Asymptotics for the Almost Sure Lyapunov Exponent for the Solution of the Parabolic Anderson Problem

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Abstract

We consider the Stochastic Partial Differential Equation

$$u_t = \kappa \Delta u + \xi(t,x)u, \quad t \geq 0, \quad x \in \mathbb{Z}^d \ .$$

The potential is assumed to be Gaussian white noise in time, stationary in space. We obtain the asymptotics of the almost sure Lyapunov exponent $\gamma(\kappa)$ for the solution as $\kappa \to 0$. Namely $\gamma(\kappa) \sim \frac{c_0}{\ln(1/\kappa)}$, where the constant c_0 is determined by the correlation function of the potential.

1 Introduction

We study the long time asymptotic behavior for the solution of the parabolic equation with random potential

$$u_t = \kappa \Delta u + \xi(t,x)u \tag{1}$$
$$t \geq 0, \quad x \in \mathbb{Z}^d$$

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with the initial data $u(0,x)$ concentrated at the point $x = 0$. The potential $\xi(t,x)$ is a mean zero stationary Gaussian field, which is white noise in time. The almost sure Lyapunov exponent of the solution is defined as the following limit

$$\gamma(\kappa) = \lim_{t\to\infty} \frac{\ln u(t,0)}{t} \ , \qquad (2)$$

provided the limit exists a.s. and is nonrandom. The existence of the Lyapunov exponent for the above type of potentials easily follows from the Kingman's subadditive ergodic theorem [4]. Indeed, the fundamental solution $q(s,x,t,y)$ satisfies

$$q(s,0,u,0) \geq q(s,0,t,0)q(t,0,u,0)$$

when $s < t < u$, while

$$\frac{\mathrm{E}\ln q(t,0,0,0)}{t} \leq \frac{\ln \mathrm{E}q(t,0,0,0)}{t} \ ,$$

which is bounded uniformly in t [2]. Thus, due to time stationarity of the potential we can apply the subadditive ergodic theorem to the process $\ln u(t,0)$, and thus prove the existence of the limit in (2). We are interested in the asymptotic behavior of the Lyapunov exponent $\gamma(\kappa)$ for small κ. In [2] the bounds on $\gamma(\kappa)$ were obtained with the lower bound being of order $\frac{1}{\ln(1/\kappa)}$, and the upper bound of order $\frac{\ln\ln(1/\kappa)}{\ln(1/\kappa)}$. In [3] the upper bound was improved to have the same order as the lower bound. In the present paper we obtain the asymptotics for the Lyapunov exponent. That is, instead of the upper and lower bounds with different constants, we show that a constant c_0 exists such that

$$\gamma(\kappa) \sim \frac{c_0}{\ln(1/\kappa)} \quad \text{as} \quad \kappa \to 0 \ .$$

We shall prove the following theorem:

Theorem 1.1 *Let $(W(t,x), t \geq 0, x \in \mathbb{Z}^d)$ be a mean zero Gaussian field with covariance $\mathrm{E}(W(t,x)W(s,y)) = \min(s,t)Q(x-y)$, where $Q(x)$ is not identically constant. Then there exists a constant $c_0 > 0$ such that the solution of the stochastic PDE*

$$du(t,x) = \kappa\Delta u(t,x)dt + u(t,x) \circ dW(t,x) \qquad (3)$$

with initial data $u(0,x) = \delta(x)$ satisfies almost surely

$$\lim_{t\to\infty} \frac{\ln u(t,0)}{t} = \gamma(\kappa) \sim \frac{c_0}{\ln 1/\kappa} \quad \text{as} \quad \kappa \to 0 \ .$$

The outline of the proof is the following: Via Feynman-Kac formula we represent the solution $u(t,0)$ as an integral over the set of paths of a continuous time random walk starting at 0 and ending at 0 at time t. The contribution from each path is a functional of a Gaussian random variable. We study a small subset of the set of all paths which gives the main contribution to the solution. In order to do that we need to study the structure of the maxima of the Gaussian field defined over the set of all paths of the random walk with a fixed number of jumps.

The paper is organized as follows. In section 2 we introduce the notations. In section 3 we prove the lower bound for the Lyapunov exponent, while in section 4 we prove the matching upper bound.

2 Notations and Preliminary Considerations

Let P_m be the set of paths of a discrete time random walk of length m starting and finishing at 0. Let $p(t,m)$ be the probability that at time t the Poisson process with intensity $2d\kappa$ is equal to m. Let $S(t,m)$ be the set of possible times of the jumps: $\widetilde{t} = (t_1, ..., t_m)$ with $0 \le t_1 \le ... \le t_m \le t$. For $\widetilde{t} \in S(t,m)$ and $\widetilde{x} = (x_0, x_1, ..., x_m) \in P_m$ we define

$$X_m(\widetilde{t}, \widetilde{x}) = W(t_1, x_0) - W(0, x_0) + W(t_2, x_1) - W(t_1, x_1) + ... + W(t, x_m) - W(t_m, x_m)\ . \quad (4)$$

By the Feynman-Kac formula the solution of equation 3 can be written as

$$u(t,0) = \sum_{m=0}^{\infty} p(t,m) \sum_{\widetilde{x} \in P_m} \frac{1}{(2d)^m} \int\limits_{\widetilde{t} \in S(t,m)} \frac{m!}{t^m} e^{X_m(\widetilde{t},\widetilde{x})} dt_1 ... dt_m =$$

$$\sum_{m=0}^{\infty} \kappa^m e^{-2d\kappa t} \sum_{\widetilde{x} \in P_m} \int\limits_{\widetilde{t} \in S(t,m)} e^{X_m(\widetilde{t},\widetilde{x})} dt_1 ... dt_m\ . \quad (5)$$

The proof of both lower and upper bounds will be a refinement of the corresponding considerations in [2] and [3]. The central idea of the lower estimation in [2] is the following: with the fixed times of the jumps $(t_1, ..., t_m)$ one selects a path, which gives a significant contribution to the integral in (5). The path is constructed as follows. With $(x_0, ..., x_{k-1})$ selected, one takes x_k to be the neighbor of x_{k-1}, which maximizes the increment $W(t_{k+1}, x_k) - W(t_k, x_k)$. Such a path follows the local maximum of the potential. Better estimations can be obtained by similar optimization with respect to pairs, triples, etc. of random jumps. In this paper, rather than maximizing the increments $W(t_{k+1}, x_k) - W(t_k, x_k)$ at each time step, we shall consider the maximum of the filed $X_m(\widetilde{t}, \widetilde{x})$ over all possible paths $\widetilde{x}$ and times of the jumps $\widetilde{t}$. We further find the optimal number of jumps as a function of t and κ.

The proof of the upper estimate is similar to that of [3]. However, now we are able to keep track of the constant, so that to make sure that the upper and lower bounds match, giving us the exact asymptotics for the Lyapunov Exponent.

For α nonnegative let $m_0(\alpha, t) = [\alpha t]$ (the integer part). Later α will be taken to be equal to $z \ln^{-2}(1/\kappa)$ where z will vary. It is the paths with the number of jumps of order $\ln^{-2}(1/\kappa) t$ that give the main contribution to the positive solution $u(t,x)$. Let us stress that for typical paths the number of jumps m_1 has the order κt, that is $m_0 >> m_1$.

Let T_α be the space of paths of the continuous time random walk starting and finishing at $x = 0$ with no more than $m_0(\alpha, t)$ jumps. Thus to specify an element of T_α we need to specify the number of jumps m, $\tilde{t} \in S(t, m)$, and $\tilde{x} \in P_m$. Then $X_m(\tilde{t}, \tilde{x})$ is a Gaussian field over T_α. Let

$$g(\alpha, t) = \sup_{(m,\tilde{t},\tilde{x}) \in T_\alpha} X_m(\tilde{t}, \tilde{x}) \ , \quad \text{and} \quad f(\alpha, t) = \mathrm{E} g(\alpha, t) \ . \tag{6}$$

Then $f(\alpha, t)$ is a superadditive nonrandom function of t. In order to see that $\lim_{t\to\infty} \frac{f(\alpha,t)}{t}$ exists we need to show that f does not grow faster than linearly in t. We shall use the following entropy estimate [1]: For a mean zero Gaussian field $X(\tau)$ over a set T one defines the canonical metric $d(\tau_1, \tau_2) = (\mathrm{E}(X(\tau_1) - X(\tau_2))^2)^{1/2}$. The metric entropy $N(\varepsilon)$ is the smallest number of closed d-balls of radius ε needed to cover T. One has the following theorem [1]

Theorem 2.1 *There exists a universal constant K such that*

$$\mathrm{E} \sup_T X(\tau) \leq K \int_0^\infty \sqrt{\ln N(\varepsilon)} d\varepsilon$$

provided that the RHS is finite.

Let us estimate the entropy function $N(\varepsilon)$ for the field $X_m(\tilde{t}, \tilde{x})$ defined over T_α. Below c_1, c_2, etc. denote constants, which depend only on the field $W(t, x)$, unless it is indicated otherwise. Note that the diameter of T_α does not exceed $2\sqrt{t}$, since $\mathrm{E}(X_m(\tilde{t}, \tilde{x}))^2 = t$. Assume that $\varepsilon \leq 2\sqrt{t}$. For $\tilde{t} = (t_1, ..., t_m)$ and $\tilde{s} = (s_1, ..., s_m)$, with m and $\tilde{x} \in P_m$ fixed, the distance between $(m, \tilde{t}, \tilde{x})$ and $(m, \tilde{s}, \tilde{x})$ in the canonical metric is estimated as follows:

$$d((m, \tilde{t}, \tilde{x}), (m, \tilde{s}, \tilde{x})) = \sqrt{\mathrm{E}(X_m(\tilde{t}, \tilde{x}) - X_m(\tilde{s}, \tilde{x}))^2} \leq c_1 \sqrt{\sum_{i=1}^m |t_i - s_i|} \ . \tag{7}$$

The inequality is the statement of Lemma 2.1 of [3], it can also be seen directly from (4). Consider the finite subset $U(t, m)$ of $S(t, m)$ defined as follows: $\tilde{t} = (t_1, ..., t_m) \in U(t, m)$ if $0 \leq t_1 \leq t_2 \leq ... \leq t_m \leq t$, and all t_i are positive integer multiples of $\frac{\varepsilon^2}{c_1^2 m}$. By (7) the elements $(m, \tilde{t}, \tilde{x})$ with $\tilde{t} \in U(t, m)$ form an ε-net in T_α. The number of elements in $U(t, m)$ is equal to $\frac{([\frac{c_1^2 tm}{\varepsilon^2}]+m-1)!}{([\frac{c_1^2 tm}{\varepsilon^2}]-1)!m!}$, which, for $\varepsilon \leq 2\sqrt{t}$ is estimated as follows

$$\frac{([\frac{c_1^2 tm}{\varepsilon^2}] + m - 1)!}{([\frac{c_1^2 tm}{\varepsilon^2}] - 1)!m!} \leq \frac{([\frac{c_1^2 tm}{\varepsilon^2}] + m - 1)^m}{m!} \leq (\frac{c_2 tm}{\varepsilon^2})^m \frac{1}{m!} \leq (\frac{c_3 t}{\varepsilon^2})^m \ ,$$

where the last inequality is due to Stirling's formula. Since there are less than $(2d)^m$ elements in P_m and $m \leq m_0$, the entropy is estimated as follows for $\varepsilon \leq 2\sqrt{t}$

$$N(\varepsilon) \leq \sum_{m=0}^{m_0} (2d)^m (\frac{c_3 t}{\varepsilon^2})^m \leq (\frac{c_4 t}{\varepsilon^2})^{m_0} \ .$$

Thus,

$$N(\varepsilon) \le \max\{1, (\frac{c_4 t}{\varepsilon^2})^{m_0}\} \ \text{ for all } \ \varepsilon \ . \tag{8}$$

Therefore

$$\int_0^\infty \sqrt{\ln N(\varepsilon)} d\varepsilon \le \sqrt{m_0} \int_0^{\sqrt{c_4 t}} \sqrt{\ln(\frac{c_4 t}{\varepsilon^2})} d\varepsilon = \sqrt{c_4 m_0 t} \int_0^1 \sqrt{\ln(\frac{1}{\varepsilon^2})} d\varepsilon \le c_5 \sqrt{m_0 t} \ .$$

The entropy estimate implies that

$$f(\alpha, t) \le c_6 \sqrt{m_0 t} = c_6 \sqrt{[\alpha t] t} \le c_7(\alpha) t \ .$$

Therefore the limit $\lim_{t\to\infty} \frac{f(\alpha,t)}{t}$ exists and is finite. It will be denoted by $F(\alpha)$. Due to the fact that $W(t,x)$ is a Wiener process in t, and is therefore self-similar,

$$\lim_{t\to\infty} \frac{1}{t} \mathrm{E} \sup_{T_\alpha} X_m(\tilde{t}, \tilde{x}) = \sqrt{\alpha} \lim_{t\to\infty} \frac{1}{t} \mathrm{E} \sup_{T_1} X_m(\tilde{t}, \tilde{x}) \ ,$$

which implies

$$F(\alpha) = \sqrt{\alpha} F(1) \ . \tag{9}$$

In order to relate the maximum of a 'typical' realization of a Gaussian field to the expectation of the maximum, controlled by the entropy estimate, one uses the Borell's inequality [1]

Theorem 2.2 *Let $X(\tau)$ be a centered Gaussian process with sample paths bounded almost surely. Then* $\mathrm{E} \sup_T X(\tau) < \infty$, *and for all* $\lambda > 0$

$$\mathrm{Prob}\{ | \sup_T X(\tau) - \mathrm{E} \sup_T X(\tau) | > \lambda \} \le 2 e^{-\frac{1}{2}\lambda^2/\sigma^2} \ , \tag{10}$$

where $\sigma^2 = \sup_T \mathrm{E} X^2(\tau)$.

We next make an intuitive argument providing us with the asymptotics for the Lyapunov exponent. The rigorous derivation will be given in the following two sections. We claim that the main contribution to the sum in the RHS of (5) comes from the terms with $m \sim z \ln^{-2}(1/\kappa) t$, where z does not depend on κ. Moreover, since we are taking the logarithm of u, only the factors κ^m and $e^{X_m(\tilde{t}, \tilde{x})}$ are important. We substitute the latter by $e^{F(z \ln^{-2}(1/\kappa))t}$. Taking the logarithm of the product of these two factors, that is ignoring the summation and integration in the RHS of (5), we arrive at the expression

$$\ln(\kappa^m e^{F(z \ln^{-2}(1/\kappa))t}) \ .$$

Dividing by t we obtain

$$\ln(\kappa^{\frac{z}{\ln^2(1/\kappa)}} e^{\frac{\sqrt{z}}{\ln(1/\kappa)} F(1)}) = \frac{F(1)\sqrt{z} - z}{\ln(1/\kappa)} \ .$$

The maximum of this function is equal to $\frac{F(1)^2}{4 \ln(1/\kappa)}$ and is achieved at $z = F(1)^2/4$. We shall prove that $\gamma(\kappa) \sim \frac{F(1)^2}{4 \ln(1/\kappa)}$. Note that $F(1)$ is greater than zero if $Q(x)$ is not identically constant.

3 Proof of the Lower Bound

We select c_0 (which is the same constant as in the statement of theorem 1.1) as follows: $c_0 = \frac{F(1)^2}{4}$. Let $\delta_0 > 0$ be given. In order to prove the estimate from below it is sufficient to show that there exists κ_0 such that

$$\gamma(\kappa) \geq \frac{c_0}{\ln(1/\kappa)}(1-\delta_0) \quad \text{for} \quad \kappa \leq \kappa_0 \ . \tag{11}$$

Let $\alpha(\kappa) = \frac{c_0}{\ln^2(1/\kappa)}$. Since we know that the Lyapunov exponent exists and is nonrandom in order to prove (11) it is sufficient to show that for $\kappa \leq \kappa_0$, for large enough t

$$\mathrm{Prob}\{\frac{1}{t}\ln(\sum_{m\leq[\alpha(\kappa)t]} \kappa^m e^{-2d\kappa t} \sum_{\widetilde{x}\in P_m} \int_{\widetilde{t}\in S(t,m)} e^{X_m(\widetilde{t},\widetilde{x})} dt_1...dt_m) \geq \frac{c_0(1-\delta_0)}{\ln(1/\kappa)}\} \geq 1/2 \ . \tag{12}$$

Since $\frac{m\ln\kappa}{t} \geq \frac{-c_0}{\ln(1/\kappa)}$ for $m \leq [\alpha(\kappa)t]$, the LHS of the first inequality in (12) can be estimated from below by the following expression

$$-\frac{c_0}{\ln(1/\kappa)} - 2d\kappa + \frac{1}{t}\ln(\sup_{m\leq[\alpha(\kappa)t]} \sup_{\widetilde{x}\in P_m} \int_{\widetilde{t}\in S(t,m)} e^{X_m(\widetilde{t},\widetilde{x})} dt_1...dt_m) \ .$$

Select κ_1 small enough so that

$$2d\kappa < \frac{c_0\delta_0}{2\ln(1/\kappa)} \quad \text{for} \quad \kappa \leq \kappa_1 \ . \tag{13}$$

Thus what we want to show is that for sufficiently small κ

$$\mathrm{Prob}\{\frac{1}{t}\ln(\sup_{m\leq[\alpha(\kappa)t]} \sup_{\widetilde{x}\in P_m} \int_{\widetilde{t}\in S(t,m)} e^{X_m(\widetilde{t},\widetilde{x})} dt_1...dt_m) \geq \frac{c_0}{\ln(1/\kappa)}(2-\frac{\delta_0}{2})\} \geq 1/2 \tag{14}$$

for large t.

We first consider the field $X_m(\widetilde{t},\widetilde{x})$ to be defined on T_α^r, a space of paths somewhat smaller than T_α. Namely T_α^r is the space of paths of the continuous time random walk starting and finishing at $x=0$ with no more than $m_0(\alpha,t) = [\alpha t]$ jumps, with the jumps separated from each other and from the endpoints of the interval $[0,t]$ by a distance of at least $2r$. Let

$$g^r(\alpha,t) = \sup_{T_\alpha^r} X_m(\widetilde{t},\widetilde{x}) \quad \text{and} \quad f^r(\alpha,t) = \mathrm{E}g^r(\alpha,t) \ .$$

Then $f^r(\alpha,t)$ is a superadditive nonrandom function of t and there exists the limit $F^r(\alpha) = \lim_{t\to\infty}\frac{f^r(\alpha,t)}{t}$. As in (9)

$$F^r(\alpha) = \sqrt{\alpha}F^{\alpha r}(1) \ .$$

We next demonstrate that $F^r(1) \to F(1)$ as $r \to 0$. Indeed, let an arbitrary $\varepsilon > 0$ be given. Select t_0 such that $F(1) - \frac{f(1,t_0)}{t_0} < \varepsilon/2$. Since $f^r(1,t_0) \to f(1,t_0)$ for fixed t_0, we can find r_0 such that $F(1) - \frac{f^r(1,t_0)}{t_0} < \varepsilon$ for $r \le r_0$. Note that $\frac{f^r(1,t)}{t}$ is increasing in t. Therefore $F(1) - \lim_{t\to\infty} \frac{f^r(1,t)}{t} < \varepsilon$ for $r \le r_0$. Thus $F(1) - F^r(1) < \varepsilon$ for $r \le r_0$, which proves our claim.

Now we estimate from below the supremum of the field $X_m(\tilde{t}, \tilde{x})$ over T^1_α.
Let $\delta_1 = \frac{\delta_0 \sqrt{c_0}}{8}$. Select κ_2 such that

$$F^{\alpha(\kappa)}(1) \ge F(1) - \frac{\delta_1}{2} \ \text{ for } \ \kappa \le \kappa_2 \ . \tag{15}$$

Then for $\kappa \le \kappa_2$ there exists $t_1 = t_1(\delta_1, \kappa)$, such that for $t \ge t_1$

$$\mathrm{E} \sup_{T^1_{\alpha(\kappa)}} X_m(\tilde{t}, \tilde{x}) \ge t(F^1(\alpha(\kappa)) - \frac{\sqrt{\alpha(\kappa)}\delta_1}{2}) =$$

$$t(\sqrt{\alpha(\kappa)} F^{\alpha(\kappa)}(1) - \frac{\sqrt{\alpha(\kappa)}\delta_1}{2}) \ge \sqrt{\alpha(\kappa)}t(F(1) - \delta_1) \ .$$

Then by Borell's inequality (10) with $\sigma^2 = t$ and $\lambda = \sqrt{\alpha(\kappa)}\delta_1 t$

$$\mathrm{Prob}\{\sup_{T^1_{\alpha(\kappa)}} X_m(\tilde{t}, \tilde{x}) < \sqrt{\alpha(\kappa)}t(F(1) - 2\delta_1)\} \le 2e^{\frac{-\delta_1^2 \alpha t}{2}} \le \frac{1}{4} \ , \tag{16}$$

where the last inequality holds for t sufficiently large. The fact that we can estimate the supremum of the field $X_m(\tilde{t}, \tilde{x})$ from below does not immediately allow us to prove (14). The problem is that the expression in the LHS of the first inequality in (14) involves integration in $t_1, ..., t_m$, and thus we need to study the fluctuations of the field $X_m(\tilde{t}, \tilde{x})$ as $\tilde{t}$ is varied. Consider the field

$$Y(\tilde{t_1}, \tilde{t_2}, m, \tilde{x}) = X_m(\tilde{t_1}, \tilde{x}) - X_m(\tilde{t_2}, \tilde{x})$$

defined on the set $U_{\alpha,\beta} \subset T_\alpha \times T_\alpha$. We say that $((\tilde{t_1}, m_1, \widetilde{x_1}), (\tilde{t_2}, m_2, \widetilde{x_2})) \in U_{\alpha,\beta}$ if and only if $m_1 = m_2 \le [\alpha t]$, $\widetilde{x_1} = \widetilde{x_2}$, and $|\tilde{t^i_1} - \tilde{t^i_2}| \le \beta$ for $1 \le i \le m_1$. Let

$$G(\alpha, \beta) = \limsup_{t\to\infty} \frac{1}{t} \mathrm{E} \sup_{U_{\alpha,\beta}} Y \ .$$

Then $G(\alpha, \beta) = \sqrt{\alpha} G(1, \alpha\beta)$. Below we are going to show that $G(1, \beta) \to 0$ as $\beta \to 0$. First, assuming that we have this statement, we prove (11). We take κ_3 so small that

$$G(1, \alpha(\kappa)) < \delta_1 \ \text{ for } \ \kappa \le \kappa_3 \ . \tag{17}$$

Then for t sufficiently large $\mathrm{E}\sup_{U_{\alpha(\kappa),1}} Y \leq \delta_1\sqrt{\alpha(\kappa)}t$. Thus by Borell's inequality with $\sigma^2 \leq 2t$ and $\lambda = \sqrt{\alpha(\kappa)}\delta_1 t$

$$\mathrm{Prob}\{\sup_{U_{\alpha(\kappa),1}} Y > 2\sqrt{\alpha(\kappa)}t\delta_1\} \leq 2e^{\frac{-\delta_1^2\alpha(\kappa)t}{4}} \leq \frac{1}{4} \tag{18}$$

where the last inequality holds for t sufficiently large. Take $\kappa_0 = \min\{\kappa_1, \kappa_2, \kappa_3\}$, so that (13), (15), and (17) hold.

From (18) and (16) it follows that for $\kappa \leq \kappa_0$ for t sufficiently large with probability of at least 1/2 one has

$$\sup_{T^1_{\alpha(\kappa)}} X > \sqrt{\alpha(\kappa)}t(F(1) - 2\delta_1) \text{ and at the same time } \sup_{U_{\alpha(\kappa),1}} Y < 2\sqrt{\alpha(\kappa)}t\delta_1 \ .$$

Therefore for large t on a set of probability of at least 1/2

$$\frac{1}{t}\ln(\sup_{m\leq[\alpha(\kappa)t]}\sup_{\widetilde{x}\in P_m}\int_{\widetilde{t}\in S(t,m)} e^{X_m(\widetilde{t},\widetilde{x})}dt_1...dt_m) \geq \frac{1}{t}\ln(e^{t\sqrt{\alpha(\kappa)}(F(1)-4\delta_1)}) = \sqrt{\alpha(\kappa)}(F(1)-4\delta_1). \tag{19}$$

Recall that $F(1) = 2\sqrt{c_0}$, $\sqrt{\alpha(\kappa)} = \frac{\sqrt{c_0}}{\ln(1/\kappa)}$, and $\delta_1 = \delta_0\sqrt{c_0}/8$. Thus the RHS of (19) is greater or equal than $\frac{c_0}{\ln(1/\kappa)}(2 - \frac{\delta_0}{2})$. Thus the estimate (14) holds. It remains to show that $G(1, \beta) \to 0$ as $\beta \to 0$.

By (8) the entropy function $N_X(\varepsilon)$ for the field $X_m(\widetilde{t}, \widetilde{x})$ defined over T_1 (that is with $m_0 = [t]$) is estimated as $N_X(\varepsilon) \leq \max\{1, (\frac{ct}{\varepsilon^2})^t\}$. The entropy function $N_Y(\varepsilon)$ for the field $Y(\widetilde{t}_1, \widetilde{t}_2, m, \widetilde{x})$ defined on the set $U_{1,\beta}$ is estimated as $N_Y(\varepsilon) \leq N_X^2(\varepsilon/2)$. The diameter of $U_{1,\beta}$ in the canonical metric associated with Y does not exceed $2\sup_{U_{1,\beta}}(\mathrm{E}Y^2)^{1/2} \leq 2\sqrt{\beta t}$. Thus for $\beta < c$ by the entropy estimate

$$\mathrm{E}\sup_{U_{1,\beta}} Y \leq K\int_0^{2\sqrt{\beta t}}\sqrt{\ln N_Y(\varepsilon)}d\varepsilon \leq K\int_0^{2\sqrt{\beta t}}\sqrt{\ln(\frac{4ct}{\varepsilon^2})^{2t}}d\varepsilon = 2\sqrt{2c}Kt\int_0^{\sqrt{\beta/c}}\sqrt{\ln(\frac{1}{\varepsilon^2})}d\varepsilon \ .$$

Then,

$$G(1, \beta) = \limsup_{t\to\infty}\frac{1}{t}\mathrm{E}\sup_{U_{1,\beta}} Y \leq 2\sqrt{2c}K\int_0^{\sqrt{\beta/c}}\sqrt{\ln(\frac{1}{\varepsilon^2})}d\varepsilon \ ,$$

which tends to 0 as $\beta \to 0$. This completes the proof of (11).

4 Proof of the Upper Bound

Let $\delta_0 > 0$ be given. In order to prove the estimate from above it is sufficient to show that there exists κ_0 such that

$$\gamma(\kappa) \leq \frac{c_0}{\ln(1/\kappa)}(1 + \delta_0) \text{ for } \kappa \leq \kappa_0 \ . \tag{20}$$

As in section 3 we ust the Feynman-Kac formula for the solution. Since we know that the Lyapunov exponent exists and is nonrandom in order to prove (20) it is sufficient to show that for $\kappa \le \kappa_0$ for large enough t

$$\mathrm{Prob}\{\frac{1}{t}\ln(\sum_{m=0}^{\infty}\kappa^m e^{-2d\kappa t}\sum_{\widetilde{x}\in P_m}\int_{\widetilde{t}\in S(t,m)} e^{X_m(\widetilde{t},\widetilde{x})}dt_1...dt_m) \le \frac{c_0(1+\delta_0)}{\ln(1/\kappa)}\} \ge 1/2 \ . \qquad (21)$$

The m summation in the LHS of the first inequality of (21) will be performed over disjoint intervals separately. For $\gamma > 0$ let us estimate the probability of the following event

$$A_{a_1,a_2}(\gamma) = \{\sum_{[\frac{a_1 t}{\ln^2(1/\kappa)}+1]\le m\le[\frac{a_2 t}{\ln^2(1/\kappa)}]}\kappa^m e^{-2d\kappa t}\sum_{\widetilde{x}\in P_m}\int_{\widetilde{t}\in S(t,m)} e^{X_m(\widetilde{t},\widetilde{x})}dt_1...dt_m > e^{\gamma t}\} \qquad (22)$$

Calculating the number of terms in each of the sums and the area of the domain of integration, we estimate the LHS of the inequality in (22) from above by

$$t(a_2-a_1)\ln^{-2}(1/\kappa)\kappa^{a_1 t}e^{-2d\kappa t}(2d)^{\frac{a_2 t}{\ln^2(1/\kappa)}}(\sup_{m\le[\frac{a_2 t}{\ln^2(1/\kappa)}]}\frac{t^m}{m!})\exp(\sup_{m\le[\frac{a_2 t}{\ln^2(1/\kappa)}]}X_m) \qquad (23)$$

Since $t(a_2-a_1)\ln^{-2}(1/\kappa) \le (2d)^{\frac{a_2 t}{\ln^2(1/\kappa)}}$ the logarithm of (23) is estimated from above by

$$\frac{2a_2 t}{\ln^2(1/\kappa)}\ln(2d) - a_1 t\ln(1/\kappa) - 2d\kappa t + \ln\sup_{m\le[\frac{a_2 t}{\ln^2(1/\kappa)}]}(\frac{t^m}{m!}) + \sup_{m\le[\frac{a_2 t}{\ln^2(1/\kappa)}]}X_m(\widetilde{t},\widetilde{x}), \qquad (24)$$

Let c_1, c_2, etc. denote constants which may depend only on the dimension d. From Stirling's formula it follows that

$$\ln\sup_{m\le[\frac{a_2 t}{\ln^2(1/\kappa)}]}(\frac{t^m}{m!}) \le \sup_{m\le[\frac{a_2 t}{\ln^2(1/\kappa)}]} m(\ln t - \ln m + c_1) \le \frac{c_2\max(1,a_2)t\ln(\ln^2(1/\kappa))}{\ln^2(1/\kappa)} \ .$$

Therefore the quantity in (24) is estimated from above by

$$\sup_{m\le[\frac{a_2 t}{\ln^2(1/\kappa)}]}X_m(\widetilde{t},\widetilde{x}) - a_1 t\ln(1/\kappa) + \frac{c_3\max(1,a_2)t\ln(\ln^2(1/\kappa))}{\ln^2(1/\kappa)} \ ,$$

Thus the probability of the event $A_{a_1,a_2}(\gamma)$ is estimated from above by

$$\mathrm{Prob}\{\sup_{m\le[\frac{a_2 t}{\ln^2(1/\kappa)}]}X_m(\widetilde{t},\widetilde{x}) - a_1 t\ln(1/\kappa) + \frac{c_3\max(1,a_2)t\ln(\ln^2(1/\kappa))}{\ln^2(1/\kappa)} > \gamma t\} =$$

$$\mathrm{Prob}\{\sup_{m\le[\frac{a_2 t}{\ln^2(1/\kappa)}]}X_m(\widetilde{t},\widetilde{x}) > t(\gamma + a_1\ln(1/\kappa) - \frac{o(\kappa)\max(1,a_2)}{\ln(1/\kappa)})\} \ . \qquad (25)$$

Recall that $\mathrm{E}\sup_{m\le[\frac{a_2 t}{\ln^2(1/\kappa)}]} X_m(\widetilde{t},\widetilde{x})$ is a superadditive function of t with the limit

$$\lim_{t\to\infty}\frac{\mathrm{E}\sup_{m\le[\frac{a_2 t}{\ln^2(1/\kappa)}]} X_m(\widetilde{t},\widetilde{x})}{t}=\frac{\sqrt{a_2}}{\ln(1/\kappa)}F(1)\ .$$

Thus $\mathrm{E}\sup_{m\le[\frac{a_2 t}{\ln^2(1/\kappa)}]} X_m(\widetilde{t},\widetilde{x})\le\frac{t\sqrt{a_2}}{\ln(1/\kappa)}F(1)$. Therefore by Borell's inequality (10) with $\sigma^2=t$ the probability in (25) is estimated from above by

$$2\exp(-\frac{1}{2}t(\gamma+a_1\ln(1/\kappa)-\frac{\sqrt{a_2}}{\ln(1/\kappa)}F(1)-\frac{o(\kappa)\max(1,a_2)}{\ln(1/\kappa)})^2) \tag{26}$$

Take

$$\gamma=\frac{c_0(1+\delta_0)}{\ln(1/\kappa)},\quad \text{and}\quad \gamma_n=\frac{c_0(1+\frac{\delta_0}{2})-n\varepsilon}{\ln(1/\kappa)},$$

where ε is a positive number to be selected below. We cover the axis $[0,\infty)$ by intervals

$$[a_1^n,a_2^n]=[(n-1)\varepsilon_1,n\varepsilon_1],\quad n\ge 1\ ,$$

where ε_1 is to be specified below. Then for large t

$$P(\sum_{m=0}^{\infty}\kappa^m e^{-2d\kappa t}\sum_{\widetilde{x}\in P_m}\int_{\widetilde{t}\in S(t,m)} e^{X_m(\widetilde{t},\widetilde{x})}dt_1...dt_m>e^{\gamma t})\le\sum_{n=1}^{\infty}P(A_{a_1^n,a_2^n}(\gamma_n)), \tag{27}$$

since

$$\sum_{n=1}^{\infty}e^{\gamma_n t}<e^{\gamma t}\quad\text{for large}\ \ t\ .$$

Each term in the RHS of (27) is estimated by the expression of the form (26). Thus, in order to demonstrate that (21) holds it is enough to show that

$$\sum_{n=1}^{\infty}2\exp(-\frac{1}{2}t(\gamma^n+\frac{a_1^n}{\ln(1/\kappa)}-\frac{\sqrt{a_2^n}}{\ln(1/\kappa)}F(1)-\frac{o(\kappa)\max(1,a_2^n)}{\ln(1/\kappa)})^2)<\frac{1}{2} \tag{28}$$

for large t. Recalling the definition of a_1^n,a_2^n, and γ^n we see that

$$\gamma^n+\frac{a_1^n}{\ln(1/\kappa)}-\frac{\sqrt{a_2^n}}{\ln(1/\kappa)}F(1)-\frac{o(\kappa)\max(1,a_2^n)}{\ln(1/\kappa)}=$$

$$\frac{1}{\ln(1/\kappa)}(c_0(1+\frac{\delta_0}{2})-n\varepsilon+n\varepsilon_1-2\sqrt{n\varepsilon_1 c_0}-\varepsilon_1-o(\kappa)\max(1,n\varepsilon_1))\ . \tag{29}$$

Since $c_0+z-2\sqrt{c_0 z}\ge 0$ for all z the expression in (29) is estimated from below by

$$\frac{1}{\ln(1/\kappa)}(c_0\frac{\delta_0}{2}-n\varepsilon+n\varepsilon_1-\varepsilon_1-o(\kappa)\max(1,n\varepsilon_1))\ .$$

Take such κ_0 that $o(\kappa) < \min(\frac{c_0\delta_0}{4}, \frac{1}{3})$ for $\kappa \leq \kappa_0$. Take $\varepsilon_1 = \frac{c_0\delta_0}{4}$ and $\varepsilon = \frac{\varepsilon_1}{3}$. Then the last expression is estimated from below by $\frac{1}{\ln(1/\kappa)} n \frac{c_0\delta_0}{12}$. Therefore the LHS of (28) is estimated from above by

$$\sum_{n=1}^{\infty} 2\exp(-\frac{t}{2\ln^2(1/\kappa)}(n\frac{c_0\delta_0}{12})^2) \ ,$$

which can be made arbitrarily small by selecting t large enough. This completes the proof of the upper bound.

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