

Branching diffusion in inhomogeneous media

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Abstract

We investigate the long-time evolution of branching diffusion processes (starting with a finite number of particles) in inhomogeneous media. The qualitative behavior of the processes depends on the intensity of the branching. In the super-critical regime, we describe the asymptotics of the number of particles in a given domain. In the sub-critical and critical regimes, we describe the distribution of the limiting number of particles.

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1 Introduction

Consider a collection of particles in \mathbb{R}^d that move diffusively and independently. Besides the diffusive motion, the particles can duplicate with the rate of duplication $\beta v(x)$, $x \in \mathbb{R}^d$, where x is the position of a given particle, v is a continuous non-negative compactly supported function and $\beta \geq 0$ is a parameter controlling the duplication rate. Both copies start moving independently immediately after the duplication.

We will give a detailed description of the behavior of the branching diffusion in \mathbb{R}^d in the case when v is a compactly supported function and t is large. The asymptotics depends on whether β is above, at, or below the critical value β_{cr} , which is the infimum of values of β for which the operator

$$L^\beta u(x) = \frac{1}{2} \Delta u(x) + \beta v(x) u(x) \tag{1}$$

has a positive eigenvalue. This is the operator in the right hand side of the equations on the particle density and higher order correlation functions, given below.

As follows from the results of [7] (see also [16], [3]), for $\beta > \beta_{\text{cr}}$ the number of particles in a given region U at time t has the asymptotics

$$n_t(U) \sim e^{\lambda_0(\beta)t} \xi \int_U \varphi(y) dy,$$

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where $\lambda_0(\beta)$ is the largest eigenvalue of L^β , ξ is a random variable that depends on the initial configuration of particles (assumed to be finite in number) and φ is a deterministic function (limiting density profile). Intuitively, the presence of the random variable ξ reflects the effect of branching at random times while the number of particles is small. After the number of particles becomes sufficiently large, it keeps growing nearly deterministically due to the fact that the bulk of particles is located near the support of v and due an ‘averaging’ effect in the branching mechanism. In this paper we will describe the distribution of ξ in terms of its moments.

When $\beta < \beta_{\text{cr}}$ (and $d \geq 3$), the effect of transience of the diffusion outweighs the branching, and all the particles will eventually wander off to infinity (assuming that initially there were finitely many particles). It will be shown that for $\beta < \beta_{\text{cr}}$ the total number of particles tends, as $t \rightarrow \infty$, to a finite random limit, whose distribution will be identified.

The case when $\beta = \beta_{\text{cr}}$ is interesting when $d \geq 3$ ($\beta_{\text{cr}} = 0$ for $d = 1, 2$). In this case the total number of particles will be shown to tend to a finite limit almost surely, although the expectation of the total number of particles tends to infinity.

Some of the corresponding results for the processes on the lattice with branching at the origin were obtained in [1], [4], [2], [17]. The main difficulty here, compared to the latter series of papers, is that the explicit formulas for the resolvent of the generator that were helpful in analyzing the processes with branching at the origin are not available now. Besides allowing the treatment of the general potential v , the techniques developed in this paper will allow us to study some of the more intricate properties of the limiting distribution: the fluctuations of the local number of particles conditioned on the total number of particles in the super-critical case, the growth of the region containing the particles and the distribution of the number of particles in the regime of large deviations (near the edge of the region containing the particles). These properties will be the subject of a subsequent paper.

The results on the large time asymptotics (Sections 4-6) will be obtained from the equations on the particle density and higher order correlation functions derived in Section 2. The analysis will be based on the spectral representation of solutions in the appropriate function spaces followed by the asymptotic analysis of integrals with integrands that depend on several parameters. Some of the techniques are related to those employed by us in the study of a polymer distribution in a mean field model ([6]). There, the asymptotics of a single equation (rather than a recursive system of equations) in the parameters t and β was examined.

Finally, let us mention a number of recent papers on the parabolic Anderson model, where v is a stationary random field (see, for example, [11], [12], [5], [10], [9], [13]). When v is random, the behavior of the solution to (2)-(3) essentially depends on nature of the tails of the distribution of v . It has been shown that in many cases the main contribution to the creation of the total number of particles is given by the isolated high peaks of the random potential. Moreover, when t is large, the bulk of the solution is located near one of those peaks with high probability. This adds to the importance of the study of

branching diffusions in the case when v is localized.

2 Equations on correlation functions

Let B_δ be a ball of radius δ in \mathbb{R}^d . For $t > 0$ and $x, y_1, y_2, \dots \in \mathbb{R}^d$ with all y_i distinct, define the particle density $\rho_1(t, x, y_1)$ and the higher order correlation functions $\rho_n(t, x, y_1, \dots, y_n)$ as the limits of probabilities of finding n distinct particles in $B_\delta(y_1), \dots, B_\delta(y_n)$, respectively, divided by $\text{Vol}^n(B_\delta)$, under the condition that there is a unique particle at $t = 0$ located at x . We extend $\rho_n(t, x, y_1, \dots, y_n)$ by continuity to allow for y_i which are not necessarily distinct. For fixed y_1 , the density satisfies the equation

$$\partial_t \rho_1(t, x, y_1) = \frac{1}{2} \Delta \rho_1(t, x, y_1) + \beta v(x) \rho_1(t, x, y_1), \quad (2)$$

$$\rho_1(0, x, y_1) = \delta_{y_1}(x). \quad (3)$$

Indeed, let $s, t > 0$. Then we can write

$$\rho_1(s+t, x, y_1) = (2\pi s)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-z|^2}{2s}} \rho_1(t, z, y_1) dz + \beta v(x) s \rho_1(t, x, y_1) + \alpha(s, t, x, y_1), \quad (4)$$

where the term with the integral on the right hand side is due to the effect of the diffusion on the interval $[0, s]$, the second term is due to the probability of branching on $[0, s]$, and α is the correction term. The correction term is present since (a) more than one instance of branching may occur before time s , and (b) even if a single branching occurs between the times 0 and s , then the original particle will be located not at x but at a nearby point and the intensity of branching there is slightly different from $\beta v(x)$. It is clear that $\lim_{s \downarrow 0} \sup_{x, y \in \mathbb{R}^d} \alpha(s, t, x, y) / s = 0$. After subtracting $\rho_1(t, x, y_1)$ from both sides of (4), dividing by s and taking the limit as $s \downarrow 0$, we obtain (2).

The equations on ρ_n , $n > 1$, are somewhat more complicated:

$$\partial_t \rho_n(t, x, y_1, \dots, y_n) = \frac{1}{2} \Delta \rho_n(t, x, y_1, \dots, y_n) + \beta v(x) (\rho_n(t, x, y_1, \dots, y_n) + H_n(t, x, y_1, \dots, y_n)), \quad (5)$$

$$\rho_n(0, x, y_1, \dots, y_n) \equiv 0. \quad (6)$$

Here

$$H_n(t, x, y_1, \dots, y_n) = \sum_{U \subset Y, U \neq \emptyset} \rho_{|U|}(t, x, U) \rho_{n-|U|}(t, x, Y \setminus U),$$

where $Y = (y_1, \dots, y_n)$, U is a proper non-empty subsequence of Y , and $|U|$ is the number of elements in this subsequence. Equation (5) is derived similarly to (2). The combinatorial term H_n appears after taking into account the event that there is a single branching on the time interval $[0, s]$, the descendants of the first particle are found at the points in U at time $s+t$, while the descendants of the second particle are found at the points of $Y \setminus U$, with the summation over all possible choices of U .

3 Analytic Properties of the Resolvent

Here we recall some basic facts about the operator $L^\beta : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ (see (1)) and its resolvent $R_\lambda^\beta = (L^\beta - \lambda)^{-1}$. We will assume that $v \geq 0$ is continuous, compactly supported and not identically equal to zero. It is well-known that the spectrum of L^β consists of the absolutely continuous part $(-\infty, 0]$ and at most a finite number of non-negative eigenvalues:

$$\sigma(L^\beta) = (-\infty, 0] \cup \{\lambda_j\}, \quad 0 \leq j \leq N, \quad \lambda_j = \lambda_j(\beta) \geq 0.$$

We enumerate the eigenvalues in the decreasing order. Thus, if $\{\lambda_j\} \neq \emptyset$, then $\lambda_0 = \max \lambda_j$. Thus the resolvent $R_\lambda^\beta : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is a meromorphic operator valued function on $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$.

Lemma 3.1. *There exists $\beta_{cr} \geq 0$ (which will be called the critical value of β) such that $\sup \sigma(L^\beta) = 0$ for $\beta \leq \beta_{cr}$ and $\sup \sigma(L^\beta) = \lambda_0(\beta) > 0$ for $\beta > \beta_{cr}$. For $\beta > \beta_{cr}$ the eigenvalue $\lambda_0(\beta)$ is a strictly increasing and continuous function of β . Moreover, $\lim_{\beta \downarrow \beta_{cr}} \lambda(\beta) = 0$ and $\lim_{\beta \uparrow \infty} \lambda(\beta) = \infty$.*

The proof of this lemma is standard (see Lemma 4.1 of [6]). Denote the kernel of R_λ^β by $R_\lambda^\beta(x, y)$. If $\beta = 0$, the kernel depends on the difference $x - y$ and will intermittently use the notations $R_\lambda^0(x, y)$ and $R_\lambda^0(x - y)$. The kernel $R_\lambda^0(x)$ can be expressed through the Hankel function $H_\nu^{(1)}$:

$$R_\lambda^0(x) = c_d k^{d-2} (k|x|)^{1-\frac{d}{2}} H_{\frac{d}{2}-1}^{(1)}(i\sqrt{2}k|x|), \quad k = \sqrt{\lambda}, \quad \text{Re} k > 0. \quad (7)$$

We shall say that $f \in C_{\text{exp}}(\mathbb{R}^d)$ (or simply C_{exp}) if f is continuous and

$$\|f\|_{C_{\text{exp}}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} (|f(x)|e^{|x|^2}) < \infty.$$

The space of bounded continuous functions on \mathbb{R}^d will be denoted by $C(\mathbb{R}^d)$ or simply C . The following lemma will be proved in the Appendix.

Lemma 3.2. *The operator $R_\lambda^\beta : C_{\text{exp}}(\mathbb{R}^d) \rightarrow C(\mathbb{R}^d)$ is meromorphic in $\lambda \in \mathbb{C}'$. Its poles are of the first order and are located at eigenvalues of the operator L^β . For each $\varepsilon > 0$ and some $\Lambda = \Lambda(\beta)$, the operator is uniformly bounded in $\lambda \in \mathbb{C}'$, $|\arg \lambda| \leq \pi - \varepsilon$, $|\lambda| \geq \Lambda$. It is of order $O(1/|\lambda|)$ as $\lambda \rightarrow \infty$, $|\arg \lambda| \leq \pi - \varepsilon$.*

If $d \geq 3$ and $\beta \in [0, \beta_{cr})$, then R_λ^β has the following asymptotic behavior as $\lambda \rightarrow 0$, $\lambda \in \mathbb{C}'$:

$$R_\lambda^\beta = Q_d^\beta \lambda^{\frac{d}{2}-1} + P_d^\beta(\lambda) + O(|\lambda|^{\frac{d}{2}}), \quad d \geq 3, \quad d - \text{odd}, \quad (8)$$

$$R_\lambda^\beta = Q_d^\beta \lambda^{\frac{d}{2}-1} \ln(1/\lambda) + P_d^\beta(\lambda) + O(|\lambda|^{\frac{d}{2}} |\ln \lambda| + |\lambda|^{d-2} |\ln \lambda|^2), \quad d \geq 4, \quad d - \text{even}, \quad (9)$$

where P_d^β are polynomials with coefficients that are bounded operators and Q_d^β are bounded operators.

The limit $P_d^\beta(0) = \lim_{\lambda \rightarrow 0, \lambda \in \mathbb{C}'} R_\lambda^\beta$ will be denoted by R_0^β . It is an operator acting from $C_{\text{exp}}(\mathbb{R}^d)$ to $C(\mathbb{R}^d)$.

It will be shown in the Appendix that if $\beta > \beta_{\text{cr}}$, then the eigenvalue $\lambda_0(\beta)$ of the operator L^β is simple and the corresponding eigenfunction does not change sign (and so can be assumed to be positive). From this and Lemma 3.2 it follows that the residue of R_λ^β at λ_0 is the integral operator with the kernel $\psi_\beta(x)\psi_\beta(y)$, where ψ_β is the positive eigenfunction normalized by the condition $\|\psi_\beta\|_{L^2(\mathbb{R}^d)} = 1$. Note that ψ_β decays exponentially at infinity. More precisely, it follows from (7) that if we write x as $(\theta, |x|)$ in polar coordinates, then there is a continuous function f_β such that

$$\psi_\beta(x) \sim f_\beta(\theta)|x|^{\frac{1-d}{2}} \exp(-\sqrt{2\lambda_0}|x|) \quad \text{as } |x| \rightarrow \infty. \quad (10)$$

If $\beta = \beta_{\text{cr}}$, then $\lambda_0 = 0$ might not be an eigenvalue of the operator L^β . As shown in Lemma 7.3, for $d \geq 3$ there is a unique (up to a multiplicative constant) positive function ψ_β (the ground state of L^β) which satisfies

$$L^\beta \psi_\beta = \frac{1}{2} \Delta \psi + \beta v \psi_\beta = 0, \quad \psi_\beta(x) = O(|x|^{2-d}), \quad \frac{\partial \psi_\beta}{\partial r}(x) = O(|x|^{1-d}) \quad \text{as } r = |x| \rightarrow \infty.$$

In fact, ψ_β is a genuine eigenvector (element of $L^2(\mathbb{R}^d)$) if and only if $d \geq 5$. We will normalize ψ_β by the condition that $\|\beta v \psi_\beta\|_{L^2(\mathbb{R}^d)} = 1$.

4 The super-critical case

Throughout this section we assume that $\beta > \beta_{\text{cr}}$. First, let us introduce some notations. Note that the dependence of some of the quantities below on β is not reflected in the notation in order to avoid overcrowded formulas. For a positive number x , we define the curve $\Gamma(x)$ in the complex plane as follows:

$$\Gamma(x) = \{\lambda : |\text{Im}\lambda| = \sqrt{4x(x - \text{Re}\lambda)}, \text{Re}\lambda \geq 0\} \cup \{\lambda : |\text{Im}\lambda| = 2x(1 - \text{Re}\lambda), \text{Re}\lambda \leq 0\}.$$

Thus $\Gamma(x)$ is a union of a piece of the parabola with the vertex in x that points in the direction of the negative real axis and two rays tangent to the parabola at the points it intersects the imaginary axis. The choice of the curve is somewhat arbitrary, yet the following properties of $\Gamma(x)$ will be important:

First, $\text{Re}\lambda \leq x$ for $\lambda \in \Gamma(x)$. Second, since the rays form a positive angle with the negative real semi-axis, we have $|\arg\lambda| \leq \pi - \varepsilon(x)$ for all $\lambda \in \Gamma(x)$ for some $\varepsilon(x) > 0$. Third, since the rays are tangent to the parabola, and the parabola is mapped into the line $\{\lambda : \text{Re}\lambda = \sqrt{x}\}$ by the mapping $\lambda \rightarrow \sqrt{\lambda}$, the image of the curve $\Gamma(x)$ under the same mapping lies in the half-plane $\{\lambda : \text{Re}\lambda \geq \sqrt{x}\}$.

The integration along the vertical lines in the complex plane and along contours $\Gamma(x)$, below, is performed in the direction of the increasing complex part.

We'll need estimates on the solutions of the following parabolic equation. Let

$$\partial_t \rho(t, x) = \frac{1}{2} \Delta \rho(t, x) + \beta v(x) \rho(t, x), \quad \rho(0, x) = g(x) \in C_{\text{exp}}. \quad (11)$$

We'll denote the Laplace transform of a function f by \tilde{f} ,

$$\tilde{f}(\lambda) = (\mathcal{L}f)(\lambda) = \int_0^\infty \exp(-\lambda t) f(t) dt.$$

Let r be the distance between λ_0 and the rest of the spectrum of the operator L^β . In the arguments that follow we'll use the symbol A to denote constants that may differ from line to line.

Lemma 4.1. *For each $\varepsilon \in (0, r)$, the solution of (11) has the form*

$$\rho(t, x) = \exp(\lambda_0 t) \langle \psi_\beta, g \rangle \psi_\beta(x) + q(t, x), \quad (12)$$

where

$$\|q(t, \cdot)\|_C \leq A(\varepsilon) \exp((\lambda_0 - \varepsilon)t) \|g\|_{C_{\text{exp}}}.$$

Proof. After the Laplace transform, the equation becomes

$$\left(\frac{1}{2} \Delta + \beta v\right) \tilde{\rho} - \lambda \tilde{\rho} = -g.$$

Thus, the solution ρ can be represented as

$$\rho(t, \cdot) = -\frac{1}{2\pi i} \int_{\text{Re}\lambda=\lambda_0+1} e^{\lambda t} R_\lambda^\beta g d\lambda. \quad (13)$$

The resolvent is meromorphic in the complex plane outside of the interval $(-\infty, \lambda_0 - r]$, with the only (simple) pole at λ_0 with the principal part of the Laurent expansion being the integral operator with the kernel $\psi_\beta(x)\psi_\beta(y)/(\lambda_0 - \lambda)$.

By Lemma 3.2, the norm of the R_λ^β does not exceed $A/|\lambda|$ near infinity to the right of $\Gamma(\lambda_0 - \varepsilon)$. Therefore, the same integral as in (13) but along the segment parallel to the real axis connecting a point $\lambda_0 + 1 + ib$ with the contour $\Gamma(\lambda_0 - \varepsilon)$ tends to zero when $b \rightarrow \infty$. Therefore, we can replace the contour of integration in (13) by $\Gamma(\lambda_0 - \varepsilon)$. The residue gives the main term, while the integral over $\Gamma(\lambda_0 - \varepsilon)$ gives the remainder term. \square

Lemma 4.2. *Let $K \subset \mathbb{R}^d$ be a compact set. For each $\varepsilon \in (0, r)$, the function $\rho_1(t, x, y)$ satisfies*

$$\rho_1(t, x, y) = \exp(\lambda_0 t) \psi_\beta(x) \psi_\beta(y) + q(t, x, y),$$

where

$$\sup_{x \in K} |q(t, x, y)| \leq A(\varepsilon) \exp((\lambda_0 - \varepsilon)t - |y| \sqrt{2(\lambda_0 - \varepsilon)})$$

for $t \geq 1/2$.

Proof. Let K' be a compact set that contains $\text{supp}(v) \cup K$ in its interior. Consider first the case when $y \in K'$. Apply (12) with t replaced by $t' = t - 1/2$ and $g = \rho_1(1/2, \cdot, y)$. In order to calculate the main term of the asymptotics, we note that $\|g\|_{C_{\text{exp}}}$ is bounded uniformly in $y \in K'$ and

$$\langle \psi_\beta, g \rangle = \exp\left(\frac{1}{2}\lambda_0\right)\psi_\beta(y).$$

The latter follows from

$$\begin{aligned} 0 &= \int_0^{1/2} \left\langle \left(\frac{\partial}{\partial t} + L^\beta\right)(\exp(-\lambda_0 t)\psi_\beta), \rho_1 \right\rangle dt = \\ &\langle \exp(-\lambda_0 t)\psi_\beta, \rho_1 \rangle|_{t=0}^{1/2} + \int_0^{1/2} \left\langle (\exp(-\lambda_0 t)\psi_\beta), \left(-\frac{\partial}{\partial t} + L^\beta\right)\rho_1 \right\rangle dt = \\ &\langle \exp\left(-\frac{1}{2}\lambda_0\right)\psi_\beta, \rho_1(1/2, \cdot, y) \rangle - \langle \psi_\beta, \rho_1(0, \cdot, y) \rangle = \\ &\exp\left(-\frac{1}{2}\lambda_0\right)\langle \psi_\beta, g \rangle - \psi_\beta(y). \end{aligned}$$

Therefore, (12) implies that

$$\rho_1(t, x, y) = \exp(\lambda_0 t)\psi_\beta(y)\psi_\beta(x) + \exp((\lambda_0 - \varepsilon)t)q(t, x, y),$$

where $\|q(t, \cdot, y)\|_C \leq A(K')$ for all $y \in K'$. It remains to consider the case when $y \notin K'$.

Let $u(t, x, y) = \rho_1(t, x, y) - p_0(t, x, y)$, where p_0 is the fundamental solution of the heat equation. Then u satisfies the non-homogeneous version of (11) with the right hand side $f = -\beta v(x)p_0(t, x, y)$ and $g \equiv 0$. Note that f is a smooth function since $y \notin K'$. Solving this equation for u using the Laplace transform, as in the proof of Lemma 4.1, we obtain

$$\begin{aligned} u(t, \cdot, y) &= -\frac{1}{2\pi i} \int_{\text{Re}\lambda=\lambda_0+1} e^{\lambda t} R_\lambda^\beta(-\beta v \tilde{p}_0(\lambda, \cdot, y)) d\lambda \\ &= -\frac{1}{2\pi i} \int_{\text{Re}\lambda=\lambda_0+1} e^{\lambda t} R_\lambda^\beta(\beta v R_\lambda^0(\cdot, y)) d\lambda \tag{14} \\ &= \exp(\lambda_0 t) \langle \psi_\beta, \beta v R_{\lambda_0}^0(\cdot, y) \rangle \psi_\beta - \frac{1}{2\pi i} \int_{\Gamma(\lambda_0-\varepsilon)} e^{\lambda t} R_\lambda^\beta(\beta v R_\lambda^0(\cdot, y)) d\lambda, \end{aligned}$$

where the first term on the right hand side is due to the residue at $\lambda = \lambda_0$. The first term can be re-written as

$$\begin{aligned} &\exp(\lambda_0 t) \langle \psi_\beta, \beta v R_{\lambda_0}^0(\cdot, y) \rangle \psi_\beta(x) = \\ &\exp(\lambda_0 t) (R_{\lambda_0}^0(\beta v \psi_\beta))(y) \psi_\beta(x) = -\exp(\lambda_0 t) \psi_\beta(y) \psi_\beta(x). \end{aligned}$$

The last equality here follows from the fact that ψ_β is an eigenfunction with eigenvalue λ_0 , that is

$$\left(\frac{1}{2}\Delta - \lambda_0\right)\psi_\beta = -\beta v \psi_\beta.$$

In order to estimate the second term on the right hand side of (14), we note that from (7) (see also (33)) it follows that

$$|R_\lambda^0(x, y)| \leq A(l) |\sqrt{\lambda}|^{\frac{d}{2}-\frac{3}{2}} |x - y|^{\frac{1}{2}-\frac{d}{2}} |\exp(-\sqrt{2\lambda}|y - x|)|$$

if $|\lambda|, |y - x| \geq l$. Thus

$$\|\beta v R_\lambda^0(\cdot, y)\|_{C_{\text{exp}}} \leq A(\varepsilon) |y|^{\frac{1}{2}-\frac{d}{2}} |\sqrt{\lambda}|^{\frac{d}{2}-\frac{3}{2}} \exp(-\sqrt{2(\lambda_0 - \varepsilon)}|y|)$$

for $y \notin K'$, $\lambda \in \Gamma(\lambda_0 - \varepsilon)$ due to the fact that $\text{Re}\sqrt{\lambda} \geq \sqrt{\lambda_0 - \varepsilon}$ for $\lambda \in \Gamma(\lambda_0 - \varepsilon)$ and $|y - x| \geq l$ for $x \in \text{supp}(v)$, $y \notin K'$.

Hence, using the estimate on the norm of $R_\lambda^\beta : C_{\text{exp}} \rightarrow C$ from Lemma 3.2, we obtain

$$\|R_\lambda^\beta(\beta v R_\lambda^0(\cdot, y))\|_C \leq A(\varepsilon) |\sqrt{\lambda}|^{\frac{d}{2}-\frac{5}{2}} \exp(-\sqrt{2(\lambda_0 - \varepsilon)}|y|), \quad \lambda \in \Gamma(\lambda_0 - \varepsilon).$$

Therefore, since $\text{Re}\lambda \leq \lambda_0 - \varepsilon$ for $\lambda \in \Gamma(\lambda_0 - \varepsilon)$ and the factor $e^{\lambda t}$ decays exponentially along $\Gamma(\lambda_0 - \varepsilon)$, the C -norm of the second term on the right hand side of (14) does not exceed $A(\varepsilon) \exp((\lambda_0 - \varepsilon)t - |y|\sqrt{2(\lambda_0 - \varepsilon)})$. The term $p_0(t, x, y)$ with $x \in K$, $y \notin K'$, $t \geq 1/2$, is estimated by the same expression, possibly with a different constant $A(\varepsilon)$. Indeed, if $t \geq 1/2$, then

$$p_0(t, x, y) \leq A \exp(-|y - x|^2/2t) \leq A \exp((\lambda_0 - \varepsilon)t - |y - x|\sqrt{2(\lambda_0 - \varepsilon)})$$

since

$$|y - x|^2/2t + (\lambda_0 - \varepsilon)t - |y - x|\sqrt{2(\lambda_0 - \varepsilon)} = (|y - x|/\sqrt{2t} - \sqrt{(\lambda_0 - \varepsilon)t})^2 \geq 0.$$

□

We'll need additional notations in order to describe the asymptotics of ρ_n with $n > 1$. Let $\alpha_\varepsilon^1(t, y) = \psi_\beta(y)$ and $\alpha_\varepsilon^2(t, y) = \exp(-\varepsilon t - |y|\sqrt{2(\lambda_0 - \varepsilon)})$. Consider all possible sequences $\sigma = (\sigma_1, \dots, \sigma_n)$ with $\sigma_i \in \{1, 2\}$. By $\Pi_\varepsilon^n(t, y_1, \dots, y_n)$ we denote the quantity

$$\Pi_\varepsilon^n(t, y_1, \dots, y_n) = \sup_{\sigma \neq (1, \dots, 1)} \alpha_\varepsilon^{\sigma_1}(t, y_1) \cdot \dots \cdot \alpha_\varepsilon^{\sigma_n}(t, y_n).$$

Let $P_t : C_{\text{exp}} \rightarrow C$ be the operator that maps the initial function g to the solution $\rho(t, \cdot)$ of equation (11). Let $P_t^0 g(x) = \exp(\lambda_0 t) \langle \psi_\beta, g \rangle \psi_\beta(x)$ and $P_t^1 = P_t - P_t^0$. Lemma 4.1 states that

$$\|P_t^1\| \leq A(\varepsilon) \exp((\lambda_0 - \varepsilon)t).$$

The particular form of P_t^0 then implies that

$$\|P_t\| \leq \|P_t^0\| + \|P_t^1\| \leq A' \exp(\lambda_0 t). \quad (15)$$

For $g \in C_{\text{exp}}$ and $n \geq 2$, we denote

$$I_n(g) := R_{n\lambda_0}^\beta g = \int_0^\infty \exp(-n\lambda_0 s) P_s g ds \in C.$$

Note that

$$\begin{aligned} \int_0^t \exp(n\lambda_0 s) P_{t-s} g ds &= \exp(n\lambda_0 t) \int_0^t \exp(-n\lambda_0 s) P_s g ds \\ &= \exp(n\lambda_0 t) (I_n(g) + O(\exp(-(n-1)\lambda_0 t))) \quad \text{as } t \rightarrow \infty. \end{aligned} \quad (16)$$

The functions f_1, f_2, \dots are defined inductively: $f_1 = \psi_\beta$ and

$$f_n = \beta \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} I_n(v f_k f_{n-k}), \quad n \geq 2. \quad (17)$$

Lemma 4.3. *Let $K \subset \mathbb{R}^d$ be a compact set. For each $\varepsilon \in (0, r)$, the function ρ_n satisfies*

$$\rho_n(t, x, y_1, \dots, y_n) = \exp(n\lambda_0 t) f_n(x) \psi_\beta(y_1) \cdot \dots \cdot \psi_\beta(y_n) + q_n(t, x, y_1, \dots, y_n), \quad (18)$$

where

$$\sup_{x \in K} |q_n(t, x, y_1, \dots, y_n)| \leq A_n(\varepsilon) \exp(n\lambda_0 t) \Pi_\varepsilon^n(t, y_1, \dots, y_n) \quad (19)$$

for $t \geq 1/2$.

Proof. For $n = 1$, the relation (18) coincides with the statement of Lemma 4.2. Let us assume that (18) holds for all natural numbers up to and including $n - 1$. A generic subsequence $U \subset Y = (y_1, \dots, y_n)$ will be written as $U = (z_1, \dots, z_{|U|})$ and its complement as $Y \setminus U = (\bar{z}_1, \dots, \bar{z}_{n-|U|})$. By the Duhamel principle applied to the equation for ρ_n , we obtain

$$\begin{aligned} \rho_n(t, \cdot, y_1, \dots, y_n) &= \int_0^t P_{t-s} (\beta v \sum_{U \subset Y, U \neq \emptyset} \rho_{|U|}(s, \cdot, z_1, \dots, z_{|U|}) \rho_{n-|U|}(s, \cdot, \bar{z}_1, \dots, \bar{z}_{n-|U|})) ds \\ &= \int_0^t P_{t-s} (\beta v \sum_{U \subset Y, U \neq \emptyset} \exp(|U|\lambda_0 s) f_{|U|}(\cdot) \psi_\beta(z_1) \cdot \dots \cdot \psi_\beta(z_{|U|}) \\ &\quad \times \exp((n-|U|)\lambda_0 s) f_{n-|U|}(\cdot) \psi_\beta(\bar{z}_1) \cdot \dots \cdot \psi_\beta(\bar{z}_{n-|U|})) ds \\ &+ 2 \int_0^t P_{t-s} (\beta v \sum_{U \subset Y, U \neq \emptyset} \exp(|U|\lambda_0 s) f_{|U|}(\cdot) \psi_\beta(z_1) \cdot \dots \cdot \psi_\beta(z_{|U|}) q_{n-|U|}(s, \cdot, \bar{z}_1, \dots, \bar{z}_{n-|U|})) ds \\ &\quad + \int_0^t P_{t-s} (\beta v \sum_{U \subset Y, U \neq \emptyset} q_{|U|}(s, \cdot, z_1, \dots, z_{|U|}) q_{n-|U|}(s, \cdot, \bar{z}_1, \dots, \bar{z}_{n-|U|})) ds. \end{aligned} \quad (20)$$

The second and third integrals on the right hand side of (20) contribute only to the remainder term. Indeed, consider the contribution to the second integral from the term with a given U :

$$\int_0^t P_{t-s} (\beta v \exp(|U|\lambda_0 s) f_{|U|}(\cdot) \psi_\beta(z_1) \cdot \dots \cdot \psi_\beta(z_{|U|}) q_{n-|U|}(s, \cdot, \bar{z}_1, \dots, \bar{z}_{n-|U|})) ds$$

$$\begin{aligned}
&\leq A\psi_\beta(z_1)\dots\psi_\beta(z_{|U|}) \int_0^t P_{t-s}(v \exp(|U|\lambda_0 s) f_{|U|}(\cdot)) \\
&\quad \times \exp((n - |U|)\lambda_0 s) \Pi_\varepsilon^{n-|U|}(s, \bar{z}_1, \dots, \bar{z}_{n-|U|}) ds \\
&\leq A\psi_\beta(z_1) \cdot \dots \cdot \psi_\beta(z_{|U|}) \int_0^t \exp(\lambda_0(t - s)) \exp(n\lambda_0 s) \Pi_\varepsilon^{n-|U|}(s, \bar{z}_1, \dots, \bar{z}_{n-|U|}) ds \\
&\leq A \exp(n\lambda_0 t) \psi_\beta(z_1) \dots \psi_\beta(z_{|U|}) \Pi_\varepsilon^{n-|U|}(t, \bar{z}_1, \dots, \bar{z}_{n-|U|}) \leq A \exp(n\lambda_0 t) \Pi_\varepsilon^n(t, y_1, \dots, y_n),
\end{aligned}$$

where the first inequality follows from the inductive assumption and the second one from (15). The third integral on the right hand side of (20) is estimated similarly. It remains to consider the first integral. It is equal to

$$\begin{aligned}
&\psi_\beta(y_1) \cdot \dots \cdot \psi_\beta(y_n) \int_0^t \exp(n\lambda_0 s) P_{t-s}(\beta v \sum_{U \subset Y, U \neq \emptyset} f_{|U|} f_{n-|U|}) ds \\
&= \psi_\beta(y_1) \cdot \dots \cdot \psi_\beta(y_n) \exp(n\lambda_0 t) (f_n(\cdot) + O(\exp(-(n-1)\lambda_0 t))),
\end{aligned}$$

where the last equality follows from (16). Thus we obtain the main term from the right hand side of (18) plus the correction

$$\psi_\beta(y_1) \cdot \dots \cdot \psi_\beta(y_n) \exp(n\lambda_0 t) O(\exp(-(n-1)\lambda_0 t))$$

for which the estimate (19) holds since $\psi_\beta(y_1) \exp(-\lambda_0 t) \leq \alpha_\varepsilon^2(t, y_1)$ due to (10). \square

We will now use Lemma 4.3 to draw conclusions about the distribution of particles at time t . First, let us observe that for each x the sequence $(\int_{\mathbb{R}^d} \psi_\beta(y) dy)^n f_n(x)$, $n \geq 1$, serves as a sequence of moments for a random variable $\xi^{\beta, x}$ whose distribution is defined uniquely. Indeed, by the Carleman theorem, it is sufficient to check that

$$\sum_{n=1}^{\infty} \left(\frac{1}{f_n(x)} \right)^{\frac{1}{2n}} = \infty. \quad (21)$$

From (15) it follows that there is a constant A such that

$$\|I_n(g)\|_C \leq \frac{A}{n-1} \|g\|_{C_{\text{exp}}}, \quad n \geq 2.$$

Therefore, from (17) it follows that for a different constant A ,

$$\|f_n\|_C \leq A \sum_{k=1}^{n-1} \frac{(n-1)!}{k!(n-k)!} \|f_k\|_C \|f_{n-k}\|_C, \quad n \geq 2, \quad \|f_1\|_C \leq A.$$

From here, by induction on n it follows that $\|f_n\|_C \leq A^{2^{n-1}} n!$, which in turn implies (21) since $n! \leq ((n+1)/2)^n$.

Let $n_t^{\beta,x}(U)$ be the number of particles in a domain $U \subseteq \mathbb{R}^d$, assuming that at $t = 0$ there was one particle located at x . We will write $n_t^{\beta,x}$ instead of $n_t^{\beta,x}(\mathbb{R}^d)$. Note that

$$\mathbb{E}(n_t^{\beta,x}(U))^n = \sum_{k=1}^n S(n, k) \int_U \dots \int_U \rho_k(t, x, y_1, \dots, y_k) dy_1 \dots dy_k, \quad (22)$$

where $S(n, k)$ is the Stirling number of the second kind (the number of ways to partition n elements into k nonempty subsets). Formula (22) easily follows if we write

$$n_t^{\beta,x}(U) = \sum_i n_t^{\beta,x}(\Delta_i),$$

where $U = \sqcup_i \Delta_i$ is the partition of U into small sub-domains, and then take the limit as $\max_i \text{diam}(\Delta_i) \rightarrow 0$.

Let $\xi^{\beta,x}$ be a random variable with the moments

$$\mathbb{E}(\xi^{\beta,x})^n = \left(\int_{\mathbb{R}^d} \psi_\beta(y) dy \right)^n f_n(x).$$

Let φ_β be the density on \mathbb{R}^d given by

$$\varphi_\beta(x) = \psi_\beta(x) / \int_{\mathbb{R}^d} \psi_\beta(y) dy.$$

Theorem 4.4. *For each $x \in \mathbb{R}^d$ and each domain $U \subseteq \mathbb{R}^d$,*

$$\lim_{t \rightarrow \infty} \left(\exp(-\lambda_0 t) n_t^{\beta,x}(U) \right) = \xi^{\beta,x} \int_U \varphi_\beta(x) dx$$

in distribution.

Proof. By the theorem of Frechet and Shohat [8], it is sufficient to prove the convergence of the moments. By (22) the n -th moment of $\exp(-\lambda_0 t) n_t^{\beta,x}(U)$ is equal to

$$\mathbb{E}(\exp(-\lambda_0 t) n_t^{\beta,x}(U))^n = \exp(-n\lambda_0 t) \sum_{k=1}^n S(n, k) \int_U \dots \int_U \rho_k(t, x, y_1, \dots, y_k) dy_1 \dots dy_k.$$

First consider the contribution to the right hand side from the term with $k = n$. Note that $S(n, n) = 1$. We use (18) for the asymptotics of ρ_n . The contribution from the term $q_n(t, x, y_1, \dots, y_n)$ tends to zero:

$$\lim_{t \rightarrow \infty} \exp(-n\lambda_0 t) \int_U \dots \int_U q_n(t, x, y_1, \dots, y_n) dy_1 \dots dy_n = 0,$$

as follows from the definition of $\Pi_\varepsilon^n(t, y_1, \dots, y_n)$. The contribution from the main term gives the desired expression:

$$\lim_{t \rightarrow \infty} \exp(-n\lambda_0 t) \int_U \dots \int_U \exp(n\lambda_0 t) f_n(x) \psi_\beta(y_1) \dots \psi_\beta(y_n) dy_1 \dots dy_n = \mathbb{E}(\xi^{\beta,x})^n \left(\int_U \varphi_\beta(x) dx \right)^n.$$

It remains to note that the contribution from each of the terms with $k < n$ tends to zero. Indeed, it is equal to

$$\exp(-(n-k)\lambda_0 t) \left(\exp(-k\lambda_0 t) S(n, k) \int_U \dots \int_U \rho_k(t, x, y_1, \dots, y_k) dy_1 \dots dy_k \right).$$

The expression inside the brackets tends to a finite limit as in the case $k = n$, while the exponential factor in front of the brackets tends to zero for $k < n$. \square

5 The sub-critical case

Throughout this section we assume that $d \geq 3$ and $\beta \in (0, \beta_{\text{cr}})$. We will show that the total number of particles $n_t^{\beta, x}$ tends to a random limit when $t \rightarrow \infty$. Denote the integrals of the correlation functions by $\bar{\rho}_n(t, x)$:

$$\bar{\rho}_n(t, x) = \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \rho_n(t, x, y_1, \dots, y_n) dy_1 \dots dy_n.$$

From (2)-(3) and (5)-(6) it follows that these quantities satisfy the equations

$$\partial_t \bar{\rho}_1(t, x) = \frac{1}{2} \Delta \bar{\rho}_1(t, x) + \beta v(x) \bar{\rho}_1(t, x),$$

$$\bar{\rho}_1(0, x) \equiv 1$$

and for $n > 1$:

$$\partial_t \bar{\rho}_n(t, x) = \frac{1}{2} \Delta \bar{\rho}_n(t, x) + \beta v(x) \left(\bar{\rho}_n(t, x) + \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} \bar{\rho}_k(t, x) \bar{\rho}_{n-k}(t, x) \right),$$

$$\bar{\rho}_n(0, x) \equiv 0.$$

As in the previous section, in order to find the asymptotics of the total number of particles, we will study the asymptotics of $\bar{\rho}_n(t, x)$ as $r \rightarrow \infty$.

Lemma 5.1. *For $\beta \in (0, \beta_{\text{cr}})$, there is a constant A such that the solution ρ of (11) can be estimated as follows*

$$\|\rho(t, \cdot)\|_C \leq A(1+t)^{-d/2} \|g\|_{C_{\text{exp}}}. \quad (23)$$

If the initial condition $g \in C_{\text{exp}}$ in (11) is replaced by $g \equiv 1$, then

$$\lim_{t \rightarrow \infty} \rho(t, x) = 1 + \varphi_\beta(x) \quad (24)$$

in $C(\mathbb{R}^d)$, where $\varphi_\beta = -R_0^\beta(\beta v)$.

Proof. Let $d \geq 3$ be odd. As in (13), we represent the solution as an integral

$$\rho(t, \cdot) = -\frac{1}{2\pi i} \int_{\text{Re}\lambda=1} e^{\lambda t} R_\lambda^\beta g d\lambda.$$

Since the integrand is analytic in \mathbb{C}' with appropriate decay at infinity and has a limit as $\lambda \rightarrow 0$, we can replace the contour of integration by

$$\gamma = \{z \in \mathbb{C} : \text{Re}z = -|\text{Im}z|\}.$$

Using the representation (8) for R_λ^β , we obtain

$$\rho(t, \cdot) = -\frac{1}{2\pi i} \int_\gamma e^{\lambda t} (Q_d^\beta \lambda^{\frac{d}{2}-1} + P_d^\beta(\lambda) + O(|\lambda|^{\frac{d}{2}})) g d\lambda.$$

The contribution to the integral from the first term is estimated as follows

$$\|Q_d^\beta g \int_\gamma e^{\lambda t} \lambda^{\frac{d}{2}-1} d\lambda\|_C = \|Q_d^\beta g t^{-\frac{d}{2}} \int_\gamma e^s s^{\frac{d}{2}-1} ds\|_C \leq A t^{-\frac{d}{2}} \|g\|_{C_{\text{exp}}},$$

where we used the change of variable $s = \lambda t$. The contribution from the second term is equal to zero since the integrand is analytic and the contour can be moved arbitrarily far to the left along the real axis. The third term can be treated in the same way as the first one, resulting in the decay in t of order $t^{-\frac{d}{2}-1}$. The obtained estimates imply (23) for $t \geq 1$. Clearly (23) holds for $t \leq 1$.

The case when $d \geq 4$ is even is treated similarly. The slight difference is that the contribution from the main term is now, up to a multiplicative constant, equal to

$$Q_d^\beta g \int_\gamma e^{\lambda t} \lambda^{\frac{d}{2}-1} \ln(1/\lambda) d\lambda.$$

After the change of variable $s = \lambda t$, the integral is seen to be equal to

$$t^{-\frac{d}{2}} \int_\gamma e^s s^{\frac{d}{2}-1} \ln(t/s) ds = t^{-\frac{d}{2}} \ln t \int_\gamma e^s s^{\frac{d}{2}-1} ds + t^{-\frac{d}{2}} \int_\gamma e^s s^{\frac{d}{2}-1} \ln(1/s) ds.$$

The first integral on the right hand side is equal to zero since the integrand is an analytic function, and the contour can therefore be moved arbitrarily far to the left along the real axis. The second term on the right hand side has the desired order in t .

It remains to prove (24). Note that $w(t, x) = \rho(t, x) - 1$ is the solution of the problem

$$\frac{\partial w(t, x)}{\partial t} = \frac{1}{2} \Delta w(t, x) + \beta v(x) w(t, x) + \beta v(x), \quad w(0, x) \equiv 0.$$

By the Duhamel formula,

$$w(t, \cdot) = \frac{-1}{2\pi i} \int_0^t \int_\gamma e^{\lambda(t-s)} R_\lambda^\beta(\beta v) d\lambda ds =$$

$$\frac{-1}{2\pi i} \int_{\gamma} \frac{e^{\lambda t} - 1}{\lambda} R_{\lambda}^{\beta}(\beta v) d\lambda = \frac{-1}{2\pi i} \int_{\gamma} \frac{e^{\lambda t}}{\lambda} R_{\lambda}^{\beta}(\beta v) d\lambda,$$

since in the domain γ^+ to the right of the contour γ , the operator R_{λ}^{β} is analytic and decays as $|\lambda|^{-1}$ at infinity. We make the change of variables $s = \lambda t$ and obtain, as $t \rightarrow \infty$,

$$\frac{-1}{2\pi i} \int_{\gamma} \frac{e^{\lambda t}}{\lambda} R_{\lambda}^{\beta}(\beta v) d\lambda = \frac{-1}{2\pi i} \int_{\gamma} \frac{e^s}{s} R_{s/t}^{\beta}(\beta v) ds \rightarrow R_0^{\beta}(\beta v) \frac{-1}{2\pi i} \int_{\gamma} \frac{e^s}{s} ds = \varphi_{\beta}.$$

□

For $g \in C_{\text{exp}}$, we define $J(g) = \int_0^{\infty} P_s g ds$. From Lemma 5.1 it follows that

$$\|J(g)\|_C \leq A \|g\|_{C_{\text{exp}}} \quad (25)$$

for some constant A . Let us define the sequence of functions f_1, f_2, \dots inductively via: $f_1 = 1 + \varphi_{\beta}$ and

$$f_n = \beta \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} J(v f_k f_{n-k}), \quad n \geq 2. \quad (26)$$

Lemma 5.2. *For each $n \geq 1$ we have*

$$\lim_{t \rightarrow \infty} \bar{\rho}_n(t, \cdot) = f_n \quad (27)$$

in $C(\mathbb{R}^d)$.

Proof. For $n = 1$, the statement coincides with the second part of Lemma 5.1. Let us assume that (27) holds for all natural numbers up to and including $n - 1$. Define the functions $c_k(t, x) = \bar{\rho}_k(t, x) - f_k(x)$. Thus $\|c_k(t, \cdot)\|_C \rightarrow 0$ as $t \rightarrow \infty$ for $k \leq n - 1$. By the Duhamel principle applied to the equation for $\bar{\rho}_n$, we obtain

$$\begin{aligned} \bar{\rho}_n(t, \cdot) &= \int_0^t P_{t-s}(\beta v \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} \bar{\rho}_k(s, \cdot) \bar{\rho}_{n-k}(s, \cdot)) ds \\ &= \int_0^t P_{t-s}(\beta v \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} f_k f_{n-k}) ds + \end{aligned} \quad (28)$$

$$2 \int_0^t P_{t-s}(\beta v \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} c_k(s, \cdot) f_{n-k}) ds + \int_0^t P_{t-s}(\beta v \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} c_k(s, \cdot) c_{n-k}(s, \cdot)) ds.$$

From Lemma 5.1 it follows that $\|P_t g\|_C \leq A(1+t)^{-d/2} \|g\|_{C_{\text{exp}}}$ for some constant A . Therefore the C -norm of the sum of the last two term on the right hand side of (28) is estimated from above by

$$\int_0^t \gamma(s) (1+t-s)^{-d/2} ds,$$

where $\gamma(s)$ is some function such that $\gamma(s) \rightarrow 0$ as $s \rightarrow \infty$. The latter integral tends to zero when $t \rightarrow \infty$ since $d \geq 3$. It remains to note that the first term on the right hand side of (28) tends to f_n in the C -norm, as immediately follows from the definition of J . \square

Theorem 5.3. *For each $x \in \mathbb{R}^d$, the total number of particles $n_t^{\beta,x}$ converges almost surely, as $t \rightarrow \infty$, to a random variable $\zeta^{\beta,x}$ with the moments $m_n(x) = \sum_{k=1}^n S(n, k) f_k(x)$.*

Proof. Since $n_t^{\beta,x}$ is monotonically increasing in t , and the moments converge to $m_n(x)$ (as follows from (22) and Lemma 5.2), we have convergence almost surely to a random variable with the specified moments. \square

6 The critical case

Throughout this section we assume that $d \geq 3$. We will show that for $\beta = \beta_{\text{cr}}$ the total number of particles tends almost surely to a finite random limit, while the expectation of the number of particles tends to infinity.

It is clear that the random variables $n_t^{\beta,x}$ can be realized on a common probability space in such a way that $n_t^{\beta,x} \leq n_{t'}^{\beta',x}$ whenever $\beta \leq \beta'$ and $t \leq t'$. Therefore, in order to show that $\text{En}_t^{\beta_{\text{cr}},x} \rightarrow \infty$ as $t \rightarrow \infty$, it is sufficient to show that

$$\lim_{\beta \uparrow \beta_{\text{cr}}} \lim_{t \rightarrow \infty} \text{En}_t^{\beta,x} = \infty.$$

This follows from Lemma 7.3 of [6], which can be stated as follows:

Lemma 6.1. *There are positive constant b_d , $d \geq 3$, such that*

$$\lim_{t \rightarrow \infty} \text{En}_t^{\beta,x} = \frac{b_d}{\beta_{\text{cr}} - \beta} \psi_{\beta_{\text{cr}}}(x) + O(1) \quad \text{as } \beta \uparrow \beta_{\text{cr}}$$

is valid in $C(\mathbb{R}^d)$, where $\psi_{\beta_{\text{cr}}}$ is the positive ground state for $L^{\beta_{\text{cr}}}$.

Now we prove the main theorem of this section.

Theorem 6.2. *For each x , the limit $\eta^x := \lim_{t \rightarrow \infty} n_t^{\beta_{\text{cr}},x}$ is finite almost surely.*

Proof. Let $S(x) = \text{P}(\eta^x = \infty)$. Let us show that $S(x)$ depends continuously on x . Indeed, let $x \in \mathbb{R}^d$ and $\varepsilon > 0$ be fixed. There exist $\delta > 0$ and $t > 0$ such that for $|y - x| \leq \delta$ the branching processes starting at x and y have the following properties:

(a) The probability that at least one branching occurs on the interval $[0, t]$ for either of the processes does not exceed $\varepsilon/3$;

(b) There is a positive function $q(z)$ such that $\int_{\mathbb{R}^d} q(z) \geq 1 - \varepsilon/3$, which serves as a lower bound for both of the heat kernels $p(t, x, z)$ and $p(t, y, z)$.

This shows that the branching processes starting at x and y can be coupled on an event of probability at least $1 - \varepsilon$, and therefore $|S(x) - S(y)| \leq \varepsilon$, proving the continuity.

Suppose that $S(x)$ is not identically equal to zero. Then $S(x) \neq 0$ on a set of positive measure, due to the continuity. By the Markov property, $S(x) \geq \int_{\mathbb{R}^d} p(1, x, z)S(z)dz$, which shows that $S(x) > 0$ for all $x \in \mathbb{R}^d$. Let

$$m = \min_{x \in \text{supp}(v)} S(x) > 0.$$

Let $\Omega^x = \{\eta^x = \infty\}$. Take $N = [4/m] + 1$. Let r^x be the probability that there are at least N particles on the support of v before time $t = 1$. Clearly, there is a positive constant r such that

$$r^x \geq r \quad \text{for all } x \in \text{supp}(v). \quad (29)$$

Define the random times $\tau_1^x, \tau_2^x, \dots$ to be the consecutive instances of branching for the process starting at x . There is at least one particle on $\text{supp}(v)$ at each of the times τ_n^x , and $\tau_n^x \rightarrow \infty$ almost surely on Ω^x . Therefore, from (29) and the Markov property it follows that almost surely on Ω^x there is a random time τ^x such that there are N particles on the support of v at the time τ^x . Therefore, there exists $T < \infty$ such that $P(\tau^x \leq T) \geq m/2$ for $x \in \text{supp}(v)$. Note that T can be taken to be independent of x using the continuity in x of the probabilities under consideration and the compactness of $\text{supp}(v)$.

Now fix an arbitrary $x \in \text{supp}(v)$. We saw that with probability at least $m/2$ there are at least N particles at time τ^x and therefore also at time T , that is $En_T^{\beta_{\text{cr}}, x} \geq Nm/2$. Applying the Markov property with respect to the stopping time τ^x and using the fact that the particles move independently, we see that $En_{2T}^{\beta_{\text{cr}}, x} \geq (Nm/2)^2$ and, continuing by induction, that $En_{kT}^{\beta_{\text{cr}}, x} \geq (Nm/2)^k \geq 2^k$.

Therefore, the expectation of the total number of particles grows at least exponentially with some exponent $\gamma > 0$. On the other hand, from the arguments in Theorem 4.4 it follows that for $\beta > \beta_{\text{cr}}$ the expectation of the number of particles grows exponentially with the exponent $\lambda_0(\beta)$. Since $\lambda_0(\beta) \downarrow 0$ as $\beta \downarrow \beta_{\text{cr}}$ and $n_t^{\beta, x}$ depends monotonically on β , we conclude that $\gamma = 0$. Thus we come to a contradiction with our assumption that $S(x)$ is not identically zero. \square

Remark 1. It is not difficult to see that the random variables $n_t^{\beta, x}$ can be realized on a common probability space in such a way that $\zeta^{\beta, x}$ from Theorem 5.3 converge almost surely, as $\beta \uparrow \beta_{\text{cr}}$, to η^x .

Remark 2. Using the spectral techniques similar to those employed above and in [6], one can get the asymptotics of the higher order moments of $n_t^{\beta_{\text{cr}}, x}$.

7 Appendix

Here we prove several statements on the analytic properties of the resolvent R_λ^β . We largely follow [6]. The new steps concern the asymptotics of the resolvent as $\lambda \rightarrow 0$. Denote

$$A_\lambda = v(x)R_\lambda^0 : C_{\text{exp}}(\mathbb{R}^d) \rightarrow C_{\text{exp}}(\mathbb{R}^d).$$

The following lemma is similar to Lemmas 5.1 and 5.2 of [6] (see also [15] for a similar statement for general elliptic operators), the difference being that we now obtain a more precise asymptotics of R_λ^0 and A_λ near the origin.

Lemma 7.1. *Consider the operators $R_\lambda^0 : C_{\text{exp}}(\mathbb{R}^d) \rightarrow C(\mathbb{R}^d)$ and $A_\lambda : C_{\text{exp}}(\mathbb{R}^d) \rightarrow C_{\text{exp}}(\mathbb{R}^d)$.*

- (1) *The operators R_λ^0 and A_λ are analytic in $\lambda \in \mathbb{C}'$.*
- (2) *The operator A_λ is compact for $\lambda \in \mathbb{C}'$.*
- (3) *For each $\varepsilon > 0$, we have $\max(\|R_\lambda^0\|, \|A_\lambda\|) = O(1/|\lambda|)$ as $\lambda \rightarrow \infty$, $|\arg \lambda| \leq \pi - \varepsilon$.*
- (4) *The operator R_λ^0 has the following asymptotic behavior as $\lambda \rightarrow 0$, $\lambda \in \mathbb{C}'$:*

$$R_\lambda^0 = Q_d \lambda^{\frac{d}{2}-1} + P_d(\lambda) + O(|\lambda|^{\frac{d}{2}}), \quad d \geq 3, \quad d - \text{odd}, \quad (30)$$

$$R_\lambda^0 = Q_d \lambda^{\frac{d}{2}-1} \ln(1/\lambda) + P_d(\lambda) + O(|\lambda|^{\frac{d}{2}} |\ln \lambda|), \quad d \geq 4, \quad d - \text{even}, \quad (31)$$

where P_d are polynomials with coefficients which are bounded operators and

$$Q_d f \equiv q_d \int_{\mathbb{R}^d} f(x) dx,$$

where $q_d \neq 0$.

Remark. The term with the coefficient Q_d is the main non-analytic term of the expansion as $\lambda \rightarrow 0$.

Proof. Let d be odd. From (7) it follows that the kernel $R_\lambda^0(x)$ is an analytic function of $\lambda \in \mathbb{C}'$ and the following estimates holds

$$|R_\lambda^0(x)| \leq C_d |\sqrt{\lambda}|^{d-2} |\sqrt{\lambda}x|^{2-d}, \quad |\sqrt{\lambda}x| \leq 1, \quad (32)$$

$$|R_\lambda^0(x)| \leq C_d |\sqrt{\lambda}|^{d-2} |e^{-\sqrt{2\lambda}|x|}| |\sqrt{\lambda}x|^{\frac{1-d}{2}}, \quad |\sqrt{\lambda}x| \geq 1. \quad (33)$$

Moreover, these estimate admits differentiation in λ and x resulting in

$$\left| \frac{\partial R_\lambda^0(x)}{\partial \lambda} \right| \leq C'_d |\sqrt{\lambda}|^{d-4} |\sqrt{\lambda}x|^{2-d}, \quad |\sqrt{\lambda}x| \leq 1, \quad (34)$$

$$\left| \frac{\partial R_\lambda^0(x)}{\partial \lambda} \right| \leq C'_d |\sqrt{\lambda}|^{d-2} |e^{-\sqrt{2\lambda}|x|}| |\sqrt{\lambda}x|^{\frac{1-d}{2}} \left(\frac{1}{|\lambda|} + \frac{|x|}{|\sqrt{\lambda}|} \right), \quad |\sqrt{\lambda}x| \geq 1, \quad (35)$$

and

$$|\nabla_x R_\lambda^0(x)| \leq C''_d |\sqrt{\lambda}|^{d-2} |\sqrt{\lambda}x|^{2-d} \frac{1}{|x|}, \quad |\sqrt{\lambda}x| \leq 1, \quad (36)$$

$$|\nabla_x R_\lambda^0(x)| \leq C''_d |\sqrt{\lambda}|^{d-2} |e^{-\sqrt{2\lambda}|x|}| |\sqrt{\lambda}x|^{\frac{1-d}{2}} \left(\frac{1}{|x|} + |\sqrt{\lambda}| \right), \quad |\sqrt{\lambda}x| \geq 1. \quad (37)$$

where C_d , C'_d and C''_d are positive constants. The estimates (32)-(33) and (34)-(35) imply the analyticity of the operators R_λ^0 and A_λ .

The estimates (32)-(33) and (36)-(37) imply that the operator $R_\lambda^0 : C_{\text{exp}}(\mathbb{R}^d) \rightarrow C^1(\mathbb{R}^d)$ is bounded. Then the standard Sobolev embedding theorem implies the compactness of the operator $A_\lambda = vR_\lambda^0$ in the space $C_{\text{exp}}(\mathbb{R}^d)$.

In order to prove the third statement of the lemma, we observe that the norm of R_λ^0 can be estimated by $\int_{\mathbb{R}^d} |R_\lambda^0(x)| dx$, which is of order $O(1/|\lambda|)$ as $\lambda \rightarrow \infty$, $|\arg \lambda| \leq \pi - \varepsilon$, due to (32)-(33).

To prove the fourth statement, we need a more detailed asymptotics of the kernel $R_\lambda^0(x)$ when $|\sqrt{\lambda}x| \downarrow 0$. Namely, it follows from the properties of the Hankel functions and (7) that there are a constant $a_d \in \mathbb{C}$ and a polynomial b_d with complex coefficients such that

$$R_\lambda^0(x) = |x|^{2-d} \left(a_d (\sqrt{\lambda}|x|)^{d-2} + b_d(\lambda|x|^2) + O(|\sqrt{\lambda}x|^d) \right) \quad \text{as } |\sqrt{\lambda}x| \downarrow 0, \quad \lambda \in \mathbb{C}'. \quad (38)$$

Combined with (33) and the definition of the space $C_{\text{exp}}(\mathbb{R}^d)$, this easily implies the fourth statement of the lemma.

The case of even d is similar. The main difference concerns formulas (32) and (38). The estimate (32) remains valid except the case $d = 2$, where it is replaced by

$$|R_\lambda^0(x)| \leq C_2 |\ln \sqrt{\lambda}x|, \quad |\sqrt{\lambda}x| \leq 1,$$

while (38) is replaced by

$$R_\lambda^0(x) = |x|^{2-d} \left(a_d (\sqrt{\lambda}|x|)^{d-2} \ln(\sqrt{\lambda}|x|) + b_d(\lambda|x|^2) + O(|\sqrt{\lambda}x|^d |\ln(\sqrt{\lambda}|x|)|) \right) \quad (39)$$

as $|\sqrt{\lambda}x| \downarrow 0$, $\lambda \in \mathbb{C}'$. The rest of the arguments proceed as earlier, but now employing (39) instead of (38). \square

The following lemma is simply a resolvent identity.

Lemma 7.2. *For $\lambda \in \mathbb{C}'$, we have the following relation between the meromorphic operator-valued functions*

$$R_\lambda^\beta = R_\lambda^0 - R_\lambda^0 (I + \beta v(x) R_\lambda^0)^{-1} (\beta v(x) R_\lambda^0) \quad (40)$$

Remark. Note that (40) can be written as

$$R_\lambda^\beta = R_\lambda^0 - R_\lambda^0 (I + \beta A_\lambda)^{-1} (\beta v(x) R_\lambda^0).$$

From here it also follows that

$$R_\lambda^\beta = R_\lambda^0 (I + \beta A_\lambda)^{-1}, \quad (41)$$

which should be understood as an identity between meromorphic in λ operators acting from $C_{\text{exp}}(\mathbb{R}^d)$ to $C(\mathbb{R}^d)$.

The kernels of the operators $I + \beta A_\lambda$, $\lambda \in \mathbb{C}'$, are described by the following lemma.

Lemma 7.3. (1) The operator-valued function $(I + \beta A_\lambda)^{-1}$ is meromorphic in \mathbb{C}' . It has a pole at $\lambda \in \mathbb{C}'$ if and only if λ is an eigenvalue of L^β . These poles are of the first order.

(2) Let $\lambda_i(\beta)$ be a positive eigenvalue of L^β . There is a one-to-one correspondence between the kernel of the operator $I + \beta A_{\lambda_i}$ and the eigenspace of the operator L^β corresponding to the eigenvalue λ_i . Namely, if $(I + \beta A_{\lambda_i})h = 0$, then $\psi = -R_{\lambda_i}^0 h$ is an eigenfunction of L^β and $h = \beta v \psi$.

(3) If $d \geq 3$, there is a one-to-one correspondence between the kernel of the operator $I + \beta A_0$ and solution space of the problem

$$L^\beta(\psi) = \frac{1}{2}\Delta\psi + \beta v(x)\psi = 0, \quad \psi(x) = O(|x|^{2-d}), \quad \frac{\partial\psi}{\partial r}(x) = O(|x|^{1-d}) \quad \text{as } r = |x| \rightarrow \infty. \quad (42)$$

Namely, if $(I + \beta A_0)h = 0$ for $h \in C_{\text{exp}}(\mathbb{R}^d)$, then $\psi = -R_0^0 h$ is a solution of (42) and $h = \beta v \psi$.

Proof. The operator A_λ , $\lambda \in \mathbb{C}'$, is analytic, compact, and tends to zero as $\lambda \rightarrow +\infty$ by Lemma 7.1. Therefore $(I + \beta A_\lambda)^{-1}$ is meromorphic by the Analytic Fredholm Theorem.

If $\lambda \in \mathbb{C}'$ is a pole of $(I + \beta A_\lambda)^{-1}$, then it is also a pole of the same order of R_λ^β as follows from (41) since the kernel of R_λ^0 is trivial. Therefore the pole is simple and coincides with one of the eigenvalues λ_i . Note that λ is a pole of $(I + \beta A_\lambda)^{-1}$ if and only if the kernel of $I + \beta A_\lambda$ is non-trivial. Let $h \in C_{\text{exp}}(\mathbb{R}^d)$ be such that $\|h\|_{C_{\text{exp}}(\mathbb{R}^d)} \neq 0$ and $(I + \beta v R_\lambda^0)h = 0$. Then $\psi := -R_\lambda^0 h \in L^2(\mathbb{R}^d)$ and $(\frac{1}{2}\Delta - \lambda + \beta v)\psi = 0$, that is ψ is an eigenfunction of L^β .

Conversely, let $\psi \in L^2(\mathbb{R}^d)$ be an eigenfunction corresponding to an eigenvalue λ_i , that is

$$\left(\frac{1}{2}\Delta - \lambda_i\right)\psi + \beta v \psi = 0. \quad (43)$$

Denote $h = \beta v \psi$. Then $(\frac{1}{2}\Delta - \lambda_i)\psi = -h$. Thus $\psi = -R_{\lambda_i}^0 h$ and (43) implies that h satisfies $(I + \beta v R_{\lambda_i}^0)h = 0$. Note that $h \in C^\infty(\mathbb{R}^d)$, h vanishes outside $\text{supp}(v)$, and therefore belongs to the kernel of $I + \beta A_{\lambda_i}$. This completes the proof of the first two statements.

Similar arguments can be used to prove the last statement. If $h \in C_{\text{exp}}(\mathbb{R}^d)$ is such that $\|h\|_{C_{\text{exp}}(\mathbb{R}^d)} \neq 0$ and $(I + \beta A_0)h = 0$, then h has compact support and the integral operator R_0^0 can be applied to h . It is clear that $\psi := -R_0^0 h$ satisfies (42).

In order to prove that any solution of (42) corresponds to an eigenvector of $I + \beta A_0$, one only needs to show that the solution ψ of the problem (42) can be represented in the form $\psi = -R_0^0 h$ with $h = \beta v \psi$. The latter follows from the Green formula

$$\psi(x) = -(R_0^0(\beta v \psi))(x) + \int_{|y|=a} [R_0^0(x-y)\psi'_r(y) - \frac{\partial}{\partial r} R_0^0(x-y)\psi(y)] ds, \quad |x| < a,$$

after passing to the limit as $a \rightarrow \infty$. □

Remark. The relations (42) are an analogue of the eigenvalue problem for zero eigenvalue and the eigenfunction ψ which does not necessarily belong to $L^2(\mathbb{R}^d)$. We shall call a non-zero solution of (42) a ground state.

Due to the monotonicity and continuity of $\lambda = \lambda_0(\beta)$ for $\beta > \beta_{cr}$, we can define the inverse function

$$\beta = \beta(\lambda) : [0, \infty) \rightarrow [\beta_{cr}, \infty).$$

We'll prove that the operator $-A_\lambda$, $\lambda > 0$, has a non-negative kernel and has a positive simple eigenvalue such that all the other eigenvalues are smaller in absolute value. Such an eigenvalue is called the principal eigenvalue.

Lemma 7.4. *The operator $-A_\lambda$, $\lambda > 0$, has the principal eigenvalue. This eigenvalue is equal to $1/\beta(\lambda)$ and the corresponding eigenfunction can be taken to be positive in the interior of $\text{supp}(v)$ and equal to zero outside of $\text{supp}(v)$. If $d \geq 3$, then the same is true for the operator $-A_0$ (in particular, $\beta_{cr} > 0$).*

Proof. The maximum principle for the operator $(\frac{1}{2}\Delta - \lambda)$, $\lambda > 0$, implies that the kernel of the operator R_λ^0 , $\lambda > 0$, is negative. Thus, by (7), for all y the kernel of $-A_\lambda$ is positive when x is in the interior of $\text{supp}(v)$ and zero otherwise. Thus $-A_\lambda$, $\lambda > 0$, has the principal eigenvalue (see [14]). On the other hand, by Lemma 7.3, $1/\beta(\lambda)$ is a positive eigenvalue of $-A_\lambda$. Note that this is the largest positive eigenvalue of $-A_\lambda$. Indeed, if $\mu = 1/\beta' > 1/\beta(\lambda)$ is an eigenvalue of $-A_\lambda$, then λ is one of the eigenvalues λ_i of $L^{\beta'}$ by Lemma 7.3. Therefore, $\lambda_i(\beta') = \lambda_0(\beta)$ for $\beta' < \beta$. This contradicts the monotonicity of $\lambda_0(\beta)$. Hence the statement of the lemma concerning the case $\lambda > 0$ holds.

For $d \geq 3$, the kernel of $-A_0$ is equal to vP_d and has the same properties as the kernel of $-A_\lambda$, $\lambda > 0$. Thus $-A_0$ has the principal eigenvalue. Since $A_\lambda \rightarrow A_0$ as $\lambda \downarrow 0$, the principal eigenvalue $1/\beta(\lambda)$ converges to the principal eigenvalue $\mu < \infty$ of $-A_0$. On the other hand, $\beta(\lambda)$ is a continuous function, and therefore $\mu = 1/\beta_{cr}$, which proves the statement concerning the case $\lambda = 0$. \square

Remark. Let $d \geq 3$. Lemmas 7.3 and 7.4 imply that the ground state of the operator L^β for $\beta = \beta_{cr}$ (defined by (42)) is defined uniquely up to a multiplicative constant and corresponds to the principal eigenvalue of A_0 . If $\beta < \beta_{cr}$, then the ground state (with $\lambda = 0$) does not exist and the operator $I + \beta A_0$ has a bounded inverse.

We can finally proceed with the proof of Lemma 3.2.

Proof of Lemma 3.2. The analytic properties of R_λ^β follow from (41) and the corresponding properties of $(I + \beta A_\lambda)^{-1}$ which are in turn due to Lemma 7.3.

By Lemma 7.1, the norm of A_λ decays at infinity when $\lambda \rightarrow \infty$, $|\arg \lambda| \leq \pi - \varepsilon$. Therefore there is $\Lambda > 0$ such that the operator $(I + \beta A_\lambda)^{-1}$ is bounded for $|\arg \lambda| \leq \pi - \varepsilon$, $|\lambda| \geq \Lambda$. The decay of the norm of R_λ^β now follows from (41) and the third part of Lemma 7.1.

Now let us prove (8). Let $d \geq 3$ be odd. We use (30) to represent $I + \beta A_\lambda$ as

$$I + \beta A_\lambda = B + C_\lambda,$$

where $B = I + \beta A_0$ and $C_\lambda = \beta v(Q_d \lambda^{\frac{d}{2}-1} + P_d(\lambda) - P_d(0) + O(|\lambda|^{\frac{d}{2}}))$. Since B is invertible (see the Remark following Lemma 7.4) and $C_\lambda \rightarrow 0$ as $\lambda \rightarrow 0$, $\lambda \in \mathbb{C}'$, we have

$$(I + \beta A_\lambda)^{-1} = (B + C_\lambda)^{-1} = (I - (B^{-1}C_\lambda) + (B^{-1}C_\lambda)^2 - \dots)B^{-1}$$

for all sufficiently small $\lambda \in \mathbb{C}'$. Combining this with (41) and using (30), we obtain (8). In order to prove (9), we can repeat the same arguments, starting with (31) instead of (30). \square

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