# An Inverse Problem for Gibbs Fields with Hard Core Potential

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**Abstract**

It is well known that for a regular stable potential of pair interaction and a small value of activity one can define the corresponding Gibbs field (a measure on the space of configurations of points in $\mathbb{R}^d$).

In this paper we consider a converse problem. Namely, we show that for a sufficiently small constant $\overline{\rho}_1$ and a sufficiently small function $\overline{\rho}_2(x)$, $x \in \mathbb{R}^d$, that is equal to zero in a neighborhood of the origin, there exist a hard core pair potential, and a value of activity, such that $\overline{\rho}_1$ is the density and $\overline{\rho}_2$ is the pair correlation function of the corresponding Gibbs field.

**Key words**: Gibbs Field, Gibbs Measure, Cluster Functions, Pair Potential, Correlation Functions, Ursell Functions.

**MSC Classification**: 60G55, 60G60.

## 1 Introduction

Let us consider a translation invariant measure $\mu$ on the space of particle configurations on the space $\mathbb{R}^d$. An $m$-point correlation function $\rho_m(x_1, ..., x_m)$ is the probability density for finding $m$ different particles at locations $x_1, ..., x_m \in \mathbb{R}^d$. The following natural question has been extensively discussed in physical and mathematical literature: given $\rho_1(x_1) \equiv \overline{\rho}_1$ and $\rho_2(x_1, x_2) = \overline{\rho}_2(x_1 - x_2)$, does there exist a measure $\mu$, for which these are the first correlation function (density) and the pair correlation function, respectively?

In the series of papers [5]-[7] Lenard provided a set of relations on the functions $\rho_m$ which are necessary and sufficient for the existence of such a measure. However, given $\rho_1$

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and $\rho_2$, it is not clear how to check if there are some $\rho_3, \rho_4, ...$ for which these relations hold.

There are several recent papers which demonstrate the existence of particular types of point processes (measures on the space of particle configurations), which correspond to given $\rho_1$ and $\rho_2$ under certain conditions on $\rho_1$ and $\rho_2$. In particular, one dimensional point processes of renewal type are considered by Costin and Lebowitz in [3], while determinantal processes are considered by Soshnikov in [10]. In [1] Ambartzumian and Sukiasian prove the existence of a point process corresponding to a sufficiently small density and correlation function. Recently Costin and Lebowitz [3], and Caglioti, Kuna, Lebowitz, and Speer [2] provided various generalizations of their results. In [11] Stillinger and Torquato consider fields over a space with finitely many points. Besides, for the lattice model, Stillinger and Torquato discuss possible existence of a pair potential for a given density and correlation function using cluster expansion without addressing the issue of convergence.

In this paper we show that if $\rho_1$ and $\rho_2$ are small (in a certain sense), and $\rho_2$ is zero in a neighborhood of the origin, there exists a measure on the space of configurations for which $\rho_1$ is the density and $\rho_2$ is the pair correlation function. Moreover, this measure is the Gibbs measure corresponding to some hard core pair potential and some value of activity. In a sense, this is the converse of the classical statement that a given potential of pair interaction and a sufficiently small value of activity determine a translation invariant Gibbs measure on the space of particle configurations in $\mathbb{R}^d$ and the sequence of infinite volume correlation functions.

In our earlier paper [4] we obtained a similar result for lattice systems. In this paper we shall demonstrate that we can extend those results to particle systems in $\mathbb{R}^d$ if we assume that $\overline{\rho}_2$ is equal to zero in a neighborhood of the origin.

# 2 Notations and Formulation of the Result

Let $\Phi(x)$, $x \in \mathbb{R}^d$ be a hard core potential of pair interaction, that is $\Phi(x)$ is a measurable real-valued function for $|x| \geq R$, and $\Phi(x) = +\infty$ if $|x| < R$, where $R > 0$. Without loss of generality we may put $R = 1$. We assume that $\Phi(x) = \Phi(-x)$ for all $x$. Let $U(x_1, ..., x_n) = \sum_{1 \leq i < j \leq n} \Phi(x_i - x_j)$ be the total potential energy of the configuration $(x_1, ..., x_n)$.

We call two pair potentials equivalent if they are equal almost everywhere. When we say that an inequality involving $\Phi$ holds, it will mean that the inequality is true for some representative of the equivalence class.

A potential of pair interaction is said to be stable if there is a constant $c$ (and an element of the equivalence class of $\Phi$) such that

$$U(x_1, ..., x_n) \geq -nc \quad \text{for all } n \text{ and } x_1, ..., x_n \in \mathbb{R}^d.$$

A potential of pair interaction is said to be regular if it is essentially bounded from below and satisfies

$$\int_{\mathbb{R}^d} |\exp(-\Phi(x)) - 1| dx < +\infty.$$

When discussing the properties of $\Phi$, it will be convenient to consider the following two classes of sets. We shall say that a set $\mathcal{X}$ belongs to the class $\mathcal{C}$ if it consists of a finite number of points in $\mathbb{R}^d$, and has the property that $|x-y| \geq 1$ if $x, y \in \mathcal{X}, x \neq y$. We shall say that $\mathcal{X}$ belongs to the class $\mathcal{C}_0$ if it consists of a finite number of points in $\mathbb{R}^d \setminus B_1$, where $B_1$ is the open unit ball centered at the origin, and has the property that $|x-y| \geq 1$ if $x, y \in \mathcal{X}, x \neq y$.

For a measurable function $f$ defined on $\mathbb{R}^d$ we have the estimates

$$\int_{\mathbb{R}^d} |f(x)| dx \leq \mathrm{Vol}(B_1) \sup_{\mathcal{X} \in \mathcal{C}} \sum_{x \in \mathcal{X}} |f(x)|, \quad \int_{\mathbb{R}^d \setminus B_1} |f(x)| dx \leq \mathrm{Vol}(B_1) \sup_{\mathcal{X} \in \mathcal{C}_0} \sum_{x \in \mathcal{X}} |f(x)|. \quad (1)$$

Let us prove the first estimate. If $|f|$ is unbounded then the right-hand side is equal to $+\infty$, and the estimate follows. Assume that $|f|$ is bounded. Let $R, \varepsilon > 0$, and $B_R$ be the ball of radius $R$ centered at the origin. Let $x_1$ be a point in $B_R$ such that $|f(x_1)| \geq \sup_{x \in B_R} |f(x)| - \varepsilon/2$. Let $B(x_1)$ be the unit ball centered at $x_1$. Now we can inductively define points $x_2, ..., x_n \in B_R$ as follows. Let $x_k$, $2 \leq k \leq n$, be a point in $B_R \setminus (B(x_1) \cup ... \cup B(x_{k-1}))$ such that

$$|f(x_k)| \geq \sup_{x \in B_R \setminus (B(x_1) \cup ... \cup B(x_{k-1}))} |f(x)| - \varepsilon/2^k.$$

Here $n$ is such that the union of the unit balls $B(x_1), ..., B(x_n)$ covers $B_R$. Then the integral of $|f|$ over $B_R$ is estimated by $\mathrm{Vol}(B_1)((|f(x_1)| + ... + |f(x_n)| + \varepsilon)$. Since $R$ and $\varepsilon$ were arbitrary, this implies the first estimate in (1). The second estimate is completely similar.

Let $\Lambda$ be a finite subset of $\mathbb{R}^d$. The grand canonical ensemble is defined by a measure on $\bigcup_{n=0}^{\infty} \Lambda^n$, whose restriction on $\Lambda^n$ has the density

$$\nu(x_1, ..., x_n) = \frac{z^n}{n!} e^{-U(x_1, ..., x_n)} .$$

The parameter $z > 0$ is called the activity. The inverse temperature, which is the factor usually present in front of the function $U$, is set to be equal to one (or, equivalently, incorporated into the function $U$). The total mass of the measure is the grand partition function

$$\Xi(\Lambda, z, \Phi) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{(x_1, ..., x_n) \in \Lambda^n} e^{-U(x_1, ..., x_n)} .$$

The $m$-point correlation function is defined as the probability density for finding $m$ different particles at positions $x_1, ..., x_m \in \Lambda$,

$$\rho_m^{\Lambda}(x_1, ..., x_m) = \Xi(\Lambda, z, \Phi)^{-1} \sum_{n=0}^{\infty} \frac{z^{m+n}}{n!} \int_{(y_1, ..., y_n) \in \Lambda^n} e^{-U(x_1, ..., x_m, y_1, ..., y_n)} .$$

The corresponding measure on the space of all configurations of particles on the set $\Lambda$ (Gibbs measure) will be denoted by $\mu^{\Lambda}$. Given another set $\Lambda_0 \subseteq \Lambda$, we can consider the measure $\mu^{\Lambda}_{\Lambda_0}$ obtained as a restriction of the measure $\mu^{\Lambda}$ to the set of particle configurations on $\Lambda_0$.

Given a potential of pair interaction $\Phi(x)$, we define $g(x) = e^{-\Phi(x)} - 1$, $x \in \mathbb{R}^d$. We shall make the following three assumptions:

$$g(x) = g(-x) \geq -a > -1 \quad \text{almost surely} \quad \text{for} \ \ |x| \geq 1. \tag{2}$$

$$g(x) = -1 \quad \text{almost surely} \quad \text{for} \ \ |x| < 1. \tag{3}$$

For some element of the equivalence class of $g$ we have

$$\sup_{\mathcal{X} \in \mathcal{C}_0} \sum_{x \in \mathcal{X}} |g(x)| \leq c < +\infty. \tag{4}$$

Clearly, any function $g(x)$ which satisfies (2)-(3) defines a hard core potential of pair interaction via

$$\Phi(x) = -\ln(g(x) + 1) \ .$$

From (4) it easily follows that $\Phi$ is regular and stable. Indeed, it is regular due to (1), while the stability follows from

$$U(x_1, ..., x_n) = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} \ln(1 + g(x_j - x_i)) \geq$$

$$-\frac{1}{2} n \sup_{\mathcal{X} \in \mathcal{C}_0} \sum_{x \in \mathcal{X}} |g(x)| \geq -\frac{1}{2} nc \quad \text{ if } |x_i - x_j| \geq 1 \quad \text{for} \ \ i \neq j,$$

and

$$U(x_1, ..., x_n) = +\infty \quad \text{ if } |x_i - x_j| < 1 \quad \text{for} \ \ i \neq j.$$

It is well known ([9], [8]) that for stable regular pair potentials the following two limits exist for sufficiently small $z$ when $\Lambda \to \mathbb{R}^d$ in a suitable manner (for example, $\Lambda = [-k, k]^d$ and $k \to \infty$):
(a) There is a probability measure $\mu^{\mathbb{R}^d}$ on the space of all locally finite configurations on $\mathbb{R}^d$, such that

$$\mu^{\Lambda}_{\Lambda_0} \to \mu^{\mathbb{R}^d}_{\Lambda_0} \quad \text{as} \ \ \Lambda \to \mathbb{R}^d \tag{5}$$

for any finite set $\Lambda_0 \subset \mathbb{R}^d$.
(b) All the correlation functions converge to the infinite volume correlation functions. Namely,

$$\text{ess sup}_{x_1, ...x_m \in \Lambda_0} |\rho^{\Lambda}_m(x_1, ..., x_m) - \rho_m(x_1, ..., x_m)| \to 0 \quad \text{as} \ \ \Lambda \to \mathbb{Z}^d \tag{6}$$

for any finite set $\Lambda_0 \subset \mathbb{R}^d$. The infinite volume correlation functions $\rho_m$ are the probability densities for finding $m$ different particles at positions $x_1, ..., x_m \in \mathbb{R}^d$ corresponding to the measure $\mu^{\mathbb{R}^d}$.

To make these statements precise we formulate them as the following lemma (here we take into account that (2)-(4) imply that the potential is regular and stable).

**Lemma 2.1.** *([9], [8]) Assuming that (2)-(4) hold, there is a positive* $\overline{z} = \overline{z}(a, c)$*, such that (5) and (6) hold for all* $0 < z \le \overline{z}$ *when* $\Lambda = [-k, k]^d$ *and* $k \to \infty$.

Thus, a pair potential defines a sequence of infinite volume correlation functions for sufficiently small values of activity. Note that all the correlation functions are translation invariant,

$$\rho_m(x_1, ..., x_m) = \rho_m(x_1 + a, ..., x_m + a) \quad \text{for any } a \in \mathbb{R}^d.$$

Thus, $\rho_1$ is a constant, $\rho_2$ can be considered as a function of one variable, etc. Let $\overline{\rho}_m$ be the function of $m - 1$ variables, such that

$$\rho_m(x_1, ..., x_m) = \overline{\rho}_m(x_2 - x_1, ..., x_m - x_1) \ . \tag{7}$$

The main result of this paper is the following theorem.

**Theorem 2.2.** *Let* $0 < r < 1$ *be a constant. Let* $\overline{\rho}_1$ *be a constant, and let* $\overline{\rho}_2(x)$, $x \in \mathbb{R}^d$, *be a function that satisfies*

$$\overline{\rho}_2(x) = 0 \quad \text{for } |x| < 1; \quad \overline{\rho}_2(x) = \overline{\rho}_2(-x), \quad \textit{and}$$

$$\sup_{\mathcal{X} \in \mathcal{C}_0} \sum_{x \in \mathcal{X}} |\overline{\rho}_2(x) - \overline{\rho}_1^2| \le r \overline{\rho}_1^2.$$

*Then for all sufficiently small values of* $\overline{\rho}_1$ *there are a potential* $\Phi(x)$*, which satisfies (2)-(4), and a value of activity* $z$*, such that* $\overline{\rho}_1$ *and* $\overline{\rho}_2(x)$ *are the first and the second correlation functions, respectively, for the system defined by* $(z, \Phi)$.

**Remark.** As will seen from the proof of the theorem, the pair potential and the activity corresponding to given $\overline{\rho}_1$ and $\overline{\rho}_2(x)$ are unique, if we restrict consideration to sufficiently small values of $\Phi$ and $z$.

The outline of the proof is the following. In Sections 3 and 4, assuming that a pair potential and a value of the activity exist, we express the correlation functions (or, rather, the cluster functions, which are closely related to the correlation functions) in terms of the pair potential and the activity. This relationship can be viewed as an equation for unknown $\Phi$ and $z$. In Section 5 we use the contracting mapping principle to demonstrate that this equation has a solution. In Section 6 we provide the technical estimates needed to prove that the right hand side of the equation on $\Phi$ and $z$ is indeed a contraction.

# 3 Cluster Functions and Ursell Functions

In this section we shall obtain a useful expression for cluster functions in terms of the pair potential. The cluster functions are closely related to the correlation functions. Some of the general well-known facts will be stated in this section without proofs. The reader is referred to Chapter 4 of [9] for a more detailed exposition.

Let $A$ be the complex vector space of sequences $\psi$,

$$\psi = (\psi_m(x_1, ..., x_m))_{m \geq 0}$$

such that, for each $m \geq 1$, $\psi_m$ is an essentially bounded measurable function on $\mathbb{R}^{md}$, and $\psi_0$ is a complex number. It will be convenient to represent a finite sequence $(x_1, ..., x_m)$ by a single letter $X = (x_1, ..., x_m)$. We shall write

$$\psi(X) = \psi_m(x_1, ..., x_m).$$

Let now $\psi^1, \psi^2 \in A$. We define

$$\psi^1 * \psi^2(X) = \sum_{Y \subseteq X} \psi^1(Y)\psi^2(X \backslash Y),$$

where the summation is over all subsequences $Y$ of $X$ and $X \backslash Y$ is the subsequence of $X$ obtained by striking out the elements of $Y$ in $X$.

**Remark on Notation.** Let us stress that the inclusion $Y \subseteq X$ means here that $Y$ is a subsequence of $X$, rather than a simple set-theoretic inclusion. Below we shall also use the notation $X \cup Y$ for the sequence obtained by adjoining $X$ and $Y$.

Let $A_+$ be the subspace of $A$ formed by the elements $\psi$ such that $\psi_0 = 0$. Let $\mathbf{1}$ be the unit element of $A$ ( $\mathbf{1}_0 = 1, \mathbf{1}_m \equiv 0$ for $m \geq 1$).

We define the mapping $\Gamma$ of $A_+$ onto $\mathbf{1} + A_+$ :

$$\Gamma\varphi = \mathbf{1} + \varphi + \frac{\varphi * \varphi}{2!} + \frac{\varphi * \varphi * \varphi}{3!} + ...$$

The mapping $\Gamma$ has an inverse $\Gamma^{-1}$ on $\mathbf{1} + A_+$:

$$\Gamma^{-1}(\mathbf{1} + \varphi') = \varphi' - \frac{\varphi' * \varphi'}{2} + \frac{\varphi' * \varphi' * \varphi'}{3} - ...$$

It is easy to see that $\Gamma\varphi(X)$ is the sum of the products $\varphi(X_1)...\varphi(X_r)$ corresponding to all the partitions of $X$ into subsequences $X_1, ..., X_r$. If $\varphi \in A_+$ and $\psi = \Gamma\varphi$, the first few components of $\psi$ are

$$\psi_0 = 1; \quad \psi_1(x_1) = \varphi_1(x_1); \quad \psi_2(x_1, x_2) = \varphi_2(x_1, x_2) + \varphi_1(x_1)\varphi_1(x_2).$$

Let $\Phi$ be a pair correlation function which satisfies (2)-(4), and let $z \leq \overline{z}(a, c)$. Note that the sequence of correlation functions $\rho = (\rho_m)_{m \geq 0}$ (with $\rho_0 = 1$) is an element of $\mathbf{1} + A_+$.

**Definition 3.1.** *The cluster functions $\omega_m(x_1, ..., x_m)$, $m \geq 1$ are defined by*

$$\omega = \Gamma^{-1}\rho.$$

Thus,

$$\omega_1(x_1) = \rho_1(x_1); \quad \omega_2(x_1, x_2) = \rho_2(x_1, x_2) - \rho_1(x_1)\rho_1(x_2),$$

or, equivalently,

$$\overline{\omega}_1 = \overline{\rho}_1; \quad \overline{\omega}_2(x) = \overline{\rho}_2(x) - \overline{\rho}_1^2$$

where $\overline{\omega}_m$ are defined by

$$\omega_m(x_1, ..., x_m) = \overline{\omega}_m(x_2 - x_1, ..., x_m - x_1) \ .$$

Let $\psi \in \mathbf{1} + A_+$ be defined by

$$\psi_0 = 1; \quad \psi_m(x_1, ..., x_m) = e^{-U(x_1,...,x_m)} \ .$$

Define also

$$\varphi = \Gamma^{-1}\psi \ .$$

**Definition 3.2.** *The functions $\psi_m$ and $\varphi_m$ are called Boltzmann factors and Ursell functions, respectively.*

**Lemma 3.3.** *([9]) There is a positive constant $\overline{\overline{z}}(a, c)$ such that for all $z \leq \overline{\overline{z}}(a, c)$ the cluster functions can be expressed in terms of the Ursell functions as follows*

$$\omega_m(x_1, ..., x_m) = z^m \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{(y_1,...,y_n)\in\mathbb{R}^{nd}} \varphi_{m+n}(x_1, ..., x_m, y_1, ..., y_n) \ .$$

We shall later need certain estimates on the Ursell functions in terms of the potential. To this end we obtain a recurrence formula on a set of functions related to the Ursell functions. Given $X = (x_1, ..., x_m)$, we define the operator $D_X : A \to A$ by

$$(D_X\psi)_n(y_1, ..., y_n) = \psi_{m+n}(x_1, ..., x_m, y_1, ..., y_n) \ .$$

Then define

$$\widetilde{\varphi}_X = \psi^{-1} * D_X\psi \ ,$$

where $\psi$ is the sequence of Boltzmann factors, and $\psi^{-1}$ is such that $\psi^{-1} * \psi = \mathbf{1}$. It can be seen that

$$\varphi_{1+n}(x_1, y_1, ..., y_n) = \widetilde{\varphi}_{x_1}(y_1, ..., y_n) \tag{8}$$

and that the functions $\widetilde{\varphi}_X$ satisfy a certain recurrence relation, which we state here as a lemma.

**Lemma 3.4.** *([9]) Let $X = (x_1, ..., x_m)$, $Y = (y_1, ..., y_n)$, and $Z = (z_1, ..., z_s)$ be a generic subsequence of $Y$, whose length will be denoted by $|Z|$. The functions $\widetilde{\varphi}_X$ satisfy the following recurrence relation*

$$\widetilde{\varphi}_X(Y) = \exp(-\sum_{i=2}^{m} \Phi(x_i - x_1)) \sum_{Z\subseteq Y} \prod_{j=1}^{|Z|} (\exp(-\Phi(z_j - x_1)) - 1)\widetilde{\varphi}_{Z\cup X\backslash x_1}(Y\backslash Z) \ , \tag{9}$$

*where $m \geq 1$, $n \geq 0$, and $\widetilde{\varphi}_X(Y) = \mathbf{1}$ if $m = 0$.*

# 4 Equations Relating the Potential, the Activity, and the Cluster Functions

In this section we shall recast the main theorem in terms of the cluster functions and examine a system of equations, which relates the first two cluster functions with the pair potential and the activity.

First, Theorem 2.2 can clearly be re-formulated as follows

**Proposition 4.1.** *Let $0 < r < 1$ be a constant. Given any sufficiently small constant $\overline{\omega}_1$ and any function $\overline{\omega}_2(x)$, such that*

$$\overline{\omega}_2(x) = -\overline{\omega}_1^2 \quad \text{for } |x| < 1; \quad \overline{\omega}_2(x) = \overline{\omega}_2(-x), \quad \textit{and}$$

$$\sup_{\mathcal{X} \in \mathcal{C}_0} \sum_{x \in \mathcal{X}} |\overline{\omega}_2(x)| \le r \overline{\omega}_1^2,$$

*there are a potential $\Phi(x)$, which satisfies (2)-(4), and a value of activity $z$, such that $\overline{\omega}_1$ and $\overline{\omega}_2(x)$ are the first and the second cluster functions respectively for the system defined by $(z, \Phi)$.*

Consider the power expansions for $\omega_1$ and $\omega_2$, which are provided by Lemma 3.3. Let us single out the first term in both expansions. Note the translation invariance of the functions $\omega_m$ and $\varphi_m$ and the fact that $\varphi(x_1, x_2) = g(x_1 - x_2)$.

$$\overline{\omega}_1 = z + z^2 \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} \int_{(y_1,\ldots,y_n) \in \mathbb{R}^{nd}} \overline{\varphi}_{1+n}(y_1, \ldots, y_n) \ , \tag{10}$$

$$\overline{\omega}_2(x) = z^2 g(x) + z^3 \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} \int_{(y_1,\ldots,y_n) \in \mathbb{R}^{nd}} \overline{\varphi}_{2+n}(x, y_1, \ldots, y_n) \ , \tag{11}$$

where $\overline{\varphi}_m$ is the function of $m-1$ variables, such that

$$\varphi_m(x_1, \ldots, x_m) = \overline{\varphi}_m(x_2 - x_1, \ldots, x_m - x_1) \ .$$

Let

$$A(z, g) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} \int_{(y_1,\ldots,y_n) \in \mathbb{R}^{nd}} \overline{\varphi}_{1+n}(y_1, \ldots, y_n) \ ,$$

$$B(z, g)(x) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} \int_{(y_1,\ldots,y_n) \in \mathbb{R}^{nd}} \overline{\varphi}_{2+n}(x, y_1, \ldots, y_n) \ .$$

Thus equations (10) and (11) can be rewritten as follows

$$z = \overline{\omega}_1 - z^2 A(z, g) \ , \tag{12}$$

$$g = \frac{\overline{\omega}_2}{z^2} - zB(z,g) \ . \tag{13}$$

Instead of looking at (12)-(13) as a formula defining $\overline{\omega}_1$ and $\overline{\omega}_2$ by a given pair potential and the activity, we can instead consider the functions $\overline{\omega}_1$ and $\overline{\omega}_2$ fixed, and $g$ and $z$ unknown. Thus, Proposition 4.1 follows from the following.

**Proposition 4.2.** *If $\overline{\omega}_1$ and $\overline{\omega}_2$ satisfy the assumptions of Proposition 4.1, then the system (12)-(13) has a solution $(z,g)$, such that the function $g$ satisfies (2)-(4) and $z \leq \overline{z}(a,c)$.*

# 5 Proof of the Main Result

This section is devoted to the proof of Proposition 4.2. We shall need the following notations. Let $\mathcal{G}$ be the space of measurable functions $g$, which satisfy (2)-(4) with some $a, c < \infty$. Let

$$||g|| = \sup_{\mathcal{X} \in \mathcal{C}_0} \sum_{x \in \mathcal{X}} |g(x)|. \tag{14}$$

(To be more precise, the space $\mathcal{G}$ consists of equivalence classes - we do not distinguish between functions which are equal almost surely. We assume that the element of the equivalence class that minimizes the expression in the right hand side of (14) is used in the definition of $||g||$).

Note that $||g||$ is not a norm, since $\mathcal{G}$ is not a linear space, however $d(g_1, g_2) = ||g_1 - g_2||$ is a metric on the space $\mathcal{G}$. Let $\mathcal{G}_c$ be the set of elements of $\mathcal{G}$ for which $||g|| \leq c$. Note that if $c < 1$ then all elements of $\mathcal{G}_c$ satisfy (2) with $a = c$.

We also define $I_{z_0}^{a_1,a_2} = [a_1 z_0, a_2 z_0]$. Let $D = I_{z_0}^{a_1,a_2} \times \mathcal{G}_c$. Note that if $c < 1$ then $(z,g) \in D$ implies that $z \leq \min(\overline{z}(c,c), \overline{\overline{z}}(c,c))$ if $z_0$ is sufficiently small. Thus, the infinite volume correlation functions and cluster functions are correctly defined for $(z,g) \in D$ if $z_0$ is sufficiently small.

Let us define an operator $Q$ on the space of pairs $(z,g) \in D$ by $Q(z,g) = (z', g')$, where

$$z' = \overline{\omega}_1 - z^2 A(z,g) \ , \tag{15}$$

$$g'(x) = \frac{\overline{\omega}_2(x)}{z^2} - zB(z,g)(x) \quad \text{for} \ \ x \geq 1; \qquad g'(x) = -1 \quad \text{for} \ \ x < 1. \tag{16}$$

We shall prove the following lemma.

**Lemma 5.1.** *Let $0 < r < 1$ be a constant. There exist positive constants $a_1 < 1$, $a_2 > 1$, and $c < 1$ such that the equation $(z,g) = Q(z,g)$ has a solution $(z,g) \in D$ for all sufficiently small $z_0$ if*

$$\overline{\omega}_1 = z_0; \quad \overline{\omega}_2(x) = -\overline{\omega}_1^2 \quad \text{for } |x| < 1; \ \ \overline{\omega}_2(x) = \overline{\omega}_2(-x), \quad \textit{and}$$

$$\sup_{\mathcal{X} \in \mathcal{C}_0} \sum_{x \in \mathcal{X}} |\overline{\omega}_2(x)| \leq r \overline{\omega}_1^2.$$

Before we prove this lemma, let us verify that it implies Proposition 4.2. Let $0 < r < 1$ be fixed, and let $\overline{\omega}_1$ and $\overline{\omega}_2$ verify the assumptions of Lemma 5.1. Let $(z, g)$ be the solution of $(z, g) = Q(z, g)$, whose existence is guaranteed by Lemma 5.1. Let $\overline{\omega}_1'$ and $\overline{\omega}_2'$ be the first two cluster functions corresponding to the pair $(z, g)$. Note that $\overline{\omega}_1$ and $\overline{\omega}_1'$ satisfy the same equation

$$z = \overline{\omega}_1 - z^2 A(z, g) \ ; \qquad z = \overline{\omega}_1' - z^2 A(z, g) \ .$$

Therefore, $\overline{\omega}_1 = \overline{\omega}_1'$. The functions $\overline{\omega}_2$ and $\overline{\omega}_2'$ also satisfy the same equation

$$g(x) = \frac{\overline{\omega}_2(x)}{z^2} - zB(z, g)(x) \ ; \qquad g(x) = \frac{\overline{\omega}_2'(x)}{z^2} - zB(z, g)(x) \ ; \qquad \text{for} \ \ x \geq 1.$$

Thus, $\overline{\omega}_2(x) = \overline{\omega}_2'(x)$ for $x \geq 1$. The fact that $\overline{\omega}_2(x) = \overline{\omega}_2'(x)$ for $x < 1$ follows from

$$\overline{\omega}_2(x) = -\overline{\omega}_1^2 = -{\overline{\omega}_1'}^2 = \overline{\omega}_2'(x) \quad \text{for} \ \ x < 1.$$

Thus it remains to prove Lemma 5.1. The proof will be based on the fact that for small $z_0$ the operator $Q : D \to D$ is a contraction in an appropriate metric. Define

$$d_{z_0}(z_1, z_2) = \frac{h|z_1 - z_2|}{z_0} \ .$$

The value of the constant $h$ will be specified later. Now the metric on $D$ is given by

$$\rho((z_1, g_1), (z_2, g_2)) = d_{z_0}(z_1, z_2) + d(g_1, g_2) \ .$$

Lemma 5.1 clearly follows from the contracting mapping principle and the following lemma

**Lemma 5.2.** *Let $0 < r < 1$ be a constant. There exist positive constants $a_1 < 1$, $a_2 > 1$, and $c < 1$ such that for all sufficiently small $z_0$ the operator $Q$ acts from the domain $D$ into itself and is uniformly contracting in the metric $\rho$ for some value of $h > 0$, provided that*

$$\overline{\omega}_1 = z_0; \quad \overline{\omega}_2(x) = -\overline{\omega}_1^2 \quad \text{for } |x| < 1; \quad \overline{\omega}_2(x) = \overline{\omega}_2(-x), \quad \textit{and}$$

$$\sup_{\mathcal{X} \in \mathcal{C}_0} \sum_{x \in \mathcal{X}} |\overline{\omega}_2(x)| \leq r\overline{\omega}_1^2.$$

*Proof.* Take $c = \frac{r+2}{3}$, $a_1 = \sqrt{\frac{2r}{r+1}}$, $a_2 = 2$. We shall need certain estimates on the values of $A(z, g)$ and $B(z, g)$ for $(z, g) \in D$. Namely, there exist universal constants $u_1, ..., u_6$, such that for sufficiently small $z_0$ we have

$$\sup_{(z,g) \in D} |A(z, g)| \leq u_1 \ . \tag{17}$$

$$\sup_{(z,g) \in D} \sup_{\mathcal{X} \in \mathcal{C}_0} \sum_{x \in \mathcal{X}} |B(z, g)(x)| \leq u_2 \ . \tag{18}$$

$$\sup_{(z_1,g),(z_2,g)\in D} |A(z_1,g) - A(z_2,g)| \le u_3|z_1 - z_2| \ . \tag{19}$$

$$\sup_{(z,g_1),(z,g_2)\in D} |A(z,g_1) - A(z,g_2)| \le u_4 d(g_1,g_2) \ . \tag{20}$$

$$\sup_{(z_1,g),(z_2,g)\in D} \sup_{\mathcal{X}\in\mathcal{C}_0} \sum_{x\in\mathcal{X}} |B(z_1,g)(x) - B(z_2,g)(x)| \le u_5|z_1 - z_2| \ . \tag{21}$$

$$\sup_{(z,g_1),(z,g_2)\in D} \sup_{\mathcal{X}\in\mathcal{C}_0} \sum_{x\in\mathcal{X}} |B(z,g_1)(x) - B(z,g_2)(x)| \le u_6 d(g_1,g_2) \ . \tag{22}$$

These estimates follow from Lemma 6.1 below. For now, assuming that they are true, we continue with the proof of Lemma 5.2. The fact that $QD \subseteq D$ is guaranteed by the inequalities

$$z_0 + (a_2 z_0)^2 u_1 \le a_2 z_0 \ , \tag{23}$$

$$z_0 - (a_2 z_0)^2 u_1 \ge a_1 z_0 \ , \tag{24}$$

$$\frac{r z_0^2}{(a_1 z_0)^2} + a_2 z_0 u_2 \le c \ . \tag{25}$$

It is clear that (23)-(25) hold for sufficiently small $z_0$. Let us now demonstrate that for some $h$ and for all sufficiently small $z_0$ we have

$$\rho(Q(z_1,g_1), Q(z_2,g_2)) \le \frac{1}{2}\rho((z_1,g_1),(z_2,g_2)) \quad \text{if} \quad (z_1,g_1),(z_2,g_2) \in D. \tag{26}$$

First, taking (17), (19), and (20) into account, we note that

$$d_{z_0}(z_1^2 A(z_1,g_1), z_2^2 A(z_2,g_2)) \le d_{z_0}(z_1^2 A(z_1,g_1), z_2^2 A(z_1,g_1)) +$$
$$d_{z_0}(z_2^2 A(z_1,g_1), z_2^2 A(z_2,g_1)) + d_{z_0}(z_2^2 A(z_2,g_1), z_2^2 A(z_2,g_2)) \le$$
$$\frac{u_1 h |z_1^2 - z_2^2|}{z_0} + \frac{u_3 h (a_2 z_0)^2 |z_1 - z_2|}{z_0} + \frac{u_4 h (a_2 z_0)^2 d(g_1,g_2)}{z_0} \ .$$

If $h$ is fixed, the right hand side of this inequality can be estimated from above, for all sufficiently small $z_0$, by

$$\frac{1}{6}(d_{z_0}(z_1,z_2) + d(g_1,g_2)).$$

Similarly,

$$\sup_{\mathcal{X}\in\mathcal{C}_0} \sum_{x\in\mathcal{X}} |z_1 B(z_1,g_1)(x) - z_2 B(z_2,g_2)(x)| \le \sup_{\mathcal{X}\in\mathcal{C}_0} \sum_{x\in\mathcal{X}} |z_1 B(z_1,g_1)(x) - z_2 B(z_1,g_1)(x)| +$$
$$\sup_{\mathcal{X}\in\mathcal{C}_0} \sum_{x\in\mathcal{X}} |z_2 B(z_1,g_1)(x) - z_2 B(z_2,g_1)(x)| + \sup_{\mathcal{X}\in\mathcal{C}_0} \sum_{x\in\mathcal{X}} |z_2 B(z_2,g_1)(x) - z_2 B(z_2,g_2)(x)| \le$$
$$u_2|z_1 - z_2| + u_5 a_2 z_0 |z_1 - z_2| + u_6 a_2 z_0 d(g_1,g_2) \ .$$

Again, if $h$ is fixed, the right hand side of this inequality can be estimated from above, for all sufficiently small $z_0$, by

$$\frac{1}{6}(d_{z_0}(z_1, z_2) + d(g_1, g_2)).$$

Finally, since $r < 1$,

$$\sup_{\mathcal{X} \in \mathcal{C}_0} \sum_{x \in \mathcal{X}} |\frac{\overline{\omega}_2(x)}{z_1^2} - \frac{\overline{\omega}_2(x)}{z_2^2}| \leq r z_0^2 |\frac{1}{z_1^2} - \frac{1}{z_2^2}| \leq \frac{2a_2|z_1 - z_2|}{a_1^4 z_0} \ .$$

We can now take $h = \frac{12a_2}{a_1^4}$, which implies that the right hand side of the last inequality can be estimated from above by $\frac{1}{6}d_{z_0}(z_1, z_2)$. We have thus demonstrated the validity of (26), which means that the operator $Q$ is uniformly contracting. This completes the proof of the lemma. □

# 6 Estimates on the Ursell Functions

In this section we shall derive certain estimates on the Ursell functions, which, in particular, will imply the inequalities (17)-(22).

**Lemma 6.1.** *Suppose that the functions $g_1(x)$ and $g_2(x)$ satisfy (2)-(4) with $a = c < 1$. Let $\varphi^k = (\varphi^k_m(x_1, ..., x_m))_{m \geq 0}$, $k = 1, 2$ be the corresponding Ursell functions. Then there exist constants $q_1$ and $q_2$ such that*

$$\sup_{\mathcal{Y}_1, ..., \mathcal{Y}_n \in \mathcal{C}} \sum_{y_1 \in \mathcal{Y}_1 ... y_n \in \mathcal{Y}_n} |\overline{\varphi}^k_{1+n}(y_1, ..., y_n)| \leq n! q_1^{n+1} \ , \quad k = 1, 2,$$

$$\sup_{\mathcal{Y}_1, ..., \mathcal{Y}_n \in \mathcal{C}} \sum_{y_1 \in \mathcal{Y}_1 ... y_n \in \mathcal{Y}_n} |\overline{\varphi}^1_{1+n}(y_1, ..., y_n) - \overline{\varphi}^2_{1+n}(y_1, ..., y_n)| \leq n! q_2^{n+1} ||g_1 - g_2|| \ .$$

Note that the inequalities (17)-(22) immediately follow from this lemma, estimate (1), and the definitions of $A(z, g)$ and $B(z, g)(x)$.

Recall that in Section 3 we introduced the functions $\widetilde{\varphi}_X(Y)$, which were closely related to the Ursell functions. Given $g_1(x)$ and $g_2(x)$ which satisfy (2)-(4) with $a = c < 1$, we now define

$$r^k(m, n) = \sup_{(x_1, ..., x_m)} \sup_{\mathcal{Y}_1, ..., \mathcal{Y}_n \in \mathcal{C}} \sum_{y_1 \in \mathcal{Y}_1 ... y_n \in \mathcal{Y}_n} |\widetilde{\varphi}^k_{(x_1, ..., x_m)}(y_1, ..., y_n)| \ , \quad k = 1, 2,$$

$$d(m, n) = \sup_{(x_1, ..., x_m)} \sup_{\mathcal{Y}_1, ..., \mathcal{Y}_n \in \mathcal{C}} \sum_{y_1 \in \mathcal{Y}_1 ... y_n \in \mathcal{Y}_n} |\widetilde{\varphi}^1_{(x_1, ..., x_m)}(y_1, ..., y_n) - \widetilde{\varphi}^2_{(x_1, ..., x_m)}(y_1, ..., y_n)| \ .$$

We shall prove the following lemma.

**Lemma 6.2.** *Suppose that the functions $g_1(x)$ and $g_2(x)$ satisfy (2)-(4) with $a = c < 1$. Then there exist constants $q_1$ and $q_2$ such that*

$$r^k(m,n) \le n! q_1^{m+n} \ , \quad k = 1,2, \tag{27}$$

$$d(m,n) \le n! q_2^{m+n} ||g_1 - g_2|| \ . \tag{28}$$

Since we can express the Ursell functions in terms of $\widetilde{\varphi}_X(Y)$ via (8), Lemma 6.2 immediately implies Lemma 6.1. It remains to prove Lemma 6.2.

*Proof of Lemma 6.2.* The estimate (27) can be obtained in the same way as (4.27) of [9], and thus we shall not prove it here. We proceed with the proof of (28).

In the definition of $d(m,n)$ we can take the supremum over a restricted set of sequences $(x_1, ...x_m)$, namely those sequences, for which $|x_j - x_i| \ge 1$ if $i \ne j$. Indeed, if $|x_j - x_i| < 1$ for $i \ne j$, then $\widetilde{\varphi}^1_{(x_1,...,x_m)}(y_1,...,y_n) = \widetilde{\varphi}^2_{(x_1,...,x_m)}(y_1,...,y_n) = 0$, as follows from the definition of $\widetilde{\varphi}_X(Y)$.

Let $f_k(x) = e^{-\Phi_k(x)} = g_k(x) + 1$ , $k = 1,2$. For a sequence $X = (x_1,...,x_m)$, let

$$F_k^X = \prod_{i=2}^m |f_k(x_i - x_1)|, \quad k = 1,2,$$

For a sequence $Y = (y_1,...,y_s)$ and a point $x_1$, let

$$G_k^{Y,x_1} = \prod_{j=1}^s |g_k(y_j - x_1)|, \quad k = 1,2.$$

Note that

$$F_k^X = \exp(\sum_{i=2}^m \ln(g_k(x_i - x_1) + 1)) \le$$

$$\le \exp(\sum_{i=2}^m g_k(x_i - x_1)) \le e^c \quad \text{ if } |x_i - x_j| \ge 1 \text{ for } i \ne j. \tag{29}$$

If $\mathcal{Y}_1,...,\mathcal{Y}_s \in \mathcal{C}$ then

$$\sum_{y_1\in\mathcal{Y}_1...y_s\in\mathcal{Y}_s} G_k^{Y,x_1} = \prod_{j=1}^s \sum_{y\in\mathcal{Y}_j} |g_k(y - x_1)| \le (c_0 + c)^s \ , \tag{30}$$

where $c_0$ is the largest number of points separated by unit distance, which can fit inside a unit ball.

The proof of (28) will proceed via an induction on $m + n$. Assume that $x_1,...,x_m$ are separated by unit distance. From the recurrence relation (9) it follows that for $m \ge 1$

$$\sum_{y_1\in\mathcal{Y}_1...y_n\in\mathcal{Y}_n} |\widetilde{\varphi}^1_{(x_1,...,x_m)}(y_1,...,y_n) - \widetilde{\varphi}^2_{(x_1,...,x_m)}(y_1,...,y_n)| =$$

$$\sum_{y_1\in\mathcal{Y}_1\ldots y_n\in\mathcal{Y}_n} |\prod_{i=2}^{m} f_1(x_i-x_1)\sum_{Z\subseteq Y}\prod_{j=1}^{|Z|} g_1(z_j-x_1)\widetilde{\varphi}^1_{Z\cup X\backslash x_1}(Y\backslash Z)-$$

$$\prod_{i=2}^{m} f_2(x_i-x_1)\sum_{Z\subseteq Y}\prod_{j=1}^{|Z|} g_2(z_j-x_1)\widetilde{\varphi}^2_{Z\cup X\backslash x_1}(Y\backslash Z)| \le I_1+I_2,$$

where

$$I_1 = \sum_{y_1\in\mathcal{Y}_1\ldots y_n\in\mathcal{Y}_n}\sum_{Z\subseteq Y} |\prod_{i=2}^{m} f_1(x_i-x_1)\prod_{j=1}^{|Z|} g_1(z_j-x_1)(\widetilde{\varphi}^1_{Z\cup X\backslash x_1}(Y\backslash Z)-\widetilde{\varphi}^2_{Z\cup X\backslash x_1}(Y\backslash Z))|\ ,$$

$$I_2 = \sum_{y_1\in\mathcal{Y}_1\ldots y_n\in\mathcal{Y}_n}\sum_{Z\subseteq Y} |[\prod_{i=2}^{m} f_1(x_i-x_1)\prod_{j=1}^{|Z|} g_1(z_j-x_1)-$$

$$\prod_{i=2}^{m} f_2(x_i-x_1)\prod_{j=1}^{|Z|} g_2(z_j-x_1)]\widetilde{\varphi}^2_{Z\cup X\backslash x_1}(Y\backslash Z)|\ .$$

Note that there are $\frac{n!}{s!(n-s)!}$ subsequences $Z$ of the sequence $Y$, which are of length $s$. Rearranging the sum, so that to take it first over all possible values of $s$, and then over all possible subsequences of length $s$, we see that

$$I_1 \le \sum_{s=0}^{n}\sum_{(k_1,\ldots,k_s)\subseteq(1,\ldots,n)}\sum_{z_1\in\mathcal{Y}_{k_1}\ldots z_s\in\mathcal{Y}_{k_s}} |\prod_{i=2}^{m} f_1(x_i-x_1)\prod_{j=1}^{s} g_1(z_j-x_1)|d(m+s-1,n-s) \le$$

$$\sum_{s=0}^{n}\frac{n!}{s!(n-s)!}\max_{(k_1,\ldots,k_s)\subseteq(1,\ldots,n)}\sum_{z_1\in\mathcal{Y}_{k_1}\ldots z_s\in\mathcal{Y}_{k_s}} |\prod_{i=2}^{m} f_1(x_i-x_1)\prod_{j=1}^{s} g_1(z_j-x_1)|d(m+s-1,n-s) \le$$

$$\sum_{s=0}^{n}\frac{n!}{s!(n-s)!}e^c(c_0+c)^s d(m+s-1,n-s)\ .$$

Similarly,

$$I_2 \le \sum_{s=0}^{n}\frac{n!}{s!(n-s)!}\max_{(k_1,\ldots,k_s)\subseteq(1,\ldots,n)}\sum_{z_1\in\mathcal{Y}_{k_1}\ldots z_s\in\mathcal{Y}_{k_s}} |\prod_{i=2}^{m} f_1(x_i-x_1)\prod_{j=1}^{s} g_1(z_j-x_1)-$$

$$\prod_{i=2}^{m} f_2(x_i-x_1)\prod_{j=1}^{s} g_2(z_j-x_1)|r(m+s-1,n-s)\ .$$

Then,

$$\sum_{z_1\in\mathcal{Y}_{k_1}\ldots z_s\in\mathcal{Y}_{k_s}} |\prod_{i=2}^{m} f_1(x_i-x_1)\prod_{j=1}^{s} g_1(z_j-x_1)-\prod_{i=2}^{m} f_2(x_i-x_1)\prod_{j=1}^{s} g_2(z_j-x_1)| \le$$

$$\sum_{z_1\in\mathcal{Y}_{k_1}\dots z_s\in\mathcal{Y}_{k_s}}[|f_1(x_2-x_1)-f_2(x_2-x_1)|F_1^{(x_3,\dots,x_m)}G_1^{(z_1,\dots,z_s),x_1}+$$

$$F_2^{(x_2)}|f_1(x_3-x_1)-f_2(x_3-x_1)|F_1^{(x_4,\dots,x_m)}G_1^{(z_1,\dots,z_s),x_1}+\dots$$

$$\dots+F_2^{(x_2,\dots,x_{m-1})}|f_1(x_m-x_1)-f_2(x_m-x_1)|G_1^{(z_1,\dots,z_s),x_1}+$$

$$F_2^{(x_2,\dots,x_m)}|g_1(z_1-x_1)-g_2(z_1-x_1)|G_1^{(z_2,\dots,z_s),x_1}+\dots$$

$$\dots+F_2^{(x_2,\dots,x_m)}G_2^{(z_1,\dots,z_{s-1}),x_1}|g_1(z_s-x_1)-g_2(z_s-x_1)|].$$

There are $m+s$ terms inside the square brackets. In addition to (29) and (30), we use the fact that

$$|f_1(x_i-x_1)-f_2(x_i-x_1)|\le||g_1-g_2||\ ,\quad 2\le i\le m,$$

$$\sum_{z_j\in\mathcal{Y}_{k_j}}|g_1(z_j-x_1)-g_2(z_j-x_1)|\le||g_1-g_2||\ ,\quad 1\le j\le s.$$

Therefore, the entire sum can be estimated from above by

$$(m+s)e^{2c}(c_0+c)^s||g_1-g_2||.$$

Therefore,

$$I_2\le\sum_{s=0}^{n}\frac{n!}{s!(n-s)!}(m+s)e^{2c}(c_0+c)^s||g_1-g_2||r(m+s-1,n-s)\le$$

$$||g_1-g_2||(m+n)e^{2c}n!q_1^{m+n-1}\sum_{s=0}^{n}\frac{(c_0+c)^s}{s!}\le||g_1-g_2||(m+n)e^{c_0+3c}n!q_1^{m+n-1}\ .$$

Combining this with the estimate on $I_1$ we see that

$$d(m,n)\le\sum_{s=0}^{n}\frac{n!}{s!(n-s)!}e^c(c_0+c)^s d(m+s-1,n-s)+$$

$$||g_1-g_2||(m+n)e^{c_0+3c}n!q_1^{m+n-1}\ .$$

Let us use induction on $m+n$ to prove that

$$d(m,n)\le n!q_2^{m+n}||g_1-g_2||(m+n)\tag{31}$$

for some value of $q_2$. The statement is true when $m=0$ since $\widetilde{\varphi}=\mathbf{1}$ in this case. Assuming that the induction hypothesis holds for all $m',n'$ with $m'+n'\le m+n-1$, we obtain for $m\ge 1$

$$d(m,n)\le\sum_{s=0}^{n}\frac{n!}{s!(n-s)!}e^c(c_0+c)^s(n-s)!q_2^{m+n-1}||g_1-g_2||(m+n-1)+$$

$$||g_1 - g_2||(m+n)e^{c_0+3c}n!q_1^{m+n-1} \leq$$
$$||g_1 - g_2||(m+n)e^{c_0+3c}n!(q_1^{m+n-1} + q_2^{m+n-1}).$$

The expression in the right hand side of this inequality is estimated from above by the right hand side of (31) if $q_2 = 2e^{c_0+3c}\max(1, q_1)$. Thus, (31) holds for all $m, n$ with this choice of $q_2$. Note that we can get rid of the factor $(m+n)$ in the right hand side of (31) by taking a larger value of $q_2$. This completes the proof of (28) and of Lemma 6.2. □

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