

# Motion in a Random Force Field

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## Abstract

We consider the motion of a particle in a random isotropic force field. Assuming that the force field arises from a Poisson field in  $\mathbb{R}^d$ ,  $d \geq 4$ , and the initial velocity of the particle is sufficiently large, we describe the asymptotic behavior of the particle.

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## 1 Introduction

Let  $F$  be a random force field on  $\mathbb{R}^d$  defined on a probability space  $(\Omega', \mathcal{F}', P')$ . The motion of a particle is described by the equation

$$\ddot{X}(t) = F(X(t)), \quad (1)$$

where  $X(t)$  denotes the position of the particle at time  $t$ . Let  $V(t) = \dot{X}(t)$  be the velocity of the particle at time  $t$ . As initial conditions we take  $X(0) = 0$  and  $V(0) = v_0$ , where  $v_0$  is a non-random vector. The force field is assumed to be stationary and isotropic. The precise form of the force field will be discussed below.

We shall be interested in the asymptotic behavior of  $X(t)$  and  $V(t)$  as  $t \rightarrow \infty$ . The process  $V(t)$  can be written in the integral form as

$$V(t) = v_0 + \int_0^t F(X(s)) ds. \quad (2)$$

Formal arguments, based on the near-independence of contributions to the integral on the right-hand side of (2) from non-intersecting sub-intervals, suggest that  $V(t)$  behaves as a diffusion process, if time is re-scaled appropriately. In fact, we shall prove that there is an event  $\Omega'_{v_0}$  in the underlying probability space  $\Omega'$ , such that  $P'(\Omega' \setminus \Omega'_{v_0}) \rightarrow 0$  as  $|v_0| \rightarrow \infty$ , and  $V(c^3 t)/c$  converges, as  $t \rightarrow \infty$ , to a diffusion process on  $\Omega'_{v_0}$ . In particular, the kinetic

energy of the particle will be shown to tend to infinity as  $t \rightarrow \infty$ . The precise formulation of these results will be provided in Section 3.

We cannot, however, expect that  $V(c^3t)/c$  converges to the diffusion process for almost all realizations of the force field, if  $v_0$  is fixed. Indeed, depending on the assumptions imposed on  $F$ , the trajectory may remain in a bounded region of space and the velocity may remain bounded with positive probability.

It must be noted that we must exclude the case  $F = \nabla H$ , where  $H$  is a stationary field, since in this case  $(X(t), V(t))$  is a Hamiltonian flow with the Hamiltonian  $\overline{H}(k, x) = |k|^2/2 - H(x)$ , and  $|V(t)|^2/2 - H(X(t))$  is constant on the solutions of (1).

Earlier papers primarily studied the behavior of  $X(t)$  and  $V(t)$  on long time intervals, whose length, however, depended on  $|v_0|$ , where the initial velocity  $v_0$  was treated as a large parameter. We shall assume that  $v_0$  is fixed and  $t$  tends to infinity. The trade-off is that we need to exclude an event of small but positive measure from the underlying probability space.

Let us mention some of the earlier results concerning the long-time behavior of  $X(t)$  and  $V(t)$ . In [4], Kesten and Papanicolaou considered the equation

$$\ddot{x}(t) = \varepsilon F(x(t)) \tag{3}$$

with the initial data  $x(0) = 0$  and  $v(0) = \tilde{v}$ . Certain mixing assumptions were imposed on the force field  $F$ . It was shown that if  $d \geq 3$ , the process  $v(t/\varepsilon^2)$  converges weakly to a diffusion process  $\bar{v}(t)$  with the initial data  $\bar{v}(0) = \tilde{v}$ . The generator of the limiting process can be written out explicitly. The process  $\varepsilon^2 x(t/\varepsilon^2)$  converges weakly to  $\bar{x}(t) = \int_0^t \bar{v}(s) ds$ .

Note that equations (1) and (3) are related via the change of variables

$$X(t) = x(t/\sqrt{\varepsilon}), \quad v_0 = \tilde{v}/\sqrt{\varepsilon}. \tag{4}$$

Therefore, the convergence result for  $v(t/\varepsilon^2)$  can be formulated in terms of  $V(t)$  as follows: the process  $V(|v_0|^3 t)/|v_0|$  converges to a diffusion process when  $v_0$  tends to infinity in such a way that  $v_0/|v_0| = \tilde{v}$  remains fixed. Similarly,  $X(|v_0|^3 t)/|v_0|^4$  converges weakly to a limiting process.

In [3], Durr, Goldstein, and Lebowitz extended the convergence results to the two-dimensional case. The field  $F$  was assumed to be a gradient of  $H(x) = \sum_i h(x - p_i)$ , where  $h$  is a smooth function with compact support, and the points  $p_i$  form a Poisson field on the plane. An additional difficulty in the two-dimensional case is that, unlike the case with  $d \geq 3$ , typical trajectories of (1) will self-intersect. In [7], Komorowski and Ryzhik proved the two-dimensional result in the case when  $H$  is sufficiently mixing, but is not necessarily generated by a Poisson field.

In [6], Komorowski and Ryzhik considered the process (1) on a longer time scale. Namely, they demonstrated that  $X(|v_0|^{3+8\alpha} t)/|v_0|^{4(1+\alpha)}$  converges to a Brownian motion for all sufficiently small  $\alpha > 0$ . It was assumed that  $F = \nabla H$ , where  $H$  is sufficiently mixing.

Unlike the above papers, we shall consider the asymptotic behavior of  $V(t)$  when  $v_0$  is fixed and  $t \rightarrow \infty$ . First, however, assume that  $v_0 \rightarrow \infty$ ,  $v_0/|v_0| = \tilde{v}$ , and let

$\bar{V}(t)$  be the limiting process for  $V(|v_0|^3 t)/|v_0|$  as  $v_0 \rightarrow \infty$ . (It satisfies the stochastic differential equation (6), below.) As has been noted by Dolgopyat and De La Llave in [2], the process  $\bar{V}(t)$  is self-similar, that is for  $c > 0$  the process  $\bar{V}(c^3 t)/c$  satisfies the same stochastic differential equation with initial condition  $\bar{V}(0)/c$ . Therefore, for  $c$  fixed,  $V(c^3 |v_0|^3 t)/(c|v_0|)$  tends to the diffusion process (6) starting at  $\tilde{v}/c$ . If, instead, we assume that  $v_0$  is large but fixed, and take the limit as  $c \rightarrow \infty$ , we formally obtain that  $V(c^3 t)/c$  tends to the diffusion process (6) starting at the origin. We remark that the diffusion processes satisfying the self-similarity property described above are well understood (see e.g. [8], Section XI). In particular the fact that this process is non-recurrent for  $d > 3$  plays a crucial role in our analysis.

## 2 The Force Field

Let  $S_{R,m}$  be the space of smooth functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  which are supported inside the ball of radius  $R$  centered at the origin and satisfy  $\|f\|_{C^2(\mathbb{R}^d)} \leq m$ . Let  $\mu$  be a probability measure on  $S_{R,m}$ . We assume that

$$\int_{S_{R,m}} \int_{\mathbb{R}^d} f(x) dx d\mu(f) = 0.$$

We also assume that  $\mu$  is isotropic, that is the vectors  $(O^{-1}f(Ox_1), \dots, O^{-1}f(Ox_n))$  and  $(f(x_1), \dots, f(x_n))$  have the same distribution for any orthogonal matrix  $O$  and points  $x_1, \dots, x_n \in \mathbb{R}^d$ . Suppose that on a probability space  $(\Omega', \mathcal{F}', P')$  we have a sequence of functions  $f_i : \Omega' \rightarrow S_{R,m}$  which are independent and identically distributed with distribution  $\mu$ . We shall consider random vector fields  $F$  of the form

$$F(x) = \sum_{i=1}^{\infty} f_i(x - r_i), \tag{5}$$

where  $r_i$  form a Poisson point field with unit intensity on  $\mathbb{R}^d$ . We assume that the Poisson field is defined over the same probability space  $(\Omega', \mathcal{F}', P')$  and is independent of the sequence  $f_i$ . Note that the force field  $F$  defined by (5) is stationary, isotropic, and has zero mean. We shall denote the  $j$ -th coordinate of the vector  $F$  by  $F^j$ .

We observe that the fact that our force is Poisson and rotation invariant (rather than a general strongly mixing force) is primarily used in subsection 6.2. An alternative approach would be to estimate the rate of convergence in the averaging theorem (our Lemma 5.5) using the techniques of [1] or [6] but this would make the proof much more complicated. Therefore in this paper to we consider the simplest possible force distribution leaving the extension to more general force fields as an open question.

### 3 Formulation of the Main Result

Let  $W_t$  be a standard  $d$ -dimensional Brownian motion. Consider the  $d$ -dimensional process  $\bar{V}(t)$  which satisfies the diffusion equation

$$d\bar{V}(t) = \frac{1}{\sqrt{|\bar{V}(t)|}} \left( \lambda dW_t + (\sigma - \lambda) \frac{\bar{V}(t)}{|\bar{V}^2(t)|} (\bar{V}(t), dW_t) \right) + \frac{((d-2)\sigma^2 - (d-1)\lambda^2)\bar{V}(t)}{2|\bar{V}(t)|^3} dt, \quad (6)$$

where

$$\sigma^2 = \int_{-\infty}^{\infty} \mathbb{E}(F^1(0)F^1(e_1t))dt, \quad \lambda^2 = \int_{-\infty}^{\infty} \mathbb{E}(F^2(0)F^2(e_1t))dt, \quad (7)$$

and  $e_1$  is the first coordinate vector. It is clear that the integrals defining  $\sigma^2$  and  $\lambda^2$  are non-negative. We shall require that

$$\int_{-\infty}^{\infty} \mathbb{E}(F^1(0)F^1(e_1t))dt > 0. \quad (8)$$

Thus, the case when  $F = \nabla H$ , where  $H$  is a stationary random field, is excluded from consideration. The generator of the process  $\bar{V}(t)$  is

$$L = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial v_i} a_{ij}(v) \frac{\partial}{\partial v_j},$$

where

$$a_{ij}(v) = \int_{-\infty}^{\infty} \mathbb{E}(F^i(0)F^j(vt))dt.$$

By examining the stochastic differential equation satisfied by  $|\bar{V}(t)|^2/2$  (see formula (9) below), it follows that the origin is an inaccessible point for the process  $\bar{V}(t)$  if  $d \geq 3$  (see [8], Section XI). Therefore the solution of (6) with initial condition  $\bar{V}(0) \neq 0$  exists for all  $t$ . By the solution with the initial condition  $\bar{V}(0) = 0$  we shall mean the limit in distribution, as  $\bar{V}(0) \rightarrow 0$ , of solutions with initial condition  $\bar{V}(0)$ . We shall prove the following theorem.

**Theorem 3.1.** *Let  $F$  be a vector field in  $\mathbb{R}^d$ ,  $d \geq 4$ , given by (5), which satisfies (8). For each sufficiently large  $v_0$  there is a set  $\Omega'_{v_0}$  such that  $\lim_{|v_0| \rightarrow \infty} \mathbb{P}'(\Omega'_{v_0}) = 1$  and if  $\Omega'_{v_0}$  is viewed as a probability space with the measure  $\mathbb{P}'_{v_0}(A) = \mathbb{P}'(A)/\mathbb{P}'(\Omega'_{v_0})$ , then*

- (a) *the processes  $X(t)$  and  $V(t)$  tend to infinity almost surely,*
- (b) *the processes  $V(c^3t)/c$  on  $\Omega'_{v_0}$  converge in distribution, as  $c \rightarrow \infty$ , to the solution of (6) with the initial condition  $\bar{V}(0) = 0$ .*

Let  $E(t) = |V(t)|^2/2$  be the kinetic energy of the particle at time  $t$ , and  $\bar{E}(t) = |\bar{V}(t)|^2/2$ , where  $\bar{V}(t)$  is the solution of (6) with initial condition  $\bar{V}(0) = 0$ . By the Ito

formula,  $\bar{E}(t)$  is the solution of

$$d\bar{E}(t) = \sigma(2\bar{E}(t))^{1/4} dB_t + \frac{\sigma^2(d-1)}{2\sqrt{2\bar{E}(t)}} dt \quad (9)$$

with the initial condition  $\bar{E}(0) = 0$ , where  $B_t$  is a standard one-dimensional Brownian motion.

**Remark.** Up to a change of time by a constant factor, the process  $\bar{E}^{3/4}(t)$  is a Bessel process with dimension  $2d/3$ . Therefore if  $d > 3$  then

$$P_{v_0} (|\bar{V}(t)| \text{ reaches } 2|v_0| \text{ before } |v_0|/2) > \frac{1}{2}. \quad (10)$$

Let

$$\bar{X}(t) = \int_0^t \bar{V}(s) ds. \quad (11)$$

Theorem 3.1 immediately implies the following.

**Corollary 3.2.** *Let  $F$  be a vector field in  $\mathbb{R}^d$ ,  $d \geq 4$ , given by (5), which satisfies (8). For each sufficiently large  $v_0$  there is a set  $\Omega'_{v_0}$  such that  $\lim_{|v_0| \rightarrow \infty} P'(\Omega'_{v_0}) = 1$  and if  $\Omega'_{v_0}$  is viewed as a probability space with the measure  $P'_{v_0}(A) = P'(A)/P'(\Omega'_{v_0})$ , then*

(a) *the processes  $E(c^3t)/c^2$  on  $\Omega'_{v_0}$  converge in distribution, as  $c \rightarrow \infty$ , to the solution of (9) with the initial condition  $\bar{E}(0) = 0$ . The processes  $X(c^3t)/c^4$  on  $\Omega'_{v_0}$  converge in distribution, as  $c \rightarrow \infty$ , to the process  $\bar{X}(t)$  defined by (11).*

(b) *There is a constant  $c$  such that  $E(t)/t^{2/3}$  converges in distribution to a random variable with density*

$$p(x) = \frac{3}{2\Gamma(d/3)} x^{\frac{d}{2}-1} \exp\left(-x^{\frac{3}{2}}\right).$$

## 4 Auxiliary Processes

### 4.1 Time Discretization

Let  $X(t)$  be the solution of (1) with initial conditions  $X(0) = 0$ ,  $V(0) = v_0$ . We assume that the field  $F$  and, consequently, the process  $X(t)$  are defined on a probability space  $(\Omega', \mathcal{F}', P')$ . Assume, momentarily, that the trajectories of  $X(t)$  always “keep exploring” new regions of  $\mathbb{R}^d$  in the sense that for any  $t \geq 0$  the tail of the trajectory  $X(s)$ ,  $s \geq t+1$ , is separated from the initial part of the trajectory  $X(s)$ ,  $s \leq t$ , by a distance larger than  $2R$ . Then, for large  $t$ , the interval  $[0, t]$  can be split into sub-intervals, such that the contribution to the integral on the right-hand side of (2) from different sub-intervals are almost independent. This fact will be helpful when proving that  $V(c^3t)/c$  converges to a diffusion process.

We shall demonstrate that with high probability the trajectories of the process  $X(t)$  indeed have the desired property if the initial velocity is large. To this end, we shall

construct an auxiliary process  $Y(t)$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The processes  $X(t)$  and  $Y(t)$  will have the same distribution if events with small probabilities are excluded from their respective probability spaces. The process  $Y(t)$  is defined as the solution of

$$\dot{Y}(t) = \tilde{F}(t, Y(t)), \quad Y(0) = 0, \quad \dot{Y}(0) = v_0, \quad (12)$$

where  $\tilde{F}(t, x)$  can be obtained from  $F(x)$  by “switching on” new versions of  $F(x)$  at stopping times  $\tau_n$ , as described below.

Let  $i, n \geq 1$ , and  $f_i^n$  be independent identically distributed functions with distribution  $\mu$ . Let  $F_0 = F$ . Define the sequence of random fields  $F_1, F_2, \dots$  as follows:

$$F_n(x) = \sum_{i=1}^{\infty} f_i^n(x - r_i^n),$$

where, for each  $n \geq 1$ ,  $r_i^n$  form a Poisson point field with unit intensity on  $\mathbb{R}^d \setminus B_{2R}(0)$  and zero intensity on  $B_{2R}(0)$ , and  $B_{2R}(0)$  is the ball of radius  $2R$  centered at the origin. The Poisson fields  $r^0, r^1, r^2, \dots$  are assumed to be independent (here  $r^0 = r$ ). We can assume that the random fields  $F_n$  are defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , which is an extension of the original probability space  $(\Omega', \mathcal{F}', \mathbb{P}')$ . Let  $\tau_0 = 0$ ,  $\tilde{F}_0 = F_0$ ,  $Y_0 = 0$ , and  $v_0$  be the initial condition for the process  $X(t)$ . Assuming that  $\tau_{n-1}$ ,  $\tilde{F}_{n-1}$ ,  $Y_{n-1}$ , and  $v_{n-1}$  have been defined for some  $n \geq 1$ , we inductively define  $\tau_n$ ,  $\tilde{F}_n$ ,  $Y_n$ , and  $v_n$ . Let  $y(t)$  be the solution of the equation

$$\ddot{y}(t) = \tilde{F}_{n-1}(y(t)), \quad t \geq \tau_{n-1}$$

with the initial conditions  $y(\tau_{n-1}) = Y_{n-1}$ ,  $\dot{y}(\tau_{n-1}) = v_{n-1}$ . Let  $l = 4R$ . Let  $\tau_n$  be the first time after  $\tau_{n-1} + l|v_{n-1}|^{-1}$  when there are no points  $r_i^{n-1}$ ,  $i \geq 1$ , within the  $2R$ -neighborhood of  $y(t) - Y_{n-1}$ , that is

$$\tau_n = \inf\{t \geq \tau_{n-1} + l|v_{n-1}|^{-1} : \inf_{i \geq 1} |y(t) - Y_{n-1} - r_i^{n-1}| \geq 2R\}.$$

If  $\tau_n = \infty$ , then  $Y_i$ ,  $v_i$  and  $\tilde{F}_i(x)$  are undefined for  $i \geq n$ . Otherwise, define  $Y_n = y(\tau_n)$ ,  $v_n = \dot{y}(\tau_n)$ , and  $\tilde{F}_n(x) = F_n(x - Y_n)$ .

Now we can set  $\tilde{F}(t, x) = \tilde{F}_{n-1}(x)$  for  $\tau_{n-1} \leq t < \tau_n$ . Then the solution  $Y(t)$  of (12) satisfies  $Y(\tau_n) = Y_n$  and  $\dot{Y}(\tau_n) = v_n$ . The relation of  $Y(t)$  to the original process  $X(t)$  is explained by the following lemma.

**Lemma 4.1.** *Let  $Y(t)$  be the solution of (12) on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . For each sufficiently large  $v_0$  there are sets  $\Omega'_{v_0} \subseteq \Omega'$  and  $\Omega_{v_0} \subseteq \Omega$  with the following properties:*

- (a)  $\lim_{|v_0| \rightarrow \infty} \mathbb{P}'(\Omega'_{v_0}) = \lim_{|v_0| \rightarrow \infty} \mathbb{P}(\Omega_{v_0}) = 1$ .
- (b) *The processes  $X(t)$  and  $Y(t)$  have the same distribution if restricted to the spaces  $\Omega'_{v_0}$  and  $\Omega_{v_0}$ , respectively.*
- (c) *If  $\Omega_{v_0}$  is viewed as a probability space with the measure  $\mathbb{P}_{v_0}(A) = \mathbb{P}(A)/\mathbb{P}(\Omega_{v_0})$ , then the processes  $\dot{Y}(c^3t)/c$  on  $\Omega_{v_0}$  converge in distribution, as  $c \rightarrow \infty$ , to the solution of (6) with the initial condition  $\bar{V}(0) = 0$ .*

It is clear that Theorem 3.1 follows from Lemma 4.1. We shall study the properties of  $Y(t)$  and prove parts (a) and (b) of Lemma 4.1 in Section 5. We then prove part (c) of Lemma 4.1 in Section 7.

## 4.2 Another Auxiliary Process

Note that the distribution of vector field  $F_0$  is slightly different from the distribution of the fields  $F_n$ ,  $n \geq 1$ . Namely,  $F_0$  is based on a Poisson field on  $\mathbb{R}^d$ , while  $F_n$ ,  $n \geq 1$ , are based on Poisson fields on  $\mathbb{R}^d \setminus B_{2R}(0)$ .

Consider the vector field  $\overline{F}$ , which is defined in the same way as  $\tilde{F}$ , except that now we assume  $F_0$  to be defined by a Poisson field with unit intensity on  $\mathbb{R}^d \setminus B_{2R}(0)$  and zero intensity on  $B_{2R}(0)$ . The process  $Z(t)$  is defined as the solution of

$$\ddot{Z}(t) = \overline{F}(t, Z(t)), \quad Z(0) = 0, \quad \dot{Z}(0) = w_0,$$

where  $w_0$  is a random vector independent of  $\overline{F}$ . The reason to consider  $Z(t)$  is the following Markov property.

Let  $\mathcal{G}_n$  be the  $\sigma$ -algebra generated by  $\tilde{F}_i$ ,  $i \leq n-1$ . For any  $n \geq 1$ ,  $A \in \mathcal{B}(\mathbb{R}^d)$ , and  $B \in \mathcal{B}(C([0, \infty)))$  we have:

$$\mathrm{P}(Y(\tau_n + \cdot) \in B | \mathcal{G}_n) \chi_{\{\dot{Y}(\tau_n) \in A\}} = \mathrm{P}(Z(\cdot) \in B) \chi_{\{w_0 \in A\}} \quad \text{in distribution,}$$

where the initial velocity vector  $w_0$  for the process  $Z(t)$  is assumed to be distributed as  $\dot{Y}(\tau_n)$ .

## 5 Preliminaries

In this section we recall some results about diffusion approximation for the process  $\dot{Y}(t)$  and provide bounds on probabilities of some unlikely events.

### 5.1 Behavior of $Y(t)$ and $\dot{Y}(t)$ on the Time Interval $[\tau_n, \tau_{n+1}]$

In this subsection we shall prove that with high probability the velocity vector does not change significantly between the times  $\tau_n$  and  $\tau_{n+1}$  if  $|v_n|$  is large. Therefore  $Y(t)$  can be well approximated by a straight line.

Let us examine the equation

$$\ddot{y}(t) = \tilde{F}_n(y(t)), \quad y(\tau_n) = Y_n, \quad \dot{y}(\tau_n) = v_n, \tag{13}$$

on the part of the probability space where  $\tau_n < \infty$ . Note that  $Y(t)$  satisfies this equation on the interval  $[\tau_n, \tau_{n+1}]$ .

Let  $z_n(t) = Y_n + (t - \tau_n)v_n$ , that is  $z_n(t)$  is the solution of

$$\ddot{z}_n(t) = 0, \quad z_n(\tau_n) = Y_n, \quad \dot{z}_n(\tau_n) = v_n.$$

Let  $-1 < \alpha < 0$ . Let  $T_n = |v_n|^\alpha$ . Let  $\eta_n$ ,  $n \geq 1$  be the first time after  $\tau_{n-1} + l|v_{n-1}|^{-1}$  when there are no points  $r_i^{n-1}$ ,  $i \geq 1$ , within the  $2R$ -neighborhood of  $z_{n-1}(t) - Y_{n-1}$ , that is

$$\eta_n = \inf\{t \geq \tau_{n-1} + l|v_{n-1}|^{-1} : \inf_{i \geq 1} |z_{n-1}(t) - Y_{n-1} - r_i^{n-1}| \geq 2R\}.$$

Let

$$\xi_n(t) = \int_{\tau_n}^t \tilde{F}_n(z_n(s)) ds, \quad \zeta_n(t) = \int_{\tau_n}^t \xi_n(s) ds, \quad \tau_n \leq t \leq \tau_n + T_n.$$

Let us first examine (13) with  $n = 0$ . We shall say that an event (which depends on  $v_0$ ) happens with high probability if for any  $N$  the probability of the complement does not exceed  $|v_0|^{-N}$  for all sufficiently large  $|v_0|$ . From the definition of  $F_0$  it easily follows that for any  $\delta > 0$

$$\|F_0\|_{C^2(B_{|v_0|}(0))} \leq \delta \ln |v_0| \quad (14)$$

with high probability. It is not difficult to show that for any  $\delta > 0$

$$\sup_{0 \leq t \leq T_0} |\xi_0(t)| \leq |v_0|^{(\alpha-1)/2+\delta} \quad (15)$$

with high probability. Therefore, for any  $\delta > 0$

$$\sup_{0 \leq t \leq T_0} |\zeta_0(t)| \leq |v_0|^{(3\alpha-1)/2+\delta} \quad (16)$$

with high probability. By (13),

$$y(t) = z_0(t) + \zeta_0(t) + \int_0^t \int_0^u (F_0(y(s)) - F_0(z_0(s))) ds du.$$

Therefore, if  $y(t) \in B_{|v_0|}(0)$  for  $0 \leq t \leq T_0$ , then

$$|y(t) - z_0(t) - \zeta_0(t)| \leq T_0 \|F_0\|_{C^1(B_{|v_0|}(0))} \int_0^t |y(s) - z_0(s)| ds \quad \text{for } 0 \leq t \leq T_0.$$

By (14), (16), and due to the Gronwall Inequality, for any  $\delta > 0$

$$\sup_{0 \leq t \leq T_0} |y(t) - z_0(t)| \leq |v_0|^{(3\alpha-1)/2+\delta} \quad (17)$$

with high probability. By (13),

$$\dot{y}(t) - v_0 = \xi_0(t) + \int_0^t (F_0(y(s)) - F_0(z_0(s))) ds.$$

Due to (15) and (17), for any  $\delta > 0$

$$\sup_{0 \leq t \leq T_0} |\dot{y}(t) - v_0| \leq |v_0|^{(\alpha-1)/2+\delta} \quad (18)$$



with high probability. By the expression  $\langle \nabla F_0, v \rangle$ , where  $v$  is a vector, we shall mean the vector  $w$  with components  $w^j = \sum_{i=1}^d F_{0x_i}^j v^i$ . If  $y(t) \in B_{|v_0|}(0)$  for  $0 \leq t \leq T_0$ , then by the Taylor formula

$$\begin{aligned} & \sup_{0 \leq t \leq T_0} |\dot{y}(t) - v_0 - \xi_0(t)| \leq \\ & \leq \sup_{0 \leq t \leq T_0} \left| \int_0^t \langle \nabla F_0(z_0(s)), (y(s) - z_0(s)) \rangle ds \right| + \sup_{0 \leq t \leq T_0} \frac{1}{2} \int_0^t \|F_0\|_{C^2(B_{|v_0|}(0))} |y(s) - z_0(s)|^2 ds. \end{aligned}$$

From (14) and (17) it follows that for any  $\delta > 0$  the second term in the right-hand side does not exceed  $|v_0|^{4\alpha-1+\delta}$  with high probability. To estimate the first term we need to use the fact that

$$\sup_{0 \leq t \leq T_0} \left| \int_0^t F_{0x_i}^j(z_0(s)) ds \right| \leq |v_0|^{(\alpha-1)/2+\delta}, \quad 1 \leq i, j \leq d,$$

with high probability. Then, after integrating by parts and using (17) and (18), we obtain that the first term in the right-hand side does not exceed  $|v_0|^{2\alpha-1+\delta}$  with high probability. Therefore, for any  $\delta > 0$

$$\sup_{0 \leq t \leq T_0} |\dot{y}(t) - v_0 - \xi_0(t)| \leq |v_0|^{2\alpha-1+\delta} \quad (19)$$

with high probability.

Using the proximity of  $y(t)$  and  $z_0(t)$  (formula (17)) it is not difficult to show that for any  $\delta > 0$

$$\tau_1 \leq T_0 \quad (20)$$

with high probability and

$$|\xi_0(\tau_1) - \int_0^{\tau_1} F_0(z_0(s)) ds| \leq |v_0|^{-1+\delta}$$

with high probability. Let  $H_n$  be the following event

$$H_n = \left\{ \left| \xi_n(\tau_{n+1}) - \int_{\tau_n}^{\tau_{n+1}} F_n(z_n(s)) ds \right| \leq |v_n|^{-1+\delta} \right\}.$$

In the Appendix we again use the proximity of  $y(t)$  and  $z_0(t)$  to prove that

$$\mathbb{E} \left( \chi_{H_0} \left| \xi_0(\tau_1) - \int_0^{\tau_1} F_0(z_0(s)) ds \right| \right) \leq |v_0|^{-3+\delta} \quad (21)$$

for all sufficiently large  $|v_0|$ . Combining (20) with (18), we obtain that for any  $\delta > 0$

$$\sup_{0 \leq t \leq \tau_1} |\dot{y}(t) - v_0| \leq |v_0|^{(\alpha-1)/2+\delta}$$

with high probability. Recalling that we could take  $\alpha$  to be arbitrarily close to  $-1$ , we can summarize the results obtained above as the following lemma.

**Lemma 5.1.** *For any  $N$  and  $\delta > 0$  we have*

$$\mathbb{P}(\tau_1 > |v_0|^{-1+\delta}) \leq |v_0|^{-N}, \quad (22)$$

$$\mathbb{P}\left(\sup_{0 \leq t \leq \tau_1} |\dot{Y}(t) - v_0| > |v_0|^{-1+\delta}\right) \leq |v_0|^{-N}, \quad (23)$$

$$\mathbb{P}\left(\sup_{0 \leq t \leq \tau_1} |\dot{Y}(t) - v_0 - \xi_0(t)| > |v_0|^{-3+\delta}\right) \leq |v_0|^{-N},$$

$$\mathbb{P}\left(|\xi_0(\tau_1) - \int_0^{\eta_1} F_0(z_0(s))ds| > |v_0|^{-1+\delta}\right) \leq |v_0|^{-N},$$

$$\mathbb{E}\left(\chi_{H_0} |\xi_0(\tau_1) - \int_0^{\eta_1} F_0(z_0(s))ds|\right) \leq |v_0|^{-3+\delta}$$

for all sufficiently large  $|v_0|$ .

**Remark.** Obviously, the same result holds if the process  $Y(t)$  is replaced by the process  $Z(t)$  with initial velocity  $v_0$ . Therefore, we have the following.

**Corollary 5.2.** *For any  $N$  and  $\delta > 0$  there is  $r > 0$  such that*

$$\mathbb{P}(\tau_{n+1} - \tau_n > |v_n|^{-1+\delta} | \mathcal{G}_n) \leq |v_n|^{-N}, \quad (24)$$

$$\mathbb{P}\left(\sup_{\tau_n \leq t \leq \tau_{n+1}} |\dot{Y}(t) - v_n| > |v_n|^{-1+\delta} | \mathcal{G}_n\right) \leq |v_n|^{-N}, \quad (25)$$

$$\mathbb{P}\left(\sup_{\tau_n \leq t \leq \tau_{n+1}} |\dot{Y}(t) - v_n - \xi_n(t)| > |v_n|^{-3+\delta} | \mathcal{G}_n\right) \leq |v_n|^{-N}, \quad (26)$$

$$\mathbb{P}\left(|\xi_n(\tau_{n+1}) - \int_{\tau_n}^{\eta_{n+1}} F_n(z_n(s))ds| > |v_n|^{-1+\delta} | \mathcal{G}_n\right) \leq |v_n|^{-N}, \quad (27)$$

$$\mathbb{E}\left(\chi_{H_n} |\xi_n(\tau_{n+1}) - \int_{\tau_n}^{\eta_{n+1}} F_n(z_n(s))ds| | \mathcal{G}_n\right) \leq |v_n|^{-3+\delta} \quad (28)$$

hold almost surely on the event  $|v_n| > r$ .

## 5.2 Behavior of $Y(t)$ and $\dot{Y}(t)$ on a Time Interval Proportional to $|v_0|^3$

Observe that for equation (3) with  $F$  of zero mean it takes time  $\varepsilon^{-2}$  for  $\dot{x}$  to change significantly. In view of the relation (4), we expect that if the initial velocity  $v_0$  is large and  $c$  is a constant, then it takes time of order  $|v_0|^3$  for the velocity to change by  $c|v_0|$ .

In this section we recall the effective equation for  $\dot{Y}$  on the scale  $|v_0|^3$  and provide estimates for the probability that  $\dot{Y}$  changes much faster or much slower than expected.

For  $a, r > 0$ ,  $b > 1$ , and  $n \geq 0$ , let

$$\begin{aligned}\tau_n^a &= \min_{k \geq n} \{\tau_k : \tau_k - \tau_n \geq ar^3\}, \\ \tau_n^b &= \min_{k \geq n} \{\tau_k : |\dot{Y}(t)| \notin (r, br) \text{ for some } \tau_n \leq t \leq \tau_k\}, \\ \bar{\tau}_n &= \min\{\tau_n^a, \tau_n^b\}.\end{aligned}$$

In what follows  $a$  and  $b$  will be fixed, but  $r$  will be allowed to vary.

Assume that  $r$  is large and  $|v_0| \in (r, br)$ . Let us first describe the behavior of the process  $Y(t)$  on the time interval  $[0, \bar{\tau}_0]$ .

**Lemma 5.3.** *For any  $N, \delta > 0$ ,  $a > 0$ , and  $b > 1$ , we have*

$$\mathbb{P}(\tau_{n+1} - \tau_n > |v_n|^{-1+\delta} \text{ for some } n \text{ such that } \tau_n < \bar{\tau}_0) \leq r^{-N}, \quad (29)$$

$$\mathbb{P}\left(\sup_{\tau_n \leq t \leq \tau_{n+1}} |\dot{Y}(t) - v_n| > |v_n|^{-1+\delta} \text{ for some } n \text{ such that } \tau_n < \bar{\tau}_0\right) \leq r^{-N}, \quad (30)$$

$$\mathbb{P}(\bar{\tau}_0 = \infty) \leq r^{-N} \quad (31)$$

for all sufficiently large  $r$  and all  $|v_0| \in (r, br)$ .

*Proof.* For fixed  $n$ , the probability  $\mathbb{P}(\tau_{n+1} - \tau_n > |v_n|^{-1+\delta}, \tau_n < \bar{\tau}_0)$  is estimated from above by  $r^{-N}$  due to (22) (if  $n = 0$ ) and (24) (if  $n \geq 1$ ). The number of  $n$  for which  $\tau_n < \bar{\tau}_0$  does not exceed  $abr^4$ . Since  $N$  was arbitrary, this implies (29). In the same way, (23) and (25) imply (30). Finally, (29) implies (31) again due to the fact that  $\tau_n \geq \bar{\tau}_0$  for  $n > abr^4$ .  $\square$

As before, by considering  $Z(t)$  instead of  $Y(t)$ , we obtain the following.

**Corollary 5.4.** *For any  $N, \delta > 0$ ,  $a > 0$ , and  $b > 1$ , we have*

$$\begin{aligned}\mathbb{P}(\tau_{k+1} - \tau_k > |v_k|^{-1+\delta} \text{ for some } k \text{ such that } \tau_n < \tau_k < \bar{\tau}_n | \mathcal{G}_n) &\leq r^{-N}, \\ \mathbb{P}\left(\sup_{\tau_k \leq t \leq \tau_{k+1}} |\dot{Z}(t) - v_k| > |v_k|^{-1+\delta} \text{ for some } k \text{ such that } \tau_n \leq \tau_k < \bar{\tau}_n | \mathcal{G}_n\right) &\leq r^{-N}, \\ \mathbb{P}(\bar{\tau}_n = \infty | \mathcal{G}_n) &\leq r^{-N}\end{aligned}$$

for all sufficiently large  $r$  almost surely on the event  $|v_n| \in (r, br)$ .

The next lemma is a slight modification of the results of [3, 4, 5, 6] to the case of the processes  $Y(t)$  and  $Z(t)$  so we omit the proof.

**Lemma 5.5.** *Assume that  $v_0 = (|v_0|, 0, \dots, 0)$ , and  $|v_0| \rightarrow \infty$ . Then both families of processes  $\dot{Y}(|v_0|^3 t)/|v_0|$  and  $\dot{Z}(|v_0|^3 t)/|v_0|$  converge weakly to the diffusion process  $\bar{V}(t)$  given by (6) starting at  $(1, 0, \dots, 0)$ .*

**Corollary 5.6.** *For any  $a > 0$  and  $b > 1$  there is  $c < 1$  such that*

$$\mathbb{P}(\tau_0^a < \tau_0^b) \leq c$$

*for all sufficiently large  $r$  and all  $|v_0| \in (r, br)$ . The same is true if  $\tau_0^a$  and  $\tau_0^b$  are defined as the stopping times for the process  $Z(t)$  with initial velocity  $v_0$ .*

Now we can replace the stopping time  $\bar{\tau}_0$  by  $\tau_0^b$  in Lemma 5.3.

**Lemma 5.7.** *For any  $N, \delta > 0$ , and  $b > 1$ , we have*

$$\mathbb{P}(\tau_{n+1} - \tau_n > |v_n|^{-1+\delta} \text{ for some } n \text{ such that } \tau_n < \tau_0^b) \leq r^{-N}, \quad (32)$$

$$\mathbb{P}\left(\sup_{\tau_n \leq t \leq \tau_{n+1}} |\dot{Y}(t) - v_n| > |v_n|^{-1+\delta} \text{ for some } n \text{ such that } \tau_n < \tau_0^b\right) \leq r^{-N}, \quad (33)$$

$$\mathbb{P}(\tau_0^b = \infty) \leq r^{-N} \quad (34)$$

*for all sufficiently large  $r$  and all  $|v_0| \in (r, br)$ .*

*Proof.* Let

$$q_Y = q_Y(r) = \sup_{v_0: |v_0| \in (r, br)} \mathbb{P}(\tau_{n+1} - \tau_n > |v_n|^{-1+\delta} \text{ for some } n \text{ such that } \tau_n < \tau_0^b).$$

Let  $q_Z = q_Z(r)$  be defined as  $q_Y$ , with the only difference that the stopping times are assumed to correspond to the process  $Z(t)$  instead of  $Y(t)$ . Take an arbitrary  $a > 0$ . Then, for  $|v_0| \in (r, br)$  we have

$$\begin{aligned} & \mathbb{P}(\tau_{n+1} - \tau_n > |v_n|^{-1+\delta} \text{ for some } n \text{ such that } \tau_n < \tau_0^b) \leq \\ & \mathbb{P}(\tau_{n+1} - \tau_n > |v_n|^{-1+\delta} \text{ for some } n \text{ such that } \tau_n < \bar{\tau}_0) + \\ & \mathbb{P}(\tau_{n+1} - \tau_n > |v_n|^{-1+\delta} \text{ for some } n \text{ such that } \bar{\tau}_0 \leq \tau_n < \tau_0^b). \end{aligned}$$

The first term in the right-hand side does not exceed  $r^{-N}$  by Lemma 5.3. The second term does not exceed  $q_Z \mathbb{P}(\tau_0^a < \tau_0^b) \leq q_Z c$ , where  $c$  is the constant from Corollary 5.6. Therefore,

$$q_Y \leq r^{-N} + cq_Z.$$

Similarly,

$$q_Z \leq r^{-N} + cq_Z.$$

Since  $c < 1$  and  $N$  is arbitrary, these two inequalities imply (32). The proof of (33) is similar. In order to prove (34), define

$$q_Y(k) = \sup_{v_0: |v_0| \in (r, br)} \mathbb{P}(\tau_0^b > 2kar^3), \quad k \geq 0.$$

Let  $q_Z(k)$  be defined as  $q_Y(k)$ , with the only difference that the stopping times are assumed to correspond to the process  $Z(t)$  instead of  $Y(t)$ . Note that for  $|v_0| \in (r, br)$  we have

$$\mathbb{P}(\tau_0^b > 2kar^3) \leq \mathbb{P}(\bar{\tau}_0 > 2ar^3) + \mathbb{P}(\bar{\tau}_0 \leq 2ar^3, \tau_0^b > 2kar^3).$$

The first term in the right-hand side can be estimated from above by  $r^{-N}$  due to (29). The second term does not exceed  $q_Z(k-1)\mathbb{P}(\tau_0^a < \tau_0^b) \leq q_Z(k-1)c$ , where  $c$  is the constant from Corollary 5.6. Therefore,

$$q_Y(k) \leq r^{-N} + cq_Z(k-1).$$

Similarly,

$$q_Z(k) \leq r^{-N} + cq_Z(k-1).$$

Since  $c < 1$  and  $N$  is arbitrary, these two inequalities imply that

$$\max(q_Y(k), q_Z(k)) \leq r^{-N} + c^k. \quad (35)$$

This implies (34) since an arbitrarily large  $k$  can be taken.  $\square$

**Corollary 5.8.** *For any  $N$ ,  $\delta > 0$ , and  $b > 1$ , we have*

$$\begin{aligned} & \mathbb{P}(\tau_{k+1} - \tau_k > |v_k|^{-1+\delta} \text{ for some } k \text{ such that } \tau_n < \tau_k < \tau_n^b | \mathcal{G}_n) \leq r^{-N}, \\ & \mathbb{P}\left(\sup_{\tau_k \leq t \leq \tau_{k+1}} |\dot{Y}(t) - v_k| > |v_k|^{-1+\delta} \text{ for some } k \text{ such that } \tau_n \leq \tau_k < \tau_n^b | \mathcal{G}_n\right) \leq r^{-N}, \\ & \mathbb{P}(\tau_n^b = \infty | \mathcal{G}_n) \leq r^{-N} \end{aligned}$$

for all sufficiently large  $r$  almost surely on the event  $|v_n| \in (r, br)$ .

**Lemma 5.9.** *For any  $N$ ,  $\delta > 0$ , and  $k > 0$  we have*

$$\mathbb{P}\left(\sup_{0 \leq t \leq |v_0|^{3-\delta}} |\dot{Y}(t) - v_0| > k|v_0|\right) \leq |v_0|^{-N}$$

for all sufficiently large  $|v_0|$ .

*Proof.* Let us write

$$\dot{Y}(t) - v_0 = (v_1 - v_0) + (v_2 - v_1) + \dots + (v_n - v_{n-1}) + \dot{Y}(t) - v_n, \quad (36)$$

where  $n = n(t)$  is the random time such that  $\tau_{n-1} \leq t < \tau_n$ . Without loss of generality we may assume that  $k \leq 1/2$ . Let  $L = L(v_0) = \lceil 2|v_0|^{4-\delta} + 1 \rceil$ . Let  $\sigma$  be the random time defined by

$$\sigma = \min\{m : |(v_1 - v_0) + (v_2 - v_1) + \dots + (v_m - v_{m-1})| \geq k|v_0|/2\} \wedge L.$$

By (36),

$$\begin{aligned} & \left\{ \sup_{0 \leq t \leq |v_0|^{3-\delta}} |\dot{Y}(t) - v_0| > k|v_0| \right\} \subseteq \\ & \subseteq \{ \sigma < L \} \cup \left( \{ \sigma = L \} \cap \bigcup_{m=1}^L \left\{ \sup_{\tau_{m-1} \leq t \leq \tau_m} |\dot{Y}(t) - v_m| \geq k|v_0|/2 \right\} \right). \end{aligned}$$

Taking the union in  $m$  from 1 to  $L$  is explained as follows: if  $|\dot{Y}(t)|$  does not exceed  $2|v_0|$  on the time interval  $[0, |v_0|^{3-\delta}]$ , then  $\tau_L \geq |v_0|^{3-\delta}$ . Define  $c_i$ ,  $i \geq 1$ , by

$$(v_i - v_{i-1}) = \int_{\tau_{i-1}}^{\eta_i} F_{i-1}(z_{i-1}(s)) ds + c_i.$$

By Corollary 5.2 (formulas (26), (27), and (28)), for each  $N$  and  $\varepsilon > 0$  the estimates

$$\mathbb{P}(|c_i| > |v_0|^{-1+\varepsilon} | \mathcal{G}_{i-1}) \leq |v_0|^{-N}, \quad (37)$$

$$\mathbb{E}(|c_i| \chi_{\{|c_i| \leq |v_0|^{-1+\varepsilon}\}} | \mathcal{G}_{i-1}) \leq |v_0|^{-3+\varepsilon} \quad (38)$$

hold for each  $i$  on  $\{\sigma \geq i\}$  if  $|v_0|$  is sufficiently large. Let

$$\begin{aligned} C_j &= \sum_{i=1}^{j \wedge \sigma} |c_i|, \\ h_j &= \sum_{i=1}^{j \wedge \sigma} (|c_i| \chi_{\{|c_i| \leq |v_0|^{-1+\varepsilon}\}} - |v_0|^{-3+\varepsilon}). \end{aligned} \quad (39)$$

By (37) and (38),  $h_j$  is a supermartingale. Let  $h_j = \alpha_j + \beta_j$  be the Doob decomposition of  $h_j$ , where

$$\beta_j = \sum_{i=1}^{j \wedge \sigma} \mathbb{E} \left( (|c_i| \chi_{\{|c_i| \leq |v_0|^{-1+\varepsilon}\}} - |v_0|^{-3+\varepsilon}) | \mathcal{G}_{i-1} \right)$$

is a non-increasing process. Let  $\langle \alpha \rangle_j$  be the quadratic variation of  $\alpha_j$ . From (37) and (39) it follows that for each  $p \in \mathbb{N}$  there is a constant  $k_p$  such that

$$\mathbb{E} |\langle \alpha \rangle_j|^p \leq k_p (j|v_0|^{-2+2\varepsilon})^p.$$

In particular, for  $j = L$  we obtain

$$\mathbb{E} |\langle \alpha \rangle_\sigma|^p \leq k_p (|v_0|^{2+2\varepsilon-\delta})^p.$$

Take  $\varepsilon = \delta/3$ . Then, by the Chebyshev Inequality and the Martingale Moment Inequality, for each  $N$  there are  $p$  and  $K_p$  such that

$$\mathbb{P}(h_\sigma \geq k|v_0|/8) \leq \mathbb{P}(\alpha_\sigma \geq k|v_0|/8) \leq \mathbb{P}(|\alpha_\sigma|^{2p} \geq (k|v_0|/8)^{2p}) \leq$$

$$\frac{\mathbf{E}|\alpha_\sigma|^{2p}}{(k|v_0|/8)^{2p}} \leq \frac{K_p \mathbf{E}|\langle \alpha \rangle_\sigma|^p}{(k|v_0|/8)^{2p}} \leq |v_0|^{-N}$$

if  $|v_0|$  is sufficiently large. From (37) and (39) it then follows that for each  $N$

$$\mathbf{P}(C_\sigma \geq k|v_0|/4) \leq |v_0|^{-N} \quad (40)$$

if  $|v_0|$  is sufficiently large. Let

$$f_j = \sum_{i=1}^{j \wedge \sigma} \int_{\tau_{i-1}}^{\eta_i} F_{i-1}(z_{i-1}(s)) ds, \quad j \geq 1. \quad (41)$$

Notice that  $f_j$  is a martingale. Its quadratic variation is

$$\langle f \rangle_j = \sum_{i=1}^{j \wedge \sigma} \mathbf{E} \left( \left( \int_{\tau_{i-1}}^{\eta_i} F_{i-1}(z_{i-1}(s)) ds \right)^2 \middle| \mathcal{G}_{i-1} \right), \quad j \geq 1.$$

Using arguments similar to those in the proof of Corollary 5.2, it is easy to show that for  $p \in \mathbb{N}$  and all  $j \leq L$  we have

$$\mathbf{E}|\langle f \rangle_j|^p \leq (j|v_0|^{-2+\delta/2})^p$$

if  $|v_0|$  is sufficiently large. In particular, for  $j = L$  we obtain

$$\mathbf{E}|\langle f \rangle_\sigma|^p \leq (3|v_0|^{2-\delta/2})^p.$$

By the Chebyshev Inequality and the Martingale Moment Inequality, for each  $N$  there are  $p \in \mathbb{N}$  and  $K_p > 0$  such that

$$\mathbf{P}(|f_\sigma| \geq k|v_0|/4) = \mathbf{P}(|f_\sigma|^{2p} \geq (k|v_0|/4)^{2p}) \leq \frac{\mathbf{E}|f_\sigma|^{2p}}{(k|v_0|/4)^{2p}} \leq \frac{K_p \mathbf{E}|\langle f \rangle_\sigma|^p}{(k|v_0|/4)^{2p}} \leq |v_0|^{-N}$$

if  $|v_0|$  is sufficiently large.

Together with (40), this implies that

$$\mathbf{P}(\sigma < L) \leq |v_0|^{-N}.$$

It easily follows from Corollary 5.2 that

$$\mathbf{P}(\{\sigma = L\} \cap \bigcup_{m=1}^L \{ \sup_{\tau_{m-1} \leq t \leq \tau_m} |\dot{Y}(t) - v_m| \geq k|v_0|/2 \}) \leq |v_0|^{-N}.$$

□

## 6 Long time behavior of $Y(t)$ .

The goal of this section is to show that the paths of  $Y(t)$  are not self-intersecting with probability close to one, and therefore the distributions of  $X$  and  $Y$  are close, as claimed. This is achieved in subsection 6.2. In subsection 6.1 we establish some *a priori* bounds on the growth of  $|\dot{Y}|$ .

### 6.1 Behavior of $|\dot{Y}(t)|$ as $t \rightarrow \infty$ .

In this section we shall demonstrate that for large  $|v_0|$  with high probability the norm of the velocity vector  $|\dot{Y}(t)|$  grows as  $t^{1/3}$  when  $t \rightarrow \infty$ .

The idea of the proof is the following. We consider  $\dot{Y}$  at the moments  $s_n$  its modulus crosses  $2^l$  (alternating odd and even  $l$ ). By Lemma 5.5 and (10),  $\ln |Y(s_n)|$  can be well approximated by a simple random walk biased to the right. It follows that  $|\dot{Y}(s_n)|$  grows exponentially and so  $\dot{Y}(t)$  spends most of the time near its maximum. Since  $s_{n+1} - s_n$  is of order  $|\dot{Y}(s_n)|^3$ , by Lemma 5.5 the statement follows. Let us now give a detailed proof.

We start by describing a discretized version of the process  $|\dot{Y}(t)|$ . Let  $2^{m-\frac{1}{2}} \leq |v_0| < 2^{m+\frac{1}{2}}$  for some  $m \in \mathbb{Z}$ . Let  $0 < \delta < 1$ . Define, inductively, a sequence of events  $\mathcal{E}_n^\delta$  and three processes  $s_n, t_n \in \mathbb{R}^+ \cup \infty$  and  $\xi_n \in \mathbb{Z}$  as follows. Let  $\mathcal{E}_0^\delta = \Omega$ ,  $s_0 = t_0 = 0$ , and  $\xi_0 = m$ . Assume that  $\mathcal{E}_{n-1}^\delta$ ,  $s_{n-1}$ ,  $t_{n-1}$ , and  $\xi_{n-1}$  have been defined for some  $n \geq 1$ . We then define

$$s_n = \inf\{t : |\dot{Y}(t)| = 2^{\xi_{n-1}-1} \text{ or } |\dot{Y}(t)| = 2^{\xi_{n-1}+1}\},$$

$$t_n = \min\{\tau_k : \tau_k \geq s_n\}, \quad \text{and} \quad \xi_n = \log_2 |\dot{Y}(s_n)|.$$

$$\mathcal{E}_n^\delta = \mathcal{E}_{n-1}^\delta \cap \{t_n < \infty\} \cap \{\tau_{k+1} - \tau_k \leq |v_k|^{-1+\delta} \text{ for all } k \text{ such that } t_{n-1} < \tau_k \leq t_n\} \cap$$

$$\cap \left\{ \sup_{\tau_k \leq t \leq \tau_{k+1}} |\dot{Y}(t) - v_k| \leq |v_k|^{-1+\delta} \text{ for all } k \text{ such that } t_{n-1} \leq \tau_k \leq t_n \right\}.$$

Let  $\mathcal{F}_n$  be the  $\sigma$ -algebra of events determined before  $t_n$ , that is  $\mathcal{F}_n = \sigma(Y(t), t \leq t_n)$ . The process  $\xi_n$  is a random walk (with memory), while  $t_n$  can be viewed as transition times for the random walk.

The following lemma describes the one-step transition times and transition probabilities.

**Lemma 6.1.** (a)  $\mathcal{E}_n^\delta$  is  $\mathcal{F}_n$ -measurable. For each  $N > 0$  there is  $M$  such that for  $m \geq M$  we have

$$P(\mathcal{E}_n^\delta | \mathcal{F}_{n-1}) \geq 1 - 2^{-Nm} \quad \text{almost surely on } \{\xi_{n-1} = m\} \cap \mathcal{E}_{n-1}^\delta. \quad (42)$$

(b) For each  $N > 0$  there exist  $M$  and  $0 < c < 1$ , such that for  $m \geq M$  we have

$$P(t_n - t_{n-1} > 2^{3m} k | \mathcal{F}_{n-1}) \leq c^k + 2^{-Nm}, \quad k \geq 1, \quad \text{almost surely on } \{\xi_{n-1} = m\} \cap \mathcal{E}_{n-1}^\delta.$$

(c) There exist  $M$  and  $0 < c < 1$ , such that for  $m \geq M$  and  $n \geq 2$  we have

$$P(t_n - t_{n-1} < 2^{3m} | \mathcal{F}_{n-1}) \leq c \quad \text{almost surely on } \{\xi_{n-1} = m\} \cap \mathcal{E}_{n-1}^\delta.$$



(d) There is  $p > 1/2$  such that for any  $\varepsilon > 0$  there exists  $M$ , such that for  $m \geq M$  and  $n \geq 2$  we have

$$|\mathbb{P}(\xi_n = \xi_{n-1} + 1 | \mathcal{F}_{n-1}) - p| \leq \varepsilon \quad \text{almost surely on } \{\xi_{n-1} = m\} \cap \mathcal{E}_{n-1}^\delta.$$

*Proof.* (a) The fact that  $\mathcal{E}_n^\delta$  is  $\mathcal{F}_n$ -measurable follows from the definition of  $\mathcal{E}_n^\delta$  and  $\mathcal{F}_n$ . To prove (42), we write

$$\mathbb{P}(\mathcal{E}_n^\delta | \mathcal{F}_{n-1}) = \sum_{k=0}^{\infty} \mathbb{P}(\mathcal{E}_n^\delta \cap \{t_{n-1} = \tau_k\} | \mathcal{F}_{n-1}) = \sum_{k=0}^{\infty} \mathbb{P}(\mathcal{E}_n^\delta | \mathcal{G}_k) \chi_{\{t_{n-1} = \tau_k\}}.$$

As follows from Corollary 5.8 with  $b = 4$  and  $r = 2^{m-1}$ , the expression in the right-hand side is estimated from below by  $1 - 2^{-Nm}$  on  $\{\xi_{n-1} = m\} \cap \mathcal{E}_{n-1}^\delta$  for all sufficiently large  $m$ .

(b) The statement follows from (35) once we notice that  $|\dot{Y}(t_{n-1})| \in (2^{m-1}, 2^{m+1})$  on  $\{\xi_{n-1} = m\} \cap \mathcal{E}_{n-1}^\delta$  if  $m$  is sufficiently large.

(c) This follows from Lemma 5.5 once we take into account that, by the definition of  $\mathcal{E}_{n-1}^\delta$ , for  $n \geq 2$  and all sufficiently large  $m$  we have

$$||\dot{Y}(t_{n-1})| - 2^m| \leq 1 \quad \text{on } \{\xi_{n-1} = m\} \cap \mathcal{E}_{n-1}^\delta. \quad (43)$$

(d) Consider the limiting process  $\bar{V}(t)$  with  $|\bar{V}(0)| = 1$ . Let  $p$  be the probability that the process  $|\bar{V}(t)|$  reaches 2 before reaching 1/2. Notice that  $p > 1/2$ . Therefore, the statement follows from Lemma 5.5 once we take (43) into account.  $\square$

**Lemma 6.2.** For  $\delta > 0$  we have

$$\lim_{|v_0| \rightarrow \infty} \mathbb{P} \left( (|v_0| + t^{1/3})^{1-\delta} \leq |\dot{Y}(t)| \leq (|v_0| + t^{1/3})^{1+\delta} \quad \text{for all } t \geq 0 \right) = 1.$$

*Proof.* Let  $\bar{\varepsilon}$  be fixed and  $\varepsilon$  be a positive constant, to be specified later. Let  $A_{v_0}$  be the following event

$$A_{v_0} = \left( \bigcap_{n=0}^{\infty} \mathcal{E}_n^\varepsilon \right) \cap \{|\xi_n - pn - \xi_0| \leq \varepsilon(n + \xi_0) \text{ for all } n\}.$$

From parts (a) and (d) of Lemma 6.1 it easily follows that we can take a large enough  $M$  such that

$$\mathbb{P}(A_{v_0}) \geq 1 - \bar{\varepsilon}/3 \quad (44)$$

if  $v_0$  is such that  $\xi_0 > M$ . By part (b) of Lemma 6.1,

$$\mathbb{P} \left( A_{v_0} \cap \{t_n - t_{n-1} \geq k(n) 2^{3(p(n-1) + \xi_0 + \varepsilon(n-1 + \xi_0))}\} \right) \leq c^{k(n)} + 2^{-N(p(n-1) + \xi_0 + \varepsilon(n-1 + \xi_0))}$$

for each  $n$ , where  $0 < c < 1$ . Take  $k(n) = 2^{\varepsilon(n + \xi_0)}$ . Let

$$B_{v_0} = \{t_n - t_{n-1} \geq 2^{\varepsilon(n + \xi_0)} 2^{3(p(n-1) + \xi_0 + \varepsilon(n-1 + \xi_0))} \text{ for some } n\}.$$

Then

$$\mathbb{P}(A_{v_0} \cap B_{v_0}) \leq \sum_{n=1}^{\infty} \left( c^{2^{\varepsilon(n+\xi_0)}} + 2^{-N(p(n-1)+\xi_0-\varepsilon(n-1+\xi_0))} \right).$$

The right-hand side of this inequality can be made smaller than  $\bar{\varepsilon}/3$  by taking sufficiently large  $M$ .

Notice that for any  $a(n)$  and  $\bar{k}(n)$

$$A_{v_0} \cap \{t_n < a(n)\} \subseteq A_{v_0} \cap \{t_n - t_{n-1} < a(n)\} \cap \dots \cap \{t_{n-\bar{k}(n)} - t_{n-\bar{k}(n)-1} < a(n)\}. \quad (45)$$

Let  $\bar{k}(n) = \varepsilon(n + \xi_0)$  and  $a(n) = 2^{3(p(n-1-\bar{k}(n))+\xi_0-\varepsilon(n-1+\xi_0))}$ . By part (c) of Lemma 6.1, the probability of the event in the right-hand side of (45) is estimated from above by  $c^{\bar{k}(n)+1}$ , where  $0 < c < 1$ . Let

$$C_{v_0} = \{t_n < 2^{3(p(n-1-\varepsilon(n+\xi_0))+\xi_0-\varepsilon(n-1+\xi_0))} \text{ for some } n\}.$$

Then

$$\mathbb{P}(A_{v_0} \cap C_{v_0}) \leq \sum_{n=1}^{\infty} c^{\varepsilon(n+\xi_0)+1}.$$

The right-hand side of this inequality can be made smaller than  $\bar{\varepsilon}/3$  by taking sufficiently large  $M$ . We have thus obtained that

$$\mathbb{P}(A_{v_0} \setminus (B_{v_0} \cup C_{v_0})) \geq 1 - \bar{\varepsilon}.$$

On the event  $A_{v_0} \setminus (B_{v_0} \cup C_{v_0})$  we have

$$|\xi_n - pn - \xi_0| \leq \varepsilon(n + \xi_0) \quad \text{for all } n;$$

$$t_n - t_{n-1} \leq 2^{3(p(n-1)+\xi_0+\varepsilon(n-1+\xi_0))+\varepsilon(n+\xi_0)} \quad \text{for all } n;$$

$$t_n \geq 2^{3(p(n-1-\varepsilon(n+\xi_0))+\xi_0-\varepsilon(n-1+\xi_0))} \quad \text{for all } n.$$

Since  $\varepsilon$  can be taken arbitrarily small, these three inequalities imply that for any  $\delta > 0$

$$(2^{\xi_0} + t_n^{\frac{1}{3}})^{1-\delta} \leq 2^{\xi_n} \leq (2^{\xi_0} + t_n^{\frac{1}{3}})^{1+\delta} \quad \text{for all } n \geq 0$$

on  $A_{v_0} \setminus (B_{v_0} \cup C_{v_0})$ , provided that  $M$  is sufficiently large. This implies the statement of the lemma since  $2^{\xi_{n-2}} \leq |\dot{Y}(t)| \leq 2^{\xi_{n+2}}$  for  $t_n \leq t \leq t_{n+1}$  on  $A_{v_0} \setminus (B_{v_0} \cup C_{v_0})$  due to (43).  $\square$

**Corollary 6.3.** *For  $\delta > 0$  we have*

$$\lim_{|v_0| \rightarrow \infty} \mathbb{P}(|v_n|^{-1} \leq \tau_{n+1} - \tau_n \leq |v_n|^{-1+\delta} \text{ for all } n \geq 0) = 1, \quad (46)$$

$$\lim_{|v_0| \rightarrow \infty} \mathbb{P}((n|v_0|^{-1} + n^{3/4})^{1-\delta} \leq \tau_n \leq (n|v_0|^{-1} + n^{3/4})^{1+\delta} \text{ for all } n \geq 0) = 1. \quad (47)$$

*Proof.* The first statement easily follows from (24). Then (47) follows from Lemma 6.2 and (46) (it is easy to consider the cases  $t < |v_0|^3$  and  $t \geq |v_0|^3$  separately).  $\square$

Let  $D_{v_0}^\delta$  be the following event

$$\begin{aligned} D_{v_0}^\delta &= \{(|v_0| + t^{1/3})^{1-\delta} \leq |\dot{Y}(t)| \leq (|v_0| + t^{1/3})^{1+\delta} \text{ for all } t \geq 0\} \cap \\ &\quad \cap \{|v_n|^{-1} \leq \tau_{n+1} - \tau_n \leq |v_n|^{-1+\delta} \text{ for all } n \geq 0\} \cap \\ &\quad \cap \{(n|v_0|^{-1} + n^{3/4})^{1-\delta} \leq \tau_n \leq (n|v_0|^{-1} + n^{3/4})^{1+\delta} \text{ for all } n \geq 0\}. \end{aligned}$$

As we saw above,

$$\lim_{|v_0| \rightarrow \infty} \mathbb{P}(D_{v_0}^\delta) = 1.$$

The next result provides a more precise information about the growth of  $\dot{Y}(t)$  but only for a fixed value of  $t$ .

**Lemma 6.4.** *We have the following limit*

$$\lim_{|v_0| \rightarrow \infty} \lim_{a \rightarrow \infty} \liminf_{t \rightarrow \infty} \mathbb{P}\left(\frac{1}{a}t^{1/3} \leq |\dot{Y}(t)| \leq at^{1/3}\right) = 1.$$

*Proof.* First let us estimate the probability that  $|\dot{Y}(t)|$  is too large. To this end, let  $0 \leq \delta \leq 1/4$ ,  $m$  be the largest integer such that  $2^m \leq at^{1/3}/4$ , and  $n^*$  be the first time when  $\xi_n = m$ . Then, by Lemma 5.5,  $\mathbb{P}(D_{v_0}^\delta \cap \{t_{n^*+1} - t_{n^*} \leq t\}) \rightarrow 0$  as  $a \rightarrow \infty$  uniformly in  $t \geq 1$ . Since  $\max_{t \leq t_{n^*+1}} |\dot{Y}(t)| \leq at^{1/3}$  on  $D_{v_0}^\delta$  for large  $|v_0|$ , we see that

$$\mathbb{P}(|\dot{Y}(t)| \geq at^{1/3})$$

can be made as small as we wish by choosing  $a$  and  $|v_0|$  large.

To estimate the probability that  $|\dot{Y}(t)|$  is too small, it is enough to show that if  $n_* = n_*(b, t)$  is the first time when  $|\dot{Y}(t_n)| \geq bt^{1/3}$ , then

$$\lim_{|v_0| \rightarrow \infty} \lim_{b \rightarrow 0} \liminf_{t \rightarrow \infty} \mathbb{P}(t_{n_*} \leq t) = 1 \tag{48}$$

since, by Lemma 5.5, for fixed  $b$

$$\mathbb{P}\left(\max_{s \in [t_{n_*}, t_{n_*} + t]} |\dot{Y}(s)| \leq \frac{t^{1/3}}{a}\right)$$

can be made as small as we wish by taking  $a$  large uniformly in  $t \geq 1$ .

Let

$$T(p) = \sum_{n=1}^{\infty} (t_{n+1} - t_n) \chi_{\{\xi_n = p\}}.$$

Since  $\xi_n \leq \log_2(bt^{1/3})$  for  $n \leq n_*$ , to establish (48) it is enough to show that for each  $N > 0$  there are  $c_0 > 0$ ,  $c < 1$  such that

$$\mathbb{P}(T(p) \geq 2^{3p}k) \leq c_0\sqrt{k} \left( c^{\sqrt{k}} + 2^{-Np} \right). \quad (49)$$

To establish (49) we note that

$$\mathbb{P}\left(\#(n : \xi_n = p) \geq \sqrt{k}\right) \leq c^{\sqrt{k}}$$

since every time  $\xi_n$  visits  $p$  it has a positive probability of never returning there. On the other hand by Lemma 6.1(b)

$$\mathbb{P}\left(\max_{n:\xi_n=p} t_{n+1} - t_n \geq 2^{3p}\sqrt{k}|\#(n : \xi_n = p)\right) \leq \sqrt{k} \left( c^{\sqrt{k}} + 2^{-Np} \right)$$

so (49) follows.  $\square$

## 6.2 Probability of a Near Self-Intersection for $Y(t)$

In this section we prove that if  $|v_0|$  is large, then with high probability the ‘tail’ of the trajectory  $Y(t)$  (the part of the trajectory corresponding to  $t \geq \tau_n$ ) leaves a neighborhood of  $Y_n$  and then never comes close to the part of the trajectory corresponding to  $t \leq \tau_n$ . This allows us to conclude that switching to a new version of the force field at each of the times  $\tau_n$  does not have a major effect on the distribution of the solution, that is the distributions of  $X(t)$  and  $Y(t)$  are the same if we throw out events of small measure from their respective probability spaces.

Let  $\gamma_n$ ,  $n \geq 1$ , be the trajectory of the process  $Y(t)$  between times  $\tau_{n-1}$  and  $\tau_n$ , that is

$$\gamma_n = \{Y(t), \tau_{n-1} \leq t \leq \tau_n\}.$$

Let  $\Gamma_n$  be the trajectory of the process after time  $\tau_n$ , that is

$$\Gamma_n = \{Y(t), \tau_n \leq t < \infty\}.$$

Let  $\gamma_n^{2R}$  be the  $2R$ -neighborhood of  $\gamma_n$  and  $\Gamma_n^R$  the  $R$ -neighborhood of  $\Gamma_n$ . We shall prove the following lemma.

**Lemma 6.5.** *There exists  $0 < \delta < 1$  such that*

$$\mathbb{P}\left(D_{v_0}^\delta \cap \gamma_n^{2R} \cap \Gamma_{n+1}^R \neq \emptyset\right) \leq (|v_0| + n^{1/4})^{-4d+12-\delta} \quad (50)$$

for all sufficiently large  $|v_0|$  and all  $n \geq 1$ .

Before we prove Lemma 6.5, let us make several remarks which will, in particular, allow us to deduce parts (a) and (b) of Lemma 4.1 from Lemma 6.5. For  $x, v \in \mathbb{R}^d$ , let  $K^+(x, v)$  and  $K^-(x, v)$  be the cones

$$K^+(x, v) = \{y \in \mathbb{R}^d : (y - x, v) \geq \frac{3}{4}|y - x||v|\},$$

$$K^-(x, v) = \{y \in \mathbb{R}^d : (y - x, -v) \geq \frac{3}{4}|y - x||v|\}.$$

From the definition of  $D_{v_0}^\delta$  it easily follows that

$$\mathbb{P} \left( D_{v_0}^\delta \cap \bigcup_n (\{\gamma_n \not\subseteq K^-(y_n, v_n)\} \cup \{\gamma_{n+1} \not\subseteq K^+(y_n, v_n)\}) \right) \leq |v_0|^{-N}$$

if  $|v_0|$  is sufficiently large. This implies that for any  $0 < \delta < 1$

$$\mathbb{P} \left( D_{v_0}^\delta \cap \bigcup_n \{\gamma_n^{2R} \cap \gamma_{n+1}^R \not\subseteq B_{2R}(Y_n)\} \right) \leq |v_0|^{-N} \quad (51)$$

for all sufficiently large  $|v_0|$ . Take  $0 < \delta < 1$  such that (50) holds. Let

$$\Omega_{v_0} = D_{v_0}^\delta \cap \{\gamma_n^{2R} \cap \Gamma_{n+1}^R = \emptyset \text{ for all } n\} \cap \{\gamma_n^{2R} \cap \gamma_{n+1}^R \subseteq B_{2R}(Y_n) \text{ for all } n\}. \quad (52)$$

In order to see that parts (a) and (b) of Lemma 4.1 hold, it remains to note that

$$\lim_{|v_0| \rightarrow \infty} \sum_{n=1}^{\infty} (|v_0| + n^{1/4})^{-4d+12-\delta} = 0 \quad (53)$$

if  $d \geq 4$ .

Again, from the definition of  $D_{v_0}^\delta$  it follows that if  $\delta' > 0$ , then  $\gamma_n \subseteq B(Y(\tau_n), v_n^{\delta'})$  for all  $n \geq 1$  if  $\delta > 0$  is sufficiently small and  $|v_0|$  is sufficiently large. Let us represent  $\Gamma_{n+1}^R$  as follows

$$\Gamma_{n+1}^R = \bar{\Gamma}_{n+1}^R(\delta) \cup \overline{\bar{\Gamma}}_{n+1}^R(\delta),$$

where  $\bar{\Gamma}_{n+1}^R(\delta)$  is the  $R$ -neighborhood of  $\bar{\Gamma}_{n+1}(\delta) = \{Y(t), \tau_{n+1} \leq t \leq \tau_n + |v_n|^{3-\delta}\}$  and  $\overline{\bar{\Gamma}}_{n+1}^R(\delta)$  is the  $R$ -neighborhood of  $\overline{\bar{\Gamma}}_{n+1}(\delta) = \{Y(t), \tau_n + |v_n|^{3-\delta} \leq t \leq \infty\}$ .

Recall that the constant  $l$  from the definition of the stopping time  $\tau_n$  is equal to  $4R$ . Since Lemma 5.9 is obviously also applicable to the process  $Z(t)$ ,

$$\mathbb{P}(D_{v_0}^\delta \cap \{\text{dist}(K^-(y_n, v_n), \bar{\Gamma}_{n+1}(\delta)) \leq 3R\}) \leq (|v_0| + n^{1/4})^{-4d+12-\delta}$$

if  $\delta > 0$  is sufficiently small and  $|v_0|$  is sufficiently large. This implies (50) with  $\bar{\Gamma}_{n+1}^R(\delta)$  instead of  $\Gamma_{n+1}^R$ . Thus, Lemma 6.5 will follow if we prove that

$$\mathbb{P} \left( D_{v_0}^\delta \cap \gamma_n^{2R} \cap \overline{\bar{\Gamma}}_{n+1}^R(\delta) \neq \emptyset \right) \leq (|v_0| + n^{1/4})^{-4d+12-\delta} \quad (54)$$

**Lemma 6.6.** *There exist  $0 < \varepsilon < 1$  and  $0 < \delta_0 < 1$  such that for any  $0 < \delta, \delta' < \delta_0$  and  $R'$  the inequality*

$$\mathbb{P} \left( D_{v_0}^\delta \cap \{|Y(\tau_n + |v_n|^{3-\delta} + t) - x| \leq R'\} | \mathcal{G}_n \right) \leq (|v_n|^{3-\delta} + t)^{-\frac{4}{3}(d-2)-\varepsilon} \quad (55)$$

*holds for all sufficiently large  $|v_0|$  uniformly in  $n \geq 0$ ,  $x \in B(Y(\tau_n), v_n^{\delta'})$  and  $t \geq 0$ .*

*Proof.* In view of Lemma 5.9 we can assume that  $t > |v_n|^{3-\delta}$ . Denote  $\tilde{t} = \tau_n + |v_n|^{3-\delta} + t$ . Let us first explain the proof of a weaker bound: for each  $\varepsilon > 0$  we have

$$\mathbb{P} \left( D_{v_0}^\delta \cap \{|Y(\tilde{t}) - x| \leq R'\} | \mathcal{G}_n \right) \leq t^{-\frac{4}{3}(d-2)+\varepsilon}. \quad (56)$$

This suffices for  $d > 4$  (see the proof of Lemma 6.5). Then we explain how to improve this estimate to get (56). The proof of (55) consists of two steps.

(I) Fix  $\varepsilon_1 > 0$ . We show that if the intersection does take place and  $D_{v_0}^\delta$  takes place then with high probability there exists a number  $k$  such that  $\tau_n + t^{1-\varepsilon_1} \leq \tau_{n+k} \leq \tilde{t}$  and the following conditions are satisfied.

- (A)  $|Y(\tau_{n+k}) - x| \geq t^{4/3-\varepsilon}$ ,
- (B)  $\frac{\pi}{4} \leq \angle((Y(\tau_{n+k}) - x), v_{n+k}) \leq \frac{3\pi}{4}$ .
- (II) By step (I) it suffices to show that

$$\mathbb{P} \left( Y(\tilde{t}) \in B(x, R') \text{ and (A) and (B) hold} \right) \leq \text{Const} t^{-\frac{4}{3}(d-2)+\varepsilon} \quad (57)$$

To prove (57), denote  $r = |Y(\tau_{n+k}) - x|$ , let  $\Pi$  be the plane passing through  $x$  orthogonal to  $v_0$  and let  $Pr$  denote the projection to  $\Pi$ . We can find a set  $S = \{x_j\}$  of cardinality at least  $cr^{d-2}$  such that  $x_1 = x$ , the balls  $B(x_j, R')$  are disjoint, and for any  $j$  there is an isometry  $\mathcal{O}_j$  leaving  $Y(\tau_{n+k})$  and  $v_{n+k}$  fixed and such that  $\mathcal{O}_j(x_j) = x_1$ . By the rotation invariance,

$$\mathbb{P} \left( Pr(Y(\tilde{t})) \in B(x_1, R') \right) \leq \frac{1}{\text{Card}(S)} \quad (58)$$

proving (57).

Thus to complete the proof of (56) it remains to justify step I. Observe that on  $D_{v_0}^\delta$  we have

$$\begin{aligned} |Y(\tau_n + t^{1-\varepsilon_1}) - Y(\tau_n)| &\leq \text{Const} t^{\frac{4}{3}-\varepsilon_1(1+\delta)} \\ |Y(\tilde{t}) - Y(\tilde{t} - t^{1-\varepsilon_1})| &\leq \text{Const} t^{\frac{4}{3}-\varepsilon_1(1+\delta)} \end{aligned}$$

On the other hand the inequality  $\angle(\dot{Y}(s), (Y(s) - x)) \leq \frac{\pi}{3}$  for all  $s \in [\tau_n + t^{1-\varepsilon_1}, \tilde{t} - t^{1-\varepsilon_1}]$  would imply

$$\left| |Y(\tilde{t} - t^{1-\varepsilon_1}) - x| - |Y(\tau_n + t^{1-\varepsilon_1}) - x| \right| \geq \text{Const} t^{4/3(1-\delta)}$$

making intersection impossible if  $\varepsilon_1 > 3\delta$ . Thus there exists  $t_1 \in [\tau_n + t^{1-\varepsilon_1}, \tilde{t} - t^{1-\varepsilon_1}]$  such that  $\angle(\dot{Y}(t_1), (Y(t_1) - x)) = \frac{\pi}{3}$ . Next with high probability the angle changes less than  $\frac{\pi}{12}$

on  $[t_1, t_1 + t^{(1-\varepsilon_1)(3-\delta)}]$ . Thus the motion on this interval is well approximated by a straight line and consequently there is  $t_2 \in [t_1, t_1 + t^{(1-\varepsilon_1)(3-\delta)}]$  such that

$$|Y(t_2) - x| \geq \text{Const} t^{(1-\varepsilon_1)(3-\delta)} t^{1/3-\delta}.$$

Taking  $k$  to be the first number such that  $\tau_{n+k} > t_2$  establishes our claim.

Now let us now indicate how to prove the lemma in full generality. We need to prove (57) with  $-\varepsilon$  instead of  $\varepsilon$  in the right-hand side. In the arguments leading to (57) we only used the projection on the plane orthogonal to  $v_{n+k}$ . Now we consider the projection of the process onto the  $v_{n+k}$  direction. During the time interval between  $\tilde{t} - t^{1/10}$  and  $\tilde{t}$  the projection of  $\dot{Y}(s)$  can be well-approximated by a martingale, and as such by a time-changed Brownian motion. The time-change is almost linear on this small time interval, and thus the projection of  $Y(s)$  is approximated well by the integral of the Brownian motion. This allows us to gain an extra factor of  $t^{-2\varepsilon}$ . Observe that the derivation of (58) only involved rotation-invariance, and thus (58) remains valid if we replace the probability in the left-hand side by conditional probability with the condition which involves the projection of the process on the direction of  $v_{n+k}$ .  $\square$

*Proof of Lemma 6.5.* Let

$$s_k^n(\delta) = \tau_{n-1} + k(|v_n| + n^{1/4})^{-3\delta}, \quad k = 0, \dots, [(\tau_n - \tau_{n-1})(|v_n| + n^{1/4})^{3\delta}].$$

As follows from the definition of  $D_{v_0}^\delta$ , for any  $R'$  these points form an  $R'$ -net in  $\gamma_n$  if  $|v_0|$  is sufficiently large. By applying (55) to  $x_k^n(\delta) = Y(s_k^n(\delta))$ , we obtain that

$$\begin{aligned} \mathbb{P} \left( D_{v_0}^\delta \cap \{\text{dist}(Y(\tau_n + |v_n|^{3-\delta} + t), \gamma_n) \leq R'\} | \mathcal{G}_n \right) &\leq \\ &\leq (|v_n|^{3-\delta} + t)^{-\frac{4}{3}(d-2)-\varepsilon} (|v_n| + n^{1/4})^{3\delta} \end{aligned}$$

holds for all sufficiently large  $|v_0|$  uniformly in  $n \geq 0$  and  $t \geq 0$ . Since  $|\dot{Y}(t)| \leq (|v_0| + t^{1/3})^{1+\delta}$  on  $D_{v_0}^\delta$ , and  $R'$  was arbitrary,

$$\begin{aligned} \mathbb{P} \left( D_{v_0}^\delta \cap \gamma_n^{2R} \cap \overline{\overline{\Gamma}}_{n+1}^R(\delta) \neq \emptyset | \mathcal{G}_n \right) &\leq \mathbb{P} \left( D_{v_0}^\delta \cap \{\text{dist}(\overline{\overline{\Gamma}}_{n+1}(\delta), \gamma_n) \leq 3R\} | \mathcal{G}_n \right) \leq \\ &\leq \int_0^\infty (|v_n|^{3-\delta} + t)^{-\frac{4}{3}(d-2)-\varepsilon} (|v_n| + n^{1/4})^{3\delta} (|v_0| + (\tau_n + |v_n|^{3-\delta} + t)^{1/3})^{1+\delta} dt \end{aligned} \quad (59)$$

holds for all sufficiently large  $|v_0|$  uniformly in  $n \geq 0$ . It follows from the definition of  $D_{v_0}^\delta$  that

$$|v_n| \leq (|v_0| + n^{1/4})^{1+3\delta}$$

on  $D_{v_0}^\delta$  for all sufficiently large  $|v_0|$ . Recall that

$$\tau_n \leq (n|v_0|^{-1} + n^{3/4})^{1+\delta}$$

on  $D_{v_0}^\delta$  for all sufficiently large  $|v_0|$ . Since  $\varepsilon$  is fixed, using these estimates, it is easy to see that right-hand side of (59) can be made smaller than the right-hand side of (54) by taking a sufficiently small  $\delta$ .

## 7 The Convergence in Distribution

Here we prove Lemma 4.1(c). Recall that  $\Omega_{v_0}$  is given by (52).

Without loss of generality we can assume that  $c$  tends to infinity along a subsequence. For fixed  $v_0$ , let us prove that the family of processes  $\dot{Y}(c^3t)/c$  is tight, when restricted to the event  $\Omega_{v_0}$ . By the Arzela-Askoli Theorem, it is sufficient to show that for any  $T, \varepsilon, \eta > 0$  there are  $c_0$  and  $\varkappa_0 > 0$  such that

$$\mathbb{P} \left( \Omega_{v_0} \cap \left\{ \sup_{0 \leq s \leq t \leq T, t-s \leq \varkappa} |\dot{Y}(c^3t)/c - \dot{Y}(c^3s)/c| > \varepsilon \right\} \right) < \eta \quad (60)$$

for  $c \geq c_0$  and  $0 < \varkappa \leq \varkappa_0$ .

Let  $T, \varepsilon, \eta > 0$  be fixed. Let  $n_* = n_*(\kappa, c)$  be the first time when  $|\dot{Y}(\tau_n)| \geq \kappa c$ . Take  $\kappa < \varepsilon/4$ . Define  $U_{\kappa,c}(t) = \dot{Y}(\tau_{n_*} + c^3t)/c$ . By Lemma 5.5, there is  $\varkappa > 0$  such that

$$\mathbb{P} \left( \Omega_{v_0} \cap \left\{ \sup_{0 \leq s \leq t \leq T, t-s \leq \varkappa} |U_{\kappa,c}(t) - U_{\kappa,c}(s)| > \frac{\varepsilon}{2} \right\} \right) < \eta$$

for large  $c$ . Now (60) follows easily.

From Lemma 6.4, the definition of  $D_{v_0}^\delta$ , and the tightness established above it follows that for any  $T, \varepsilon, \eta > 0$  there is  $\kappa > 0$  such that

$$\mathbb{P} \left( \Omega_{v_0} \cap \left\{ \sup_{t \in [0, T]} \left| U_{\kappa,c}(t) - \frac{\dot{Y}(tc^3)}{c} \right| \geq \varepsilon \right\} \right) < \eta$$

for all sufficiently large  $c$ . Likewise, if  $\bar{\tau}_\kappa$  is the first time when  $|\bar{V}(\bar{\tau})| = \kappa$ , define  $\bar{U}_\kappa(t) = \bar{V}(\bar{\tau}_\kappa + t)$ . Then for any  $T, \varepsilon, \eta > 0$  there is  $\kappa > 0$  such that

$$\mathbb{P} \left( \sup_{t \in [0, T]} |\bar{U}_\kappa(t) - \bar{V}(t)| \geq \varepsilon \right) < \eta.$$

Finally, from Lemma 5.5 and the definition of  $\Omega_{v_0}$  it follows that the distribution of  $U_{\kappa,c}$ , considered over the space  $\Omega_{v_0}$  with the normalized measure, is close to the distribution of  $\bar{U}_\kappa$  if  $c$  is large enough. This completes the proof of Lemma 4.1.

## 8 Appendix

Here we sketch the proof of (21). It is clear that  $\eta_1 \leq T_0 = |v_0|^\alpha$  with high probability. Therefore, due to (20) and the proximity of  $y(t)$  and  $z_0(t)$  (formula (17)), (21) will follow if only we show that for any  $\delta > 0$  one can choose  $\alpha > 0$  such that

$$\mathbb{E}(|\tau_1 - \eta_1| \chi_{\{\max(\tau_1, \eta_1) \leq T_0\}}) \leq |v_0|^{-3+\delta}$$



for all sufficiently large  $|v_0|$ . We shall only prove that

$$\mathbb{E}((\tau_1 - \eta_1)^+ \chi_{\{\max(\tau_1, \eta_1) \leq T_0\}}) \leq |v_0|^{-3+\delta} \quad (61)$$

since the inequality with  $\eta_1 - \tau_1$  instead of  $\tau_1 - \eta_1$  can be proved similarly.

Let  $\gamma > 0$  and  $0 \leq q \leq 2$ . We shall specify these constants later. For simplicity of notation, assume that  $v_0$  is directed along the  $x_1$ -axis, in the positive direction. Let  $S_{q,\gamma}$  and  $S_{q,\gamma}^+$  be the following random sets:

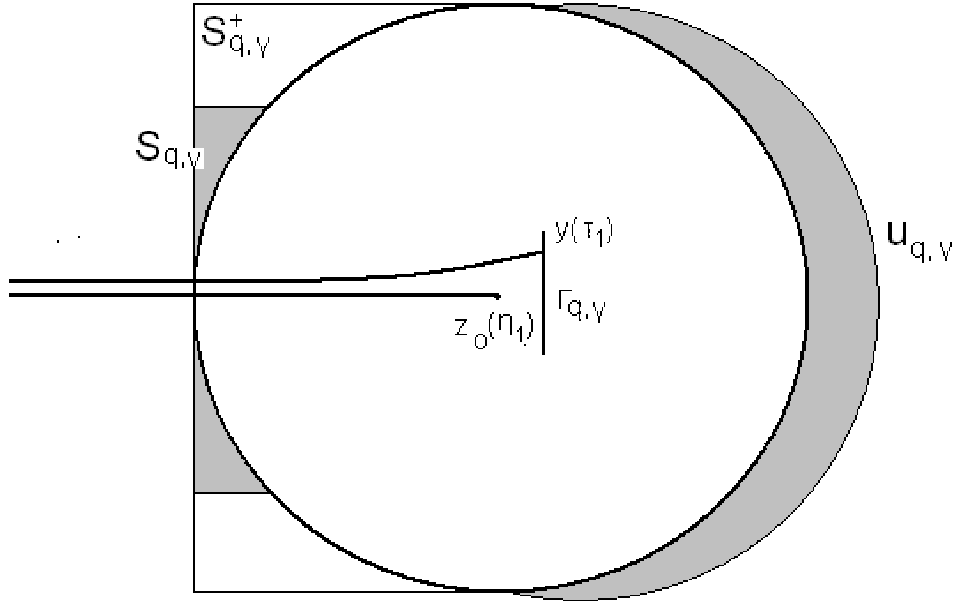
$$\begin{aligned} S_{q,\gamma} &= \{x \in \mathbb{R}^d : \text{dist}(x, z_0(\eta_1)) \geq 2R, v_0\eta_1 - 2R \leq x_1 \leq v_0\eta_1, \\ &\quad 2R - |v_0|^{-q+\gamma} \leq \sqrt{x_2^2 + \dots x_d^2} \leq 2R - |v_0|^{-q} + |v_0|^{-2}\}, \\ S_{q,\gamma}^+ &= \{x \in \mathbb{R}^d : \text{dist}(x, z_0(\eta_1)) \geq 2R, v_0\eta_1 - 2R \leq x_1 \leq v_0\eta_1, \\ &\quad 2R - |v_0|^{-q} + |v_0|^{-2} < \sqrt{x_2^2 + \dots x_d^2} \leq 2R\}. \end{aligned}$$

Let  $\Gamma_{q,\gamma}$  be the following random set:

$$\Gamma_{q,\gamma} = \{x \in \mathbb{R}^d : x_1 = v_0\eta_1 + |v_0|^{-2+2\gamma+\frac{q}{2}}, \sqrt{x_2^2 + \dots x_d^2} \leq |v_0|^{-2+\gamma}\}.$$

Let  $U_{q,\gamma}$  be the following random set:

$$U_{q,\gamma} = \{x \in \mathbb{R}^d : \text{dist}(x, z_0(\eta_1)) \geq 2R, \text{dist}(x, \Gamma_{q,\gamma}) \leq 2R\}.$$



Let  $\mathcal{E}_{q,\gamma}^S$  be the event that at least one of the points  $r_1, r_2, \dots$  belongs to  $S_{q,\gamma}$  but none belong to  $S_{q,\gamma}^+$ . Let  $\mathcal{E}_{q,\gamma}^U$  be the event that at least one of the points  $r_1, r_2, \dots$  belongs to  $U_{q,\gamma}$ . Let  $A$  be a point on the semi-axis  $\{x \in \mathbb{R}^d : x_1 \geq 0, x_2 = \dots = x_d = 0\}$ . Note that  $\mathcal{E}_{q,\gamma}^S$  and  $\mathcal{E}_{q,\gamma}^U$  are independent when conditioned on  $\{z_0(\eta_1) = A\}$ . The respective conditional probabilities can be estimated from above by  $|v_0|^{-q+2\gamma}$  and  $|v_0|^{-2+3\gamma+\frac{q}{2}}$  for all sufficiently large  $|v_0|$ . Therefore,  $\mathbb{P}(\mathcal{E}_{q,\gamma}^S \cap \mathcal{E}_{q,\gamma}^U) \leq |v_0|^{-2+5\gamma-\frac{q}{2}}$ .

Let us examine the contribution to the expectation (61) from the event  $\mathcal{E}_{q,\gamma}^S$ . First,

$$\mathbb{E}(\chi_{\mathcal{E}_{q,\gamma}^S \cap \mathcal{E}_{q,\gamma}^U}(\tau_1 - \eta_1)^+ \chi_{\{\max(\tau_1, \eta_1) \leq T_0\}}) \leq T_0 \mathbb{P}(\mathcal{E}_{q,\gamma}^S \cap \mathcal{E}_{q,\gamma}^U) \leq |v_0|^{\alpha-2+5\gamma-\frac{q}{2}}.$$

Note that the power  $\alpha - 2 + 5\gamma - \frac{q}{2}$  can be made less than  $-3 + \delta$  by selecting small  $\gamma$  and  $\alpha$  close to  $-1$ . Next, note that with high probability the trajectory  $y(t)$  reaches the set  $\Gamma_{q,\gamma}$  between times  $\eta_1$  and  $\eta_1 + |v_0|^{-3+3\gamma+\frac{q}{2}}$  due to the proximity of  $y(t)$  and  $z_0(t)$ . Note that the distance between  $\Gamma_{q,\gamma}$  and  $S_{q,\gamma}^+$  is greater than  $2R$ . Therefore, on  $\mathcal{E}_{q,\gamma}^S \setminus \mathcal{E}_{q,\gamma}^U$ , none of the points  $r_1, r_2, \dots$  belongs to the  $2R$ -neighborhood of the point where  $y(t)$  first intersects  $\Gamma_{q,\gamma}$ . Therefore, for any  $N > 0$ ,

$$\mathbb{E}(\chi_{\mathcal{E}_{q,\gamma}^S \setminus \mathcal{E}_{q,\gamma}^U}(\tau_1 - \eta_1)^+ \chi_{\{\max(\tau_1, \eta_1) \leq T_0\}}) \leq |v_0|^{-3+3\gamma+\frac{q}{2}} \mathbb{P}(\mathcal{E}_{q,\gamma}^S) + |v_0|^{-N} \leq |v_0|^{-3+6\gamma-\frac{q}{2}}.$$

Again, the power  $-3 + 6\gamma - \frac{q}{2}$  can be made less than  $-3 + \delta$  by selecting small  $\gamma$ . We have thus obtained that

$$\mathbb{E}(\chi_{\mathcal{E}_{q,\gamma}^S}(\tau_1 - \eta_1)^+ \chi_{\{\max(\tau_1, \eta_1) \leq T_0\}}) \leq |v_0|^{-3+\delta}. \quad (62)$$

Note that for fixed  $\gamma$  one can find finitely many numbers  $q_1, \dots, q_n \in [0, 2]$  such that  $\mathbb{P}(\bigcup_{i=1}^n \mathcal{E}_{q_i, \gamma}^S) = 1$ . Therefore, (62) implies (61).

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