}(1,2)$, then for every point $y \in V$ which is not a fixed point of $\{\psi_j^t\}$,

$$\lim_{t \to \pm\infty} ||\psi_j^t.y|| = \infty$$

(The norm is the Euclidean one). It follows that $\tilde{S}_j^\pm$ comprises the fixed points of $\{\psi_j^t\}$ in V, hence has Lebesgue measure zero. The same is true for $S_j^\pm$.

If $\{\psi_j^t\}$ is an $\mathbf{R}$-split 1-parameter subgroup of $G_0 \cong \mathrm{CO}(1,2)$, then, after conjugating in G_0, it is of the form

$$\psi_j^t : (x,y,z) \mapsto (e^{t(b-a)}x, e^{-at}y, e^{t(-a-b)}z) \text{ for some } a, b \in \mathbf{R}$$

If this subgroup were balanced or contracting—meaning monotone, unbounded sequences are balanced or contracting as in definition 4.2—, then (M,g) would be conformally flat by proposition 4.8. Hence, after possibly replacing X by $-X$, we may assume that $0 \leq a < b$. For points $y \in \tilde{S}_j^\pm$, $||\psi_j^t.y||$ does not tend to infinity as $t \to \pm\infty$, respectively. In particular, points of $\tilde{S}_j^+ \cup \tilde{S}_j^-$ satisfy $x = 0$ or $z = 0$, proving that $\tilde{S}_j^+ \cup \tilde{S}_j^-$ has Lebesgue measure zero. The same is thus true for $S_j^+ \cup S_j^-$.

For any 1-dimensional Z_X-orbit α, the closure $\overline{\alpha}$ is of the form $\overline{\alpha} = \{x_0\} \cup \alpha \cup \{x_1\}$, where x_0, x_1 are fixed points, not necessarily distinct. This is proved in [31, Lem 5.4], based on the reasonable topological properties of the orbit closure $\overline{\alpha}$ guaranteed by semi-analyticity as in theorem 3.3.

Thus any y for which $Z_X.y$ accumulates on a point $x_0 \in F_X$ satisfies $\lim_{t\to\infty} \varphi_X^t.y = x_0$ as $t \to \infty$ or $t \to -\infty$. It follows that the set of points accumulating on F_X can be written as the countable union $\bigcup_{j=1}^{s} \bigcup_{n\in\mathbf{N}} \varphi_X^{-n}(S_j^+) \cup \varphi_X^n(S_j^-)$, which has Lebesgue measure zero. The lemma follows. $\diamondsuit$

9. Appendix: Proof of theorem 4.12

This appendix is devoted to the proof of theorem 4.12. We will actually prove a more general statement, for smooth manifolds and sequences of local conformal transformations.

Theorem 9.1. *Let (M, g) be a smooth, compact, 3-dimensional Lorentzian manifold. Let $\{f_k\} \subset Conf^{loc}(M, [g])$ be an unbounded sequence defined on a common neighborhood U of $x \in M$. If $\{f_k\}$ admits a holonomy sequence at x contained in P^+, then there exists a nonempty open subset $U \subset M$ which is conformally flat.*

It is clear that theorem 4.12 follows directly from theorem 9.1

9.1. **Conformal geodesic segments.** The strategy to prove theorem 9.1 is to exhibit dynamical properties of the sequence $\{f_k\}$ which force conformal flatness on an open subset. The dynamical behavior of $\{f_k\}$ around a point x is understood via the action of its holonomy sequences on *conformal geodesics* in the model space, introduced below.

The 3-dimensional Minkowski space will be taken to have the quadratic form $q(x) := 2x_1x_3 + x_2^2$, and will be denoted by $\mathbf{R}^{1,2}$. A conformal immersion $j^o : \mathbf{R}^{1,2} \to \mathrm{Ein}^{1,2}$ is given in homogeneous coordinates on $\mathbf{RP}^4$ by the formula

$$j^o : x \mapsto \left[1 : x_1 : x_2 : x_3 : -\frac{q(x)}{2}\right],$$

mapping the origin in $\mathbf{R}^{1,2}$ to $o \in \mathrm{Ein}^{1,2}$. For us a *conformal geodesic segment* of $\mathrm{Ein}^{1,2}$ emanating from o will be a curve $\alpha : [0, 1] \to \mathrm{Ein}^{1,2}$ of the form

$$s \mapsto p.j^o(sw),$$

where $p \in P$ and $w \in \mathbf{R}^{1,2}$.

9.2. **Local dynamics via conformal geodesic segments.** Let $\{p_k\}$ be a holonomy sequence for $\{f_k\} \subset \mathrm{Conf}^{loc}(M)$ at $x \in M$. It is an unbounded sequence of conformal transformations of $\mathrm{Ein}^{1,2}$ fixing o, which in turn admits holonomy sequences at other points of $\mathrm{Ein}^{1,2}$. The following proposition,

borrowed from [17], explains how conformal geodesic segments relate holonomy sequences of $\{f_k\}$ and $\{p_k\}$.

Proposition 9.2 (see [17] Prop 6.3)**.** *Let (M, g) be a smooth Lorentzian manifold. Let $\{f_k\} < Conf(M, [g])$ with holonomy sequence $\{p_k\}$ along $\hat{x}_k \to \hat{x}$ in $\hat{M}$. Assume that there exists a conformal geodesic segment $\beta : [0, 1] \to \mathrm{Ein}^{1,n-1}$ emanating from o such that $\lim_{k\to\infty} p_k.[\beta] = o$. Then any pointwise holonomy sequence of $\{p_k\}$ at $\beta(1)$ admits a subsequence which is a holonomy sequence for $\{f_k\}$ with respect to some converging sequence $\hat{y}_k \to \hat{y}$ in $\hat{M}$.*

This proposition, together with proposition 4.8, brings us closer to theorem 9.1, through the following corollary:

Corollary 9.3. *Let (M, g) be a smooth, 3-dimensional, Lorentzian manifold. Let $\{p_k\}$ be a holonomy sequence for $\{f_k\}$ along $\hat{x}_k \to \hat{x}$ in $\hat{M}$. Suppose there exists a conformal geodesic segment $\beta : [0, 1] \to \mathrm{Ein}^{1,2}$ emanating from o such that $\lim_{k\to\infty} p_k.[\beta] = o$. If $\{p_k\}$ admits a pointwise holonomy sequence at $\beta(1)$ which is stable, then a nonempty open subset $U \subset M$ is conformally flat.*

Recall definition 4.2 for stable holonomy sequences.

9.3. **Lemma ensuring stable holonomy sequence.** Theorem 9.1 is a direct consequence of Corollary 9.3 and Lemma 9.4 below, which is the main technical result of this section.

Lemma 9.4. *Let $\{p_k\}$ be a sequence of P^+. After passing to a subsequence, there exists a conformal geodesic segment $\beta : [0, 1] \to \mathrm{Ein}^{1,2}$ emanating from o such that $\lim_{k\to\infty} p_k.[\beta] = o$, and such that $\{p_k\}$ admits a pointwise holonomy sequence at $\beta(1)$ which is stable.*

Proof: Denote the Euclidean norm on $\mathbf{R}^3$ by $\|\cdot\|$. Write

$$p_k = \begin{pmatrix} 1 & t_k v_k^* & -\frac{t_k^2 q(v_k)}{2} \\ 0 & I_3 & -t_k v_k \\ 0 & 0 & 1 \end{pmatrix} \in P^+ < \mathrm{O}(2, 3).$$

Here v_k is a sequence of $\mathbf{R}^{1,2}$ satisfying $\|v_k\| = 1$, and $t_k \geq 0$. The expression v_k^* stands for $v_k^t \mathbb{I}$, where $\mathbb{I}$ is as in section 4.1. Observe that $\{p_k\}$, hence $\{t_k\}$, is unbounded, because $\{f_k\}$ is unbounded. After taking a subsequence, we assume that $t_k \to \infty$, and that there is a vector $v = \lim v_k$.

Recall the conformal immersion $j^o : \mathbf{R}^{1,2} \to \mathrm{Ein}^{1,2}$ from section 9.1 above. Let $x_k \to x \in \mathbf{R}^{1,2}$. For $u \in [0, 1]$, and $k \in \mathbf{N}$, define $\beta_k(u) := j^o(ux_k)$, and $\beta(u) := j^o(ux)$. Observe that each β_k and β are conformal geodesic segments emanating from o. Define $x_k(u) := (j^o)^{-1}(p_k.\beta_k(u))$. From the matrix expression of p_k, and the formula for j^o,

(1)
$$x_k(u) = \left(1 + t_k u\langle x_k, v_k\rangle + \frac{t_k^2 u^2 q(v_k)q(x_k)}{4}\right)^{-1} \cdot \left(ux_k + \frac{t_k u^2 q(x_k)}{2} v_k\right).$$

After possibly taking a further subsequence of $\{p_k\}$, we may assume that $t_k|q(v_k)|$ converges in $[0,\infty]$. There are two subcases:

First case: $t_k|q(v_k)| \to \infty$. In this case, $q(v_k)$ is nonzero for k large enough, so that after perhaps taking a subsequence of $\{p_k\}$, we may assume that the sign of $q(v_k)$ is constant. Choose $\epsilon = \pm 1$ so that $\epsilon q(v_k) \geq 0$ for all sufficiently large k. Choose x such that

(a) $\epsilon q(x)$ is positive.
(b) $\langle x, v\rangle$ is positive.

Let $u \in (0,1]$ and write:

$$1+t_k u\langle x_k, v_k\rangle+\frac{t_k^2 u^2 q(v_k)q(x_k)}{4} = u^2 t_k^2 \epsilon q(v_k)\left(\frac{\epsilon q(x_k)}{4} + \frac{\langle x_k, v_k\rangle}{u t_k \epsilon q(v_k)} + \frac{1}{u^2 t_k^2 \epsilon q(v_k)}\right).$$

Under the assumption $t_k|q(v_k)| \to \infty$,

$$\lim_{k\to\infty}\left(\frac{\epsilon q(x_k)}{4} + \frac{\langle x_k, v_k\rangle}{u t_k \epsilon q(v_k)} + \frac{1}{u^2 t_k^2 \epsilon q(v_k)}\right) = \frac{\epsilon q(x)}{4},$$

so that for k big enough,

$$\left(\frac{\epsilon q(x_k)}{4} + \frac{\langle x_k, v_k\rangle}{u t_k \epsilon q(v_k)} + \frac{1}{u^2 t_k^2 \epsilon q(v_k)}\right) \geq \frac{\epsilon q(x_k)}{8}.$$

If $t_k u \geq 1$, we can infer from (1) and the previous inequalities that

$$\|x_k(u)\| \leq \frac{8}{t_k q(v_k) q(x_k)}\left(\|x_k\| + \frac{|q(x_k)|}{2}\right). \tag{2}$$

Observe that (2) also holds trivially if $u = 0$.

Suppose $t_k u \leq 1$, and note that conditions (a) and (b) on x imply that for k big enough,

$$1 + t_k u\langle x_k, v_k\rangle + \frac{t_k^2 u^2 q(v_k)q(x_k)}{4} \geq 1.$$

Then for k large enough,

$$\|x_k(u)\| \leq u\|x_k\| + \frac{u^2 t_k |q(x_k)|}{2} \leq \frac{1}{t_k}\left(\|x_k\| + \frac{|q(x_k)|}{2}\right) \tag{3}$$

From (3), we infer:

$$\lim_{k\to\infty} \sup_{u\in[0,1]} \|x_k(u)\| = 0. \tag{4}$$

Taking $x_k \equiv x$ gives $\lim_{k\to\infty} p_k([\beta]) = o$. Moreover, (4) shows that

$$\lim_{k\to\infty} p_k(j^o(x_k)) = o$$

for any $x_k \to x$. In particular, $\{p_k\}$ is stable at $j^o(x) = \beta(1)$ in the sense of [16, Def. 4.1]. It was proved in [16] Lemma 4.3, that if $\{p_k\}$ has this stability property at some point z, it admits a pointwise stable holonomy sequence at z. Lemma 9.4 is thus proved in this case.

Second case: $\lim_{k\to\infty} t_k q(v_k) = a \in \mathbf{R}$. This time, take $x \in \mathbf{R}^{1,2}$ such that:

(a) $q(x) = 0$.
(b) $\langle x, v\rangle > 0$

Let $u \in (0, 1]$, and write

$$1 + t_k u\langle x_k, v_k\rangle + \frac{t_k^2 u^2 q(v_k)q(x_k)}{4} = 1 + t_k u\left(\langle x_k, v_k\rangle + \frac{u}{4} t_k q(v_k)q(x_k)\right).$$

Because $\langle x_k, v_k\rangle + \frac{u}{4} t_k q(v_k)q(x_k)$ tends to $\langle x, v\rangle$ as $k \to \infty$,

$$1 + t_k u\langle x_k, v_k\rangle + \frac{t_k^2 u^2 q(v_k)q(x_k)}{4} \geq \frac{1}{2} t_k u\langle x, v\rangle \qquad \text{for } k \text{ large enough.}$$

We infer from (1) that:

$$\|x_k(u)\| \leq \frac{2}{\langle x, v\rangle}\left(\frac{\|x_k\|}{t_k} + \frac{|q(x_k)|}{2}\right). \tag{5}$$

This inequality holds trivially for $u = 0$, giving

$$\lim_{k\to\infty} \sup_{u\in[0,1]} \|x_k(u)\| = 0.$$

Taking $x_k \equiv x$ gives $\lim_{k\to\infty} p_k([\beta]) = o$. Moreover, if $\{x_k\}$ is any sequence converging to x, inequality (5) shows that $\lim_{k\to\infty} p_k(j^o(x_k)) = o$. As in the first case, this implies that $\{p_k\}$ admits a pointwise, stable holonomy sequence at $j^o(x) = \beta(1)$, and Lemma 9.4 is proved. $\diamondsuit$

References

1. A. M. Amores, *Vector fields of a finite type G-structure*, J. Diff. Geom. **14** (1979), no. 1, 1–16.
2. U. Bader and A. Nevo, *Conformal actions of simple Lie groups on compact pseudo-Riemannian manifolds*, J. Differential Geom. **60** (2002), no. 3, 355–387.
3. E. Bierstone and P.D. Milman, *Semianalytic and subanalytic sets*, IHES Publ. Math. **67** (1988), 5–42.
4. G. Calvaruso and O. Kowalski, *On the Ricci operator of locally homogeneous Lorentzian 3-manifolds*, Cent. Eur. J. Math. **7** (2009), no. 1, 124–139.
5. A. Čap and K. Melnick, *Essential Killing fields of parabolic geometries*, Indiana Univ. Math. J. **62** (2013), no. 6, 1917–1953.

6. A. Čap and J. Slovák, *Parabolic geometries I*, Mathematical Surveys and Monographs, vol. 154, American Mathematical Society, Providence, RI, 2009.
7. A. Coley, S. Hervik, and N. Pelavas, *Lorentzian spacetimes with constant curvature invariants in three dimensions*, Class. Quantum Grav. **25** (2008), no. 2.
8. G. D'Ambra, *Isometry groups of Lorentz manifolds*, Invent. Math. **92** (1988), no. 3, 555–565.
9. G. D'Ambra and M. Gromov, *Lectures on transformation groups: geometry and dynamics*, Surveys in Differential Geometry (Lehigh University, Bethlehem, PA, 1990) (Cambrige, MA), 1991, pp. 19–111.
10. S. Dumitrescu and A. Zeghib, *Géométries lorentziennes de dimension 3: classification et complétude*, Geom. Dedicata **149** (2010), 243–273.
11. L.P. Eisenhart, *Riemannian geometry*, 2nd ed., Princeton Univ. Press, 1949.
12. J. Ferrand, *Histoire de la réductibilité du groupe conforme des variétés riemanniennes (1964-1994)*, Séminaire de théorie spectrale et géométrie, vol. 17, Institut Fourier, 1998-99, pp. 9–25.
13. C. Frances, *Sur les variétés lorentziennes dont le groupe conforme est essentiel*, Math. Ann. **332** (2005), no. 1, 103–119.
14. ———, *Causal conformal vector fields, and singularities of twistor spinors*, Ann. Global Anal. Geom. **32** (2007), no. 3, 277–295.
15. ———, *Sur le groupe d'automorphismes des géométries paraboliques de rang 1*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 5, 741–764.
16. ———, *Dégénerescence locale des transformations conformes pseudo-riemanniennes*, Ann. Inst. Fourier **62** (2012), no. 5, 1627–1669.
17. ———, *Local dynamics of conformal vector fields*, Geom. Ded. **158** (2012), 35–59.
18. ———, *About pseudo-Riemannian Lichnerowicz conjecture*, Transform. Groups **20** (2015), no. 4, 1015–1022.
19. ———, *Lorentz dynamics on closed 3-manifolds*, arxiv.org/abs/1804.08695, 2018.
20. ———, *Variations on Gromov's open-dense orbit theorem*, Bull. Soc. Math. France **146** (2018), no. 4, 713–744.
21. C. Frances and K. Melnick, *Nilpotent groups of conformal flows on compact pseudo-Riemannian manifolds*, Duke Math. J. **153** (2010), no. 3, 511–550.
22. ———, *Formes normales pour les champs conformes pseudo-riemanniens*, Bull. Soc. Math. France **141** (2013), no. 3, 377–421.
23. C. Frances and A. Zeghib, *Some remarks on conformal pseudo-Riemannian actions of simple Lie groups*, Math. Res. Lett. **12** (2005), 49–56.
24. M. Gromov, *Rigid transformations groups*, Géométrie Différentielle (Paris, 1986) (D. Bernard and Y. Choquet-Bruhat, eds.), Hermann, Paris, 1988, pp. 65–139.
25. S. Kobayashi, *Transformation groups in differential geometry*, Springer, Berlin, 1995.
26. J. Lelong-Ferrand, *Transformations conformes et quasiconformes des variétés riemanniennes; application à la démonstration d'une conjecture de A. Lichnerowicz*, Acad. Roy. Belg. Cl. Sci. Mém. **39** (1971), no. 5, 5–44.
27. ———, *The action of conformal transformations on a Riemannian manifold*, Math. Ann. **304** (1996), 277–291.
28. A. Lichnerowicz, *Sur les transformations conformes d'une variété riemannieinne compacte*, C.R. Acad. Sci. Paris **259** (1964), 697–700.
29. S. Lojasiewicz, *Ensembles semi-analytiques*, IHES Lecture notes, 1965, https://perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf.

30. K. Melnick, *A Frobenius theorem for Cartan geometries, with applications*, Enseign. Math. Sér. II **57** (2011), no. 1-2, 57–89.
31. K. Melnick and V. Pecastaing, *The conformal group of a compact simply connected Lorentzian manifold*, J. Am. Math. Soc. **35** (2022), no. 1, 81–122.
32. D. Witte Morris, *Ratner's theorems on unipotent flows*, University of Chicago Press, Chicago, 2005.
33. M. Obata, *The conjectures on conformal transformations of Riemannian manifolds*, J. Differential Geom. **6** (1971), 247–258.
34. V. Pecastaing, *On two theorems about local automorphisms of geometric structures*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 1, 175–208.
35. ———, *Lorentzian manifolds with a conformal action of SL(2,**R**)*, Comentarii Mathematici Helvetici **93** (2018), no. 2, 401–439.
36. N. Rahmani and S. Rahmani, *Lorentz geometry of the Heisenberg group*, Geom.Dedicata **118** (2006), 133–140.
37. R. Schoen, *On the conformal and CR automorphism groups*, Geom. Funct. Anal. **5** (1995), no. 2, 464–481.
38. K. Sekigawa, *On some 3-dimensional curvature homogeneous spaces*, Tensor N.S. **31** (1977), 87–97.
39. R.W. Sharpe, *Differential geometry : Cartan's generalization of Klein's Erlangen program*, Springer, New York, 1996.
40. A. Zeghib, *Killing fields in compact Lorentz 3-manifolds*, J. Differential Geom. **43** (1996), 859–894.
41. ———, *Isometry groups and geodesic foliations of Lorentz manifolds I: Foundations of Lorentz dynamics*, Geom. Funct. Anal. **9** (1999), no. 4, 775–822.
42. R.J. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, Boston, 1984.
43. ———, *Split rank and semisimple automorphism groups of G-structures*, J. Differential Geom. **26** (1987), no. 1, 169–173.

Karin Melnick
Department of Mathematics
4176 Campus Drive
University of Maryland
College Park, MD 20742
USA
karin@math.umd.edu

Charles Frances
Institut de Recherche Mathématique Avancée
7 rue René-Descartes
Université de Strasbourg
67085 Strasbourg Cedex
France
frances@math.unistra.fr