On mathematical foundation of the Brownian motor theory.

L. Koralov[*], S. Molchanov[†], B. Vainberg[‡]

Abstract

The paper contains mathematical justification of basic facts concerning the Brownian motor theory. The homogenization theorems are proved for the Brownian motion in periodic tubes with a constant drift. The study is based on an application of the Bloch decomposition. The effective drift and effective diffusivity are expressed in terms of the principal eigenvalue of the Bloch spectral problem on the cell of periodicity as well as in terms of the harmonic coordinate and the density of the invariant measure. We apply the formulas for the effective parameters to study the motion in periodic tubes with nearly separated dead zones.

Keywords. Brownian motors, diffusion, effective drift, effective diffusivity, Bloch decomposition.

AMS MSC. 35Q92, 35Q70, 35K10.

1 Introduction

The paper is devoted to mathematical theory of Brownian (molecular) motors. The concept of a Brownian motor has fundamental applications in the study of transport processes in living cells and (in a slightly different form) in the porous media theory. There are thousands of publications in the area of Brownian motors in the applied literature. For example, the review by P. Reimann [16] contains 729 references. Most of these publications are in physics or biology journals and are not mathematically rigorous (although many of them are based on the mathematical model described below, see, for example [4], [5], [10], [20]). Some of them are based on numerical computations.

Consider a set of particles with an electrical charge performing Brownian motion in a tube Ω with periodic (or stationary random) cross section (see Fig. 1).

[*]Dept. of Mathematics, University of Maryland, College Park, MD 20742, koralov@math.umd.edu

[†]Dept of Mathematics and Statistics, University of North Carolina, Charlotte, NC 28223, smolchan@uncc.edu; corresponding author, phone 704-6874573

[‡]Dept of Mathematics and Statistics, University of North Carolina, Charlotte, NC 28223, brvainbe@uncc.edu

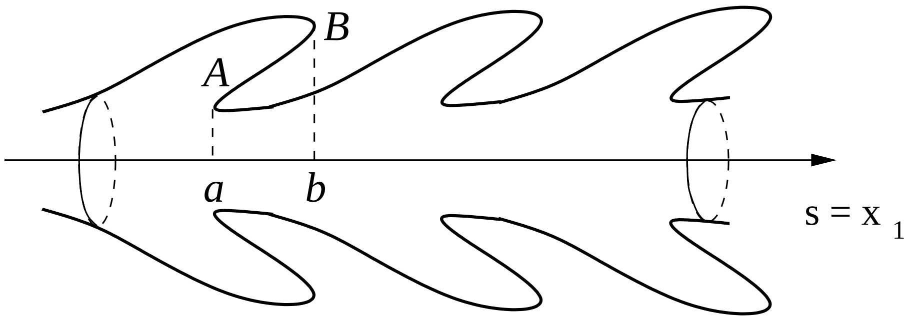

Figure 1: A periodic tube Ω with “fingers”. One can expect that $V_{\text{eff}} < |V_{\text{eff}}^-|$.

We will assume that the axis of Ω is directed along the x_1-axis. Let’s apply a constant external electric field E along the axis of Ω. Then the motion of the particles consists of the diffusion and the Stocks drift, and the corresponding generator has the form

$$Lu = \Delta u + V \frac{\partial u}{\partial x_1}, \quad V = V(E),$$

complemented by the Neumann boundary condition on $\partial\Omega$ (we assume the normal reflection at the boundary).

One can expect that the displacement of the particles on the large time scale may be approximated (due to homogenization) by a one-dimensional diffusion process $\overline{x}_1(t)$ along the x_1-axis with an effective drift V_{eff} and an effective diffusivity σ^2, which depend on V and the geometry of the tube Ω. If we reverse the direction of the external field from E to $-E$, then the corresponding effective drift V_{eff}^- and the corresponding effective diffusivity $(\sigma^-)^2$ will be, generally, different from V_{eff} and σ^2 when Ω is not symmetric with respect to the reflection $x_1 \to -x_1$. For example, one can expect that $V_{\text{eff}} < |V_{\text{eff}}^-|$ for Ω shown in Fig. 1. The difference between the effective parameters can be significant. Then by changing the direction of the exterior electric field E periodically in time, one could construct a constant drift in, say, the negative direction of x_1 and create a device (motor) producing energy from Brownian motion.

The idea of molecular motors goes back to M. Smoluchowski, R. Feynman, L. Brillouin. Starting from the 1990-s, it became a hot topic in chemical physics, molecular biology, and thermodynamics. The following natural problem must be solved by mathematicians:

1. The homogenization procedure (reduction to a one-dimensional problem) must be justified.

2. Expressions for effective parameters V_{eff}, σ^2 in terms of an appropriate PDE or a spectral problem on the period of Ω (in the case of periodic tubes) must be found.

3. For some natural geometries of Ω that include a small parameter, asymptotic expressions for effective parameters need to be obtained.

We solve the first two problems here using an analytic approach that justifies the homogenization procedure and allows us to express the effective parameters in terms of the principal eigenvalue of a spectral problem on one period of Ω. We will also provide a couple of simple consequences of the obtained formulas. We show that $dV_{\text{eff}}/dV = \sigma^2 > 0$ when $V = 0$. Thus, since V_{eff} is analytic in V, it is a strictly monotone function of V when $|V|$ is small enough. Simple asymptotic formulas for the effective parameters will be justified for periodic tubes with nearly separated dead zones. Note that the first two problems listed above also can be solved with probabilistic techniques similar to those used for homogenization of periodic operators in $\mathbb{R}^d$, as we'll discuss in a forthcoming paper.

There is immense literature on homogenization and its applications. Here we only mention the monographs of some of the founders of the method ([2], [3], [17], [19]) as well as some recent papers where other references can be found [6], [9], [11], [13], [15], [18]). There are various probabilistic and PDE approaches to justifying homogenization. Our approach is based solely on the Bloch decomposition. Recently several papers appeared (see [7, 8] and references there), where the Bloch decomposition was used to study homogenization problems in periodic media. It was shown that this approach has many advantages (when it is applicable). So far, this approach was applied to self-adjoint elliptic equations in the whole space or in a domain with a finite boundary. The symmetry and the existence of a bounded inverse operator were essential there.

We consider a parabolic problem. It is non-self-adjoint due to the drift, which can not be neglected since it is essential for applications. Besides, the homogenization is applied only with respect to one variable (compare with [6]). The combination of these features makes the problem under consideration essentially different from the applications of the Bloch decomposition in earlier work. The main difficulty is due to the presence of infinite boundary and a drift that is transversal to the boundary.

We would like to note the presence of the boundary integral term in the formula for the effective drift (see Theorem 2.5). In probabilistic terms, it arises from the re-normalized time spent by the Brownian motion with the drift on the boundary (local time). Our next paper based on a probabilistic approach to the problem will contain the detailed analysis of the asymptotic behavior of the effective parameters with respect to the parameters describing the geometry of the domain.

The plan of the paper is as follows. Lemma 2.2 describes the properties of the principal eigenvalue $\lambda_0 = \lambda_0(\theta)$ of the Bloch spectral problem on the cell of periodicity of Ω. The main results on homogenization are obtained in Theorems 2.1 and 2.3. In particular, it is shown there that the effective drift V_{eff} and the effective diffusivity σ^2 are given by the coefficients of the Taylor expansion of the eigenvalue $\lambda_0(\theta)$. To be more exact, $\lambda_0(\theta) = iV_{\text{eff}}\theta - \sigma^2\theta^2 + O|\theta|^3$, $\theta \to 0$. Expressions for the effective parameters through the harmonic coordinate and the invariant measure are given in Theorem 2.5. The latter formulas are applied to a particular class of domains in the last section. Periodic tubes with nearly separated dead zones are considered there. We rigorously justified the asymptotic formula for the effective diffusivity, which was found earlier for domains with somewhat simpler geometry in [5], [10].

We also note that the results of this paper can be modified to allow for L to be an elliptic operator with smooth variable coefficients without the potential term.

2 Description of the model and main results.

Consider a tube $\Omega \subset \mathbb{R}^d$, $d \geq 2$, periodic in x_1 with period 1, with a smooth boundary $\partial\Omega$ (see Fig. 1). Denote by S_{x_1} the cross-section of Ω by the plane $\mathbb{R}^{d-1}$ orthogonal to the x_1-axis at the point x_1 We assume that S_{x_1} is bounded (and periodic with respect to x_1).

In the simplest case, the boundary of the cross-section is a function of the angular variables. For instance, S_{x_1} is the ball of radius $R(x_1)$ if $\partial\Omega$ is the surface of revolution. However, in general it may have a very complicated form. For example, Ω in Fig.1 contains "fingers" ("dead ends" in the terminology of [7], [8]), and S_{x_1} is not connected when $a < x_1 < b$.

Consider the cell of periodicity in Ω defined as $\Omega' = \Omega \bigcap \{0 < x_1 < 1\}$. It is assumed that the compact manifold obtained from Ω' by gluing S_0 to S_1 is connected. For transparency, one can assume that Ω' itself is connected. For example, Ω' in Fig.1 is connected if the origin is slightly to the left of a. It is not connected if the origin is the midpoint between a and b, however, it still forms a connected manifold after gluing S_0 to S_1. One also can define a connected cell of periodicity Ω' in Fig.1 using planes $\mathbb{R}^{d-1}$ through $(a+b)/2$ and $(a+b)/2+1$ by cutting only the central narrow part of Ω, but not cutting the fingers. In the latter case, we use the notation S_0, S_1 not for the whole cross-sections of Ω, but for the connected parts of the cross-sections that do not intersect with the fingers.

Denote the exterior normal to $\partial\Omega$ by n. Let $x = (s, z)$, where $s = x_1,\ z = (x_2, ..., x_d)$. Consider the following parabolic problem in the tube Ω:

$$\frac{\partial u}{\partial t} = \Delta u + V\frac{\partial u}{\partial s},\ x \in \Omega;\ \ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0;\ \ u(0, x) = \varphi(x), \quad x \in \Omega, \quad \varphi \in L_1(\Omega). \tag{1}$$

The main results of the first part of the paper are stated in the following two theorems. The first theorem specifies the asymptotic behavior, as $t \to \infty$, of the solution u of the problem (1) in terms of the solution w of the one-dimensional problem

$$\frac{\partial w}{\partial t} = \sigma^2\frac{\partial^2 w}{\partial s^2} + V_{\text{eff}}\frac{\partial w}{\partial s},\ s \in \mathbb{R};\ \ w|_{t=0} = w_0(s). \tag{2}$$

Here σ^2 is the effective diffusivity, V_{eff} is the effective drift, and the initial function $w_0(s)$ is a certain average of φ in z-variable and will be defined later in (7). The second theorem describes the effective parameters σ^2 and V_{eff} in terms of the problem on the cell Ω'.

Theorem 2.1. *1) There are constants V_{eff}, σ, and C independent of φ, such that*

$$u = w + \alpha, \tag{3}$$

where w is the solution of (2), w_0 is defined in (7), and the remainder term $\alpha = \alpha(t, x)$ satisfies

$$\sup_{x\in\Omega} |\alpha(t, x)| \leq Ct^{-1}||\varphi||_{L_1(\Omega)}, \quad t \geq 1.$$

2) If $\mathrm{supp}(\varphi) \subseteq K$*, where* K *is compact, then (3) holds with* w *being the solution of (2) with the initial data* $c\delta_0$*, where* δ_0 *is the delta function at the origin,* $c = \int_{\mathbb{R}} w_0(s)ds$*, and the remainder satisfying*

$$\sup_{x\in\Omega} |\alpha(t,x)| \leq C(K)t^{-1}||\varphi||_{L_1(\Omega)}, \quad t \geq 1.$$

We need to state one important lemma before we can formulate the second theorem. Denote by S^l the lateral part of the boundary $\partial\Omega'$ of Ω', i.e., $S^l = \partial\Omega' \backslash [S_0 \bigcup S_1]$. Consider the following eigenvalue problem on the cell Ω':

$$\Delta v + V\frac{\partial v}{\partial s} = \lambda v, \ x \in \Omega'; \ \ \frac{\partial v}{\partial n}|_{S^l} = 0; \ \ v|_{S_1} = e^{i\theta} v|_{S_0}, \ v'_s|_{S_1} = e^{i\theta} v'_s|_{S_0}. \tag{4}$$

This is an elliptic problem in a bounded domain, and for each θ the spectrum of this problem consists of a discrete set of eigenvalues $\lambda = \lambda_j(\theta), \ j \geq 0$, of finite multiplicity.

Lemma 2.2. *Let* $\theta \in [-\pi, \pi]$*. There exists a simple eigenvalue* $\lambda = \lambda_0(\theta)$ *of problem (4) in a neighborhood* $|\theta| \leq \delta$ *of the origin* $\theta = 0$ *and real constants* V_{eff}, σ *such that*

1) $\lambda_0 = iV_{\text{eff}}\theta - \sigma^2\theta^2 + O(\theta^3), \quad \theta \to 0,$ *where* $\sigma^2 > 0$,

2) $\mathrm{Re}\lambda_0(\theta) < 0, \quad 0 < |\theta| \leq \delta,$

3) $\mathrm{Re}\lambda_j(\theta) < -\gamma_1, \quad |\theta| \leq \delta, \quad j \geq 1, \quad \gamma_1 = \gamma_1(\delta) > 0,$

4) $\mathrm{Re}\lambda_j(\theta) < -\gamma_2, \quad \delta \leq |\theta| < \pi, \quad j \geq 0, \quad \gamma_2 = \gamma_2(\delta) > 0.$

Remark. Lemma 2.2 states the existence of the asymptotic expansion of λ_0 with some coefficients V_{eff} and σ^2, which, a priori, are not related to equation (2). However, in the proof of Theorem 2.1 it will be shown that the effective coefficients V_{eff} and σ^2 in equation (2) coincide with the coefficients in the Taylor expansion of λ_0. Thus the following theorem holds.

Theorem 2.3. *The effective diffusivity* σ^2 *and effective drift* V_{eff} *defined in (2) coincide with the constants introduced in Lemma 2.2.*

Let us describe the initial function w_0. We need the problem adjoint to problem (4). This problem has the form:

$$\Delta v - V\frac{\partial v}{\partial s} = \lambda v, \ x \in \Omega'; \ \ (\frac{\partial}{\partial n} - Vn_1)v|_{S^l} = 0; \ \ v|_{S_1} = e^{i\theta} v|_{S_0}, \ v'_s|_{S_1} = e^{i\theta} v'_s|_{S_0}, \tag{5}$$

where $n_1 = n_1(x)$ is the first component of the normal vector n. Then $\lambda = \overline{\lambda}_0(\theta)$ is an eigenvalue of problem (5). Let $v(\theta, x)$ be the corresponding eigenfunction, and let $\pi(x) = v(0, x)$ be the eigenfunction of (5) when $\theta = 0$ (and $\lambda_0 = \lambda_0(0) = 0$) normalized by the condition

$$\int_{\Omega'} \pi(x)dx = 1. \tag{6}$$

Then w_0 is defined by

$$w_0(s) = \int_{S_s} \pi(s,z)\varphi(s,z)dz, \tag{7}$$

where S_a is the cross-section of Ω by the plane through $s = a$.

The following lemma will be needed in order to prove the theorems above. Consider the non-homogeneous problem (4) in the Sobolev space $H^2(\Omega')$:

$$(\Delta + V\frac{\partial}{\partial s} - \lambda)v = f,\ x \in \Omega';\quad \frac{\partial v}{\partial n}|_{S^l} = 0;\quad v|_{S_1} = e^{i\theta}v|_{S_0},\ v'_s|_{S_1} = e^{i\theta}v'_s|_{S_0}. \tag{8}$$

This is an elliptic problem, and the resolvent

$$R^\theta_\lambda = (\Delta + V\frac{\partial}{\partial s} - \lambda)^{-1} : L_2(\Omega') \to H^2(\Omega')$$

is a meromorphic in λ operator with poles at a discrete set of eigenvalues $\lambda = \lambda_j(\theta)$. Denote by Q_α the sector in the complex λ-plane that does not contain the negative semi-axis and is defined by inequalities $-\pi + \alpha \le \arg\lambda \le \pi - \alpha$.

Lemma 2.4. *For every $\alpha > 0$ there exist constants $R = R(\alpha)$ and $C = C(\alpha)$ such that the region $Q_{\alpha,R} = Q_\alpha \bigcap \{|\lambda| > R\}$ does not contain eigenvalues $\lambda_j(\theta)$, and the following estimate is valid for the solution $v = R^\theta_\lambda f$ of the problem (8):*

$$|\lambda| \|v\|_{L_2(\Omega')} \le C(\alpha)\|f\|_{L_2(\Omega')}, \quad \lambda \in Q_{\alpha,R}. \tag{9}$$

Proof. This lemma can be proved by referencing standard a priori estimates for parameter-elliptic problems [14]. We will provide an independent proof. We multiply equation (8) by $\overline{v}$ and integrate over Ω'. This leads to

$$-\int_{\Omega'} (|\nabla v|^2 + \lambda |v|^2)dx + \int_{\Omega'} V v'_s \overline{v} dx = \int_{\Omega'} f\overline{v}dx. \tag{10}$$

The terms on the left in (10) can be estimated as follows

$$|\int_{\Omega'} (|\nabla v|^2 + \lambda|v|^2)dx| \ge c_1(\alpha)(\|\nabla v\|^2_{L_2(\Omega')} + |\lambda|\|v\|^2_{L_2(\Omega')}), \quad \lambda \in Q_\alpha,$$

and

$$|\int_{\Omega'} V v'_s \overline{v} dx| \le \frac{c_1(\alpha)}{2}\|v'_s\|^2_{L_2(\Omega')} + \frac{V^2}{2c_1(\alpha)}\|v\|^2_{L_2(\Omega')}.$$

We put $R = V^2/(c_1(\alpha))^2$. Then the absolute value of the left-hand side in (10) is estimated from below by $\frac{c_1(\alpha)}{2}(\|\nabla v\|^2_{L_2(\Omega')} + |\lambda|\|v\|^2_{L_2(\Omega')})$ when $\lambda \in Q_{\alpha,R}$, and (10) implies that

$$\frac{c_1(\alpha)}{2}(\|\nabla v\|^2_{L_2(\Omega')} + |\lambda|\|v\|^2_{L_2(\Omega')}) \le \|f\|_{L_2(\Omega')}\|v\|_{L_2(\Omega')}, \quad \lambda \in Q_{\alpha,R}. \tag{11}$$

We omit the first term on the left and obtain (9). □

Proof of Lemma 2.2. Denote by $H_\theta = \Delta + V\frac{\partial}{\partial s}$ the operator in $L_2(\Omega')$ defined on the functions from the Sobolev space $H^2(\Omega')$ satisfying the boundary conditions (4). One can define the parabolic semi-group e^{tH_θ}. Its integral kernel is the fundamental solution of the corresponding parabolic boundary value problem in Ω'. When $\theta = 0$, this kernel is real and positive. Thus Perron-Frobenius theorem is applicable to the operator e^{tH_0}, i.e., the operator e^{tH_0} has a unique maximal eigenvalue μ such that $\mu > 0$, μ is simple, the corresponding eigenfunction is positive, all the other eigenvalues μ_j (perhaps complex) are strictly less than μ in absolute value ($|\mu_j| < \mu$), and the operator does not have strictly positive eigenfunctions with eigenvalues different from μ.

We note that $v_0 \equiv 1$ is an eigenfunction of H_0 with the eigenvalue $\lambda_0 = 0$. Thus v_0 is an eigenfunction of e^{tH_0} with the eigenvalue $\mu = 1$. Since $v_0 > 0$, from the Perron-Frobenius theorem it follows that $\mu = 1$ is the maximal eigenvalue of e^{tH_0}. This implies that $\mathrm{Re}\lambda_j < \lambda_0 = 0$ for all the eigenvalues $\lambda_j \neq 0$ of the operator H_0. Since the eigenvalues λ_j form a discrete set, from Lemma 2.4 it follows that there exists a $\gamma > 0$ such that $\mathrm{Re}\lambda_j < -\gamma$ for all $\lambda_j \neq 0$.

Since H_θ depends analytically on θ, the location of the eigenvalues λ_j when $\theta = 0$ and Lemma 2.4 imply that there exists $\delta > 0$ such that the operator H_θ has the following structure of the spectrum when $|\theta| \leq \delta$: the operator has a simple, analytic in θ eigenvalue $\lambda = \lambda_0(\theta)$, $\lambda_0(0) = 0$, and $\mathrm{Re}\lambda_j(\theta) < -\gamma_1 = -\gamma/2$ for all other eigenvalues $\lambda_j(\theta)$ of H_θ. The statement 3 of Lemma 2.2 is proved. Consider now the eigenvalue $\lambda = \lambda_0(\theta)$ for purely imaginary $\theta = iz$, $z > 0$. If v belongs to the domain of the operator H_{iz}, then $\overline{v}$ also belongs to the domain of the operator H_{iz}. From here it follows that both $\lambda = \lambda_0(iz)$ and $\lambda = \overline{\lambda_0(iz)}$ are eigenvalues of H_{iz}. Since, $\lambda = \lambda_0(\theta)$ is the unique eigenvalue in a neighborhood of the point $\lambda = 0$, it follows that $\lambda_0(iz)$ is real. This implies statement 1 of Lemma 2.2, except the inequality $\sigma^2 > 0$. The latter inequality will be proved below (in Theorem 2.5) by a direct calculation of σ^2 (see formula (32)). It is important to note that while the proofs of Theorems 2.1 and 2.3 essentially rely on the fact that $\sigma^2 > 0$, the proof of Theorem 2.5 is independent of Theorems 2.1 and 2.3, so there is no circular argument here. It remains to justify statements 2 and 4. This will be done if we prove that for every $\delta_1 > 0$, there exists $\gamma = \gamma(\delta_1) > 0$ such that $\mathrm{Re}\lambda_j(\theta) < -\gamma$ for all the eigenvalues of the operator H_θ when θ is real, $\delta_1 \leq |\theta| \leq \pi$.

Let us prove the latter estimate for the eigenvalues $\lambda_j(\theta)$, $\delta_1 \leq |\theta| \leq \pi$. We fix an arbitrary rational $\theta = \theta' = \frac{\pm m}{n}$ in the set $\delta_1 \leq |\theta| \leq \pi$. Consider the domain $\widehat{\Omega} = \Omega \bigcap \{0 < x_1 < n\}$, which consists of n elementary cells of periodicity $\Omega^{(j)} = \Omega \bigcap \{j < x_1 < j+1\}$, $0 \leq j \leq n-1$ ($\Omega^{(0)}$ coincides with the cell Ω' introduced earlier). The lateral side of the boundary of the domain $\widehat{\Omega}$ will be denoted by $\widehat{S^l}$, and the parts of the boundary located in the planes through the points $x_1 = 0$ and $x_1 = n$ will be denoted by S_0 and S_n, respectively. Let $H_\theta^{(n)} = \Delta + V\frac{\partial}{\partial s}$ be the operator in $L_2(\widehat{\Omega})$ that corresponds to the following analogue of the problem (4):

$$\Delta v+V\frac{\partial v}{\partial s}=\alpha v,\ x\in\widehat{\Omega};\ \ \frac{\partial v}{\partial n}|_{\widehat{S}^l}=0;\ \ v|_{S_n}=e^{in\theta}v|_{S_0},\ v'_s|_{S_n}=e^{in\theta}v'_s|_{S_0}. \tag{12}$$

We denote its eigenvalues by $\alpha=\alpha_j$, and we keep the notation $\lambda=\lambda_j$ for the eigenvalues of this operator when $n=1$ (and $\widehat{\Omega}=\Omega'$).

When $\theta=\theta'$, the conditions on S_0,S_n become the periodicity condition. Thus the spectrum of $H^{(n)}_{\theta'}$ has the same structure as the spectrum of H_0, i.e., $\alpha_0=0$ is an eigenvalue with the eigenfunction $v_0=1$, and

$$\mathrm{Re}\alpha_j<-\gamma<0 \tag{13}$$

for all other eigenvalues α_j of $H^{(n)}_{\theta'}$.

We compare the set $\{\lambda_j\}$ of the eigenvalues of the operator $H_{\theta'}$ and the set $\{\alpha_j\}$ of the eigenvalues of the operator $H^{(n)}_{\theta'}$. The following inclusion holds: $\{\lambda_j\}\subset\{\alpha_j\}$. Indeed, if v is an eigenfunction of the operator $H_{\theta'}$, then one can construct the corresponding eigenfunction of $H^{(n)}_{\theta'}$ with the same eigenvalue by defining it as $e^{ij\theta}v$ in each elementary cell $\Omega^{(j)}\subset\widehat{\Omega},\ 0\le j\le n-1$. However, these two sets of eigenvalues do not coincide. In particular, $\lambda=0$ is not an eigenvalue of $H_{\theta'}$ (while $\alpha=0$ is an eigenvalue of $H^{(n)}_{\theta'}$). Indeed, from the simplicity of the eigenvalue $\alpha=0$ it follows that $\lambda=0$ could be an eigenvalue of $H_{\theta'}$ only if $v_0=1$ is its eigenfunction, but v_0 does not satisfy the boundary conditions (4). Thus, (13) implies that $\mathrm{Re}\lambda_j<-\gamma<0$ for the set of eigenvalues of the operator $H_{\theta'}$. Then the same estimate with $\gamma/2$ instead of γ is valid for the eigenvalues of H_θ when θ is in a small enough neighborhood of $\theta'=\frac{\pm m}{n}$. One can find a finite covering of the set $\{\theta:\delta_1\le|\theta|\le\pi\}$ by some of these neighborhoods. Thus, the desired estimate of eigenvalues $\lambda_j(\theta)$ is valid for all θ of the set $\{\theta:\delta_1\le|\theta|\le\pi\}$.

□

Proof of Theorems 2.1 and 2.3. In Part A of the proof we solve problem (1) via the Bloch decomposition, followed by a straightforward evaluation of the resulting integral. This leads to a slightly weaker statement of Theorem 2.1, where an additional assumption is used. Namely, let us assume that $\varphi\in L_1(\Omega)\cap L_2(\Omega)$. We'll show that

$$\sup_{(s,z)\in\Omega}|u(t,(s,z))-w(t,s)|\le Ct^{-1}(\|\varphi\|_{L_1(\Omega)}+\|\varphi\|_{L_2(\Omega)}),\quad t\ge 1. \tag{14}$$

In Part B of the proof, we'll show that the assumption $\varphi\in L_2(\Omega)$ is not needed and the second term on the right hand side of (14) can be omitted. We'll also prove the second statement of Theorem 2.1 and Theorem 2.3.

Part A. Let us assume here that $\varphi\in L_1(\Omega)\cap L_2(\Omega)$. The solution u of problem (1) can be found using the Laplace transform in t:

$$u=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}v(\lambda,x)e^{\lambda t}d\lambda,\quad a>0,$$

where $v \in L_2(\Omega)$ is the solution of the corresponding stationary problem:

$$\Delta v + V\frac{\partial v}{\partial s} - \lambda v = -\varphi, \; x \in \Omega; \;\; \frac{\partial v}{\partial n}|_{\partial\Omega} = 0. \tag{15}$$

In order to solve (15), we apply the Bloch transform in the variable $s = x_1$:

$$w(x) \to \widehat{w}(\theta, x) = \sum_{-\infty}^{\infty} w(s-n, z)e^{in\theta}, \quad \theta \in [-\pi, \pi], \quad x \in \Omega.$$

This map is unitary (up to the factor $1/\sqrt{2\pi}$) from $L_2(\Omega)$ to $L_2(\Omega' \times [-\pi, \pi])$, and the inverse transform is given by

$$w(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{w}(\theta, x)d\theta, \quad x \in \Omega.$$

Note that the function $e^{-i\theta s}\widehat{w}(\theta, x)$ is periodic in s. Thus the knowledge of $\widehat{w}$ on Ω' allows one to recover $\widehat{w}$ on the whole tube Ω.

The Bloch transform reduces problem (15) to a problem for $\widehat{v}$ on the cell of periodicity Ω'. The latter problem has form (8) with $f = -\widehat{\varphi}$, i.e., $\widehat{v} = -R_\lambda^\theta\widehat{\varphi}$. Thus

$$u(t, x) = \frac{-1}{4\pi^2 i}\int_{a-i\infty}^{a+i\infty}\int_{-\pi}^{\pi}(R_\lambda^\theta\widehat{\varphi})(x)e^{\lambda t}d\theta d\lambda, \quad x \in \Omega. \tag{16}$$

The function $R_\lambda^\theta\widehat{\varphi}$ here is defined in Ω', but it is extended to the whole tube Ω in such a way that $e^{-i\theta s}R_\lambda^\theta\widehat{\varphi}$ is periodic in s.

Now that formula (16) for u is established, we are going to study the asymptotic behavior of u as $t \to \infty$. We split u as $u = u_1 + u_2$, where the terms u_1, u_2 are given by (16) with integration in θ over sets $|\theta| \leq \delta$ and $\delta \leq |\theta| \leq \pi$, respectively. We choose δ small enough, so that Lemma 2.2 holds, and then make it even smaller (if needed) to guarantee that $-\frac{\gamma_1}{2} < \mathrm{Re}\lambda_0(\theta) \leq 0, \; |\theta| \leq \delta$.

Let us show that u_2 does not contribute to the main term of asymptotics of u. Lemmas 2.4 and 2.2 allow us to rewrite u_2 in the form

$$u_2(t, x) = \frac{-1}{4\pi^2 i}\int_\Gamma\int_{\delta\leq|\theta|\leq\pi}(R_\lambda^\theta\widehat{\varphi})(x)e^{\lambda t}d\theta d\lambda, \quad x \in \Omega, \tag{17}$$

where Γ is the contour shown in Fig. 2 with $\gamma = \gamma_2$. From Lemmas 2.4 and 2.2 it follows that

$$\|R_\lambda^\theta\widehat{\varphi}\|_{L_2(\Omega')} \leq \frac{c_1}{1+|\lambda|}\|\widehat{\varphi}\|_{L_2(\Omega')}, \quad \lambda \in \Gamma, \quad \delta \leq |\theta| \leq \pi.$$

Thus

$$\int_{\delta\leq|\theta|\leq\pi}\|R_\lambda^\theta\widehat{\varphi}\|_{L_2(\Omega')}d\theta \leq \frac{c_1}{1+|\lambda|}\|\widehat{\varphi}\|_{L_2(\Omega'\times[-\pi,\pi])} = \frac{c_2}{1+|\lambda|}\|\varphi\|_{L_2(\Omega)}, \quad \lambda \in \Gamma. \tag{18}$$

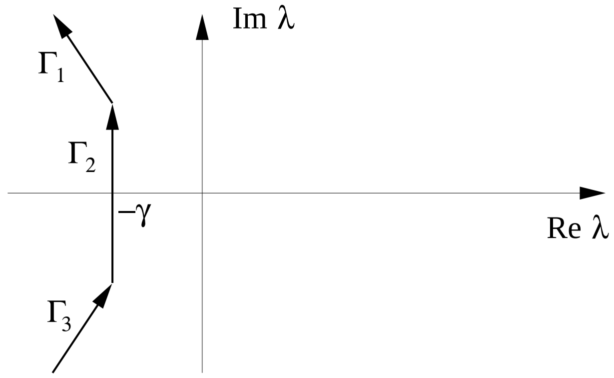

Figure 2: Contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$, where γ coincides with γ_1 or γ_2 defined in Lemma 2.2, and Γ_2 is long enough, so that the estimate (9) is valid on Γ_1 and Γ_3.

One can replace Ω' here by a shifted cell of periodicity $\Omega^{(j)}$ defined by the inequalities $j \le x_1 \le j+1$, since the function $e^{-i\theta x_1} R^\theta_\lambda \widehat{\varphi}$ is periodic in x_1. We put (18) with Ω' replaced by $\Omega^{(j)}$ into (17) and estimate the integral over Γ. This gives

$$\|u_2(t,x)\|_{L_2(\Omega^{(j)})} \le ce^{-\gamma t}\|\varphi(x)\|_{L_2(\Omega)},$$

where c does not depend on j.

In the end of this section, in Lemma 2.7, we show that there are positive constants a and b such that

$$|G(1,x,y)| \le ae^{-b(x_1-y_1)^2}, \tag{19}$$

where $G(t,x,y)$ is the Green function of problem (1). Using (19), we obtain a uniform estimate of u_2, namely

$$|u_2(t,x)| \le \int_\Omega |G(1,x,y)u_2(t-1,y)|dy \le Ae^{-\gamma t}\sum_{j=-\infty}^{\infty} e^{-b(x_1-j)^2}\|\varphi\|_{L_2(\Omega)}$$
$$\le Ce^{-\gamma t}\|\varphi\|_{L_2(\Omega)}, \quad x \in \Omega,\ t \ge 1.$$

Hence u_2 does not contribute to the main term of the asymptotics of u.

In order to study u_1, we also shift the contour of integration (in λ) to Γ. Now we take $\gamma = \gamma_1$, and in this case the simple pole of R^θ_λ at $\lambda = \lambda_0(\theta)$ must be taken into account. The residue of R^θ_λ at this pole is the integral operator with the kernel $-\psi(\theta,x)\overline{\psi^*}(\theta,y)$, where ψ is an eigenfunction of the problem (4), ψ^* is an eigenfunction of the problem (5), and they are normalized by the biorthogonality condition

$$\int_{\Omega'} \psi(\theta,x)\overline{\psi^*}(\theta,x)dx = 1. \tag{20}$$

Note that the minus sign in the expression for the kernel is needed since the residue is a negative operator. Thus

$$u_1(t,x) = \frac{1}{2\pi}\int_{|\theta|\le\delta}\int_{\Omega'}\psi(\theta,x)\overline{\psi^*}(\theta,y)e^{\lambda_0(\theta)t}\widehat{\varphi}dyd\theta - \frac{1}{4\pi^2 i}\int_{\Gamma}\int_{|\theta|\le\delta}R_\lambda^\theta\widehat{\varphi}e^{\lambda t}d\lambda, \quad x\in\Omega, \tag{21}$$

where the functions ψ, ψ^* are extended to Ω using the Bloch periodicity condition, i.e.,

$$\psi(\theta,x) = e^{i\theta s}\psi_0(\theta,x), \quad \psi^*(\theta,x) = e^{i\theta s}\psi_0^*(\theta,x), \tag{22}$$

where the functions ψ_0, ψ_0^* are periodic with respect to s. The second term in (21) can be estimated similarly to u_2. Thus

$$u(t,x) = \frac{1}{2\pi}\int_{|\theta|\le\delta}\int_{\Omega'}\psi(\theta,x)\overline{\psi^*}(\theta,y)e^{\lambda_0(\theta)t}\widehat{\varphi}dyd\theta + \alpha_1(t,x),$$

where $x\in\Omega,\ t\ge 1$, and

$$|\alpha_1| \le Ce^{-\gamma t}\|\varphi(x)\|_{L_2(\Omega)}. \tag{23}$$

Let $y=(s',z')$. Then

$$\widehat{\varphi} = \sum_{-\infty}^{\infty}\varphi(s'-n,z')e^{in\theta}.$$

Hence (with (22) taken into account),

$$\begin{aligned} u(t,x) &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{|\theta|\le\delta}\int_{\Omega'}e^{i\theta(s-s'+n)}\psi_0(\theta,x)\overline{\psi_0^*}(\theta,y)e^{\lambda_0(\theta)t}\varphi(s'-n,z')dyd\theta + \alpha_1(t,x) \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{|\theta|\le\delta}\int_{\Omega^{(n)}}e^{i\theta(s-s')}\psi_0(\theta,x)\overline{\psi_0^*}(\theta,y)e^{\lambda_0(\theta)t}\varphi(s',z')dyd\theta + \alpha_1(t,x) \\ &= \frac{1}{2\pi}\int_{|\theta|\le\delta}\int_{\Omega}e^{i\theta(s-s')}\psi_0(\theta,x)\overline{\psi_0^*}(\theta,y)e^{\lambda_0(\theta)t}\varphi(s',z')dyd\theta + \alpha_1(t,x). \end{aligned} \tag{24}$$

We split the double integral I above in two parts $I=I_1+I_2$, where the integration in θ in the part I_1 extends only over the segment $|\theta|\le\min(\delta,t^{-1/3})$ and $I_2=I-I_1$. From Lemma 2.2 it follows that $\mathrm{Re}\lambda_0(\theta) < -\frac{\sigma^2}{2}\theta^2,\ |\theta|\le\delta$, if δ is small enough. Since we can take δ as small as we please, and the functions ψ_0, ψ_0^* are bounded, it follows that

$$I_2 \le C\int_{|\theta|\in[t^{-1/3},\delta]}\int_{\Omega}e^{-\frac{\sigma^2}{2}\theta^2 t}|\varphi(y)|dyd\theta \le C\|\varphi\|_{L_1(\Omega)}e^{-\frac{\sigma^2}{2}t^{1/3}}.$$

We replace $\lambda_0(\theta)$ in the integral I_1 by its quadratic approximation found in Lemma 2.2:

$$e^{\lambda_0(\theta)t} = e^{(iV_{\mathrm{eff}}\theta-\sigma^2\theta^2)t}(1+O(|\theta|^3t)).$$

If only the remainder term is taken into account, then the corresponding integral can be estimated by

$$C\int_{|\theta|\le t^{-1/3}}\int_{\Omega}|\theta|^3te^{-\sigma^2\theta^2t}|\varphi(y)|dyd\theta\le C\|\varphi\|_{L_1(\Omega)}\int_{-\infty}^{\infty}|\theta|^3te^{-\sigma^2\theta^2t}d\theta=\frac{C}{t}\|\varphi(y)\|_{L_1(\Omega)}.$$

From here, (24), and (23) it follows that

$$u(t,x)=\frac{1}{2\pi}\int_{|\theta|\le t^{-1/3}}\int_{\Omega}e^{i\theta(s-s')}\psi_0(\theta,x)\overline{\psi_0^*}(\theta,y)e^{(iV_{\text{eff}}\theta-\sigma^2\theta^2)t}\varphi(s',z')dyd\theta+\alpha_2(t,x)),$$

where

$$|\alpha_2|\le Ct^{-1}(\|\varphi(x)\|_{L_1(\Omega)}+\|\varphi(x)\|_{L_2(\Omega)}),\quad t\ge 1.$$

The functions ψ_0 and $\overline{\psi_0^*}$ can be chosen to be smooth in θ at $\theta=0$. We put $\theta=0$ in the arguments of these functions. Choose $\psi_0(0,x)\equiv 1$ and take into account that $\psi_0^*(0,y)$ is real. Then $\psi_0^*(0,y)$ will be the eigenfunction of problem (5) with $\lambda=\theta=0$, normalized, due to (20), by the condition

$$\int_{\Omega'}\psi_0^*(0,x)dx=1.$$

Thus $\psi_0^*(0,y)=\pi(y)$ and $\psi_0(\theta,x)\overline{\psi_0^*}(\theta,y)=\pi(y)+O(|\theta|)$. We put this relation into the formula above and note that the integral with the term $O(|\theta|)$ does not exceed

$$C\int_{|\theta|\le t^{-1/3}}\int_{\Omega}|\theta|e^{-\sigma^2\theta^2t}|\varphi(y)|dyd\theta\le Ct^{-1}\|\varphi\|_{L_1(\Omega)}.$$

Hence

$$u(t,x)=\frac{1}{2\pi}\int_{|\theta|\le t^{-1/3}}\int_{\Omega}e^{i\theta(s-s')}\pi(s',z')e^{(iV_{\text{eff}}\theta-\sigma^2\theta^2)t}\varphi(s',z')dyd\theta+\alpha_3(t,x),\quad y=(s',z'),$$

with the same estimate for α_3 (and α_4 below) as for α_2. We can replace here the integration in θ over the interval $|\theta|\le t^{-1/3}$ by the integration over the whole line since the difference between the corresponding integrals decays exponentially as $t\to\infty$. Thus

$$u(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\Omega}e^{i\theta(s-s')}\pi(s',z')e^{(iV_{\text{eff}}\theta-\sigma^2\theta^2)t}\varphi(s',z')dyd\theta+\alpha_4(t,x),\quad t\to\infty.$$

The integral above, denoted by $w=w(t,s)$ is the main term of the asymptotics. It does not depend on z. By simple differentiation one can check that it satisfies equation (2). Function $w(0,s)$ is the Fourier transform in s' followed by its inverse (in θ), i.e.,

$$w(0,s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{S_{s'}}e^{i\theta(s-s')}\pi(s',z')\varphi(s',z')dz'ds'd\theta=\int_{S_s}\pi(s,z')\varphi(s,z')dz',\quad (25)$$

where S_a is the cross-section of Ω by the plane through $s=a$. Thus w is the solution of problem (2) with w_0 given in (7). Therefore, (14) holds.

Part B. We do, however, need to prove the statement under the assumption that $\varphi \in L_1(\Omega)$ and without the L_2-norm in the right hand side. To prepare for this task, let us make several simple observations.

(a) Obviously, in the estimates of all the remainder terms above one can replace the restriction $t \geq 1$ by $t \geq 1/2$. Similarly, one can use (19) with $t = 1/2$.

(b) Suppose that a measurable function f satisfies

$$|f(s)| \leq ae^{-bs^2}, \quad s \in \mathbb{R}.$$

Then there is a constant K that depends on b (and also on σ) such that

$$\sup_{s\in\mathbb{R}} \Big| \int_{\mathbb{R}} \frac{1}{2\sigma\sqrt{\pi t}} \exp(-\frac{(s-s'+V_{\mathrm{eff}}t)^2}{4\sigma^2 t}) f(s')ds' - \frac{1}{2\sigma\sqrt{\pi t}} \exp(-\frac{(s+V_{\mathrm{eff}}t)^2}{4\sigma^2 t}) \int_{\mathbb{R}} f(s')ds' \Big|$$

$$\leq K\frac{a}{t}$$

for $t \geq 1/2$. The first integral on the left hand side is the solution of (2) with initial data f, while the factor before the second integral is the fundamental solution of the same equation.

(c) There is a constant K such that

$$\sup_{s\in\mathbb{R}} \Big| \frac{1}{2\sigma\sqrt{\pi t}} \exp(-\frac{(s+V_{\mathrm{eff}}t)^2}{4\sigma^2 t}) - \frac{1}{2\sigma\sqrt{\pi(t-\frac{1}{2})}} \exp(-\frac{(s+V_{\mathrm{eff}}t)^2}{4\sigma^2 (t-\frac{1}{2})}) \Big| \leq \frac{K}{t},$$

for $t \geq 1$, i.e., we have a bound on the difference of the values of the fundamental solution at times t and $t - 1/2$.

(d) Let $\varphi \in L_1(\Omega)$ and suppose that $\mathrm{supp}(\varphi) \subseteq \overline{\Omega'}$ (closure of Ω'). Let $\widetilde{\varphi}(x) = u(1/2, x)$, where u is the solution of (1) with initial data φ. From (19) with $t = 1/2$ it follows that there is a constant c such that

$$\|\widetilde{\varphi}\|_{L_1(\Omega)} + \|\widetilde{\varphi}\|_{L_2(\Omega)} \leq c\|\varphi\|_{L_1(\Omega)}.$$

(e) Again, suppose that $\varphi \in L_1(\Omega)$ and $\mathrm{supp}(\varphi) \subseteq \overline{\Omega'}$. Let

$$\widetilde{w}_0(s) = \int_{S_s} \pi(s,z)\widetilde{\varphi}(s,z)dz.$$

Then

$$\int_{\mathbb{R}} \widetilde{w}_0(s)ds = \int_{\mathbb{R}} w_0(s)ds,$$

since π is the stationary solution to the adjoint problem.

(f) From (19) with $t = 1/2$ it follows that there are constants c and b such that

$$|\widetilde{w}_0(s)| \leq c\|\varphi\|_{L_1(\Omega)} e^{-bs^2}, \quad s \in \mathbb{R},$$

provided that $\operatorname{supp}(\varphi) \subseteq \overline{\Omega'}$.

(g) Let $\overline{w}_0(s) = w(1/2, s)$. Then

$$\int_{\mathbb{R}} \overline{w}_0(s)ds = \int_{\mathbb{R}} w_0(s)ds$$

and there are constants c and b such that

$$|\overline{w}_0(s)| \leq c\|\varphi\|_{L_1(\Omega)} e^{-bs^2}, \quad s \in \mathbb{R},$$

provided that $\operatorname{supp}(\varphi) \subseteq \overline{\Omega'}$.

Having made these observations, we return to the proof of Theorem 2.1. First, we claim that

$$\sup_{(s,z)\in\Omega} |u(t,(s,z)) - \frac{1}{2\sigma\sqrt{\pi t}} \exp(-\frac{(s+V_{\text{eff}}t)^2}{4\sigma^2 t}) \int_{\Omega} \varphi(x)\pi(x)dx| \leq Ct^{-1}\|\varphi\|_{L_1(\Omega)}, \quad t \geq 1, \tag{26}$$

provided that $\varphi \in L_1(\Omega)$ and $\operatorname{supp}(\varphi) \subseteq \overline{\Omega'}$. Indeed, the solution u to equation (1) with initial data φ can be viewed as the solution with initial data $\widetilde{\varphi}$ (see (d)), but evaluated at time $t - 1/2$ instead of time t. By (14) and observation (d), $u(t,(s,z))$ is close to the solution, at time $t-1/2$, of the one-dimensional equation (2) with the initial function $\widetilde{w}_0(s)$. By observations (b) and (f), the solution of (2) with the initial data $\widetilde{w}_0(s)$ can be, in turn, approximated by the fundamental solution (with the coefficient $\int_{\mathbb{R}} \widetilde{w}_0(s)ds$) of the same equation. By observation (c), the fundamental solution at time $t-1/2$ is close to the fundamental solution at time t. The coefficient $\int_{\mathbb{R}} \widetilde{w}_0(s)ds$ can be replaced by $\int_{\mathbb{R}} w_0(s)ds = \int_{\Omega} \varphi(x)\pi(x)dx$ by the observation (e).

We have thus justified (26). In other words, we proved that the solution of the original PDE is close to the fundamental solution of the one-dimensional equation provided that the initial data is supported in Ω'. This proves part 2 of Theorem 2.1 provided that $K \subset \Omega'$. If K is not contained in Ω', we can take a sufficiently large n so that $K \subset \Omega \cap \{x : x_1 \in [-n,n]\}$, and treat $\Omega \cap \{x : x_1 \in [-n,n]\}$ as the unit of periodicity instead of Ω'.

By observations (b) and (g), the fundamental solution (multiplied by $\int_{\Omega} \varphi(x)\pi(x)dx$) at time $t-1/2$ is close to the solution of (2) with the initial data $\overline{w}_0$. The latter is equal to w evaluated at time t. Therefore, we can replace the fundamental solution by w in the estimate (26), thus obtaining part 1 of Theorem 2.1 under the additional assumption that $\operatorname{supp}(\varphi) \subseteq \overline{\Omega'}$.

This assumption is not essential, however, since every initial function φ can be written as

$$\varphi = \sum_{i=-\infty}^{\infty} \varphi \chi_{\{x : x_1 \in [i, i+1]\}}.$$

Since the statement hold for each of the terms in the series, it holds for the function φ as well. Thus Theorem 2.1 is proved.

Finally, Theorem 2.3 holds true since the expressions for V_{eff} and σ^2 were introduced in the course of the proof of Theorem 2.1 as the coefficients in the expansion of λ_0 from Lemma 2.2.

$\square$

In conclusion of this section we will provide some formulas that can be useful for practical evaluation of the effective drift V_{eff} and the effective diffusivity σ^2.

Recall that $\lambda_0(\theta)$ is a simple eigenvalue of problem (4) due to Lemma 2.2. Therefore, the eigenvalue and the corresponding eigenfunction v are analytic in θ. We can choose v in such a way that $v = 1$ when $\theta = 0$. Then it is defined uniquely up to a scalar factor that is analytic in θ. Let

$$v = 1 + i\theta v_1(x) - \theta^2 v_2(x) + O(\theta^3), \quad \theta \to 0, \tag{27}$$

be the Taylor expansion of v at $\theta = 0$. We plug (27) and the expansion for $\lambda_0(\theta)$ from Lemma 2.2 into (4) and take into account that conditions on S_0, S_1 in (4) can be rewritten as periodicity in $s = x_1$ of the function $e^{-ix\theta}v$. Then we obtain the following problems for the coefficients v_1, v_2:

$$\Delta v_1 + V\frac{\partial v_1}{\partial s} = V_{\text{eff}}, \quad x \in \Omega'; \ \ \frac{\partial v_1}{\partial n}|_{S^l} = 0; \ \ v_1 = s + \psi_1(x), \tag{28}$$

$$\Delta v_2 + V\frac{\partial v_2}{\partial s} = V_{\text{eff}} v_1 + \sigma^2, \quad x \in \Omega'; \ \ \frac{\partial v_2}{\partial n}|_{S^l} = 0; \ \ v_2 = \frac{s^2}{2} + s\psi_1(x) + \psi_2(x), \tag{29}$$

where ψ_1, ψ_2 are some periodic functions and V_{eff}, σ^2 are defined in Lemma 2.2 (see also Theorem 2.3). Function v_1 is called the *harmonic coordinate* (it is defined up to an additive constant). Usually the harmonic coordinate satisfies a homogeneous equation, but this equation (see (28)) becomes inhomogeneous in the presence of a drift in the problem. Recall that $\pi = \pi(x)$ is the principal eigenfunction (normalized by (6)) with eigenvalue $\lambda = 0$ for the adjoint problem (5) with $\theta = 0$, i.e.,

$$\Delta \pi - V\frac{\partial \pi}{\partial s} = 0, \ x \in \Omega'; \ \ (\frac{\partial}{\partial n} - Vn_1)\pi|_{S^l} = 0; \ \ \pi|_{S_1} = \pi|_{S_0}, \ \pi'_s|_{S_1} = \pi'_s|_{S_0}. \tag{30}$$

This is a positive function due to the Perron-Frobenius theorem (see details in the proof of Lemma 2.2). This function is *the density of the invariant measure.* Note that $\pi(x) \equiv 1/|\Omega'|$ if $V = 0$.

Theorem 2.5. *The effective drift V_{eff} can be found from either of the following three formulas:*

$$V_{\text{eff}} = \int_S (V\pi - \pi'_s)dS = V - \int_{\Omega'} \pi'_s dx = V - \int_{\partial S^l} n_1 \pi dS, \tag{31}$$

where S is a cross-section of the domain Ω by an arbitrary hyperplane $s =$ const., n_1 is the first component of the outward normal vector n, and $S^l = \partial\Omega' \setminus [S_0 \bigcup S_1]$ is the lateral part of the boundary of Ω'. The effective diffusivity is given by

$$\sigma^2 = \int_{\Omega'} |\nabla v_1|^2 \pi dx. \tag{32}$$

Remark. Note that $\pi \equiv 1/|\Omega'|$ if $V = 0$. Formula (31) then implies that $V_{\text{eff}} = 0$. We'll also show that in this case

$$\sigma^2 = 1 - \frac{\int_{\Omega'} |\nabla \psi_1|^2 dx}{|\Omega'|}.$$

The latter formula shows that the effective diffusivity is always smaller than in the free space if Ω is not a cylinder (i.e., if ψ_1 is not identically zero).

Proof of Theorem 2.5. Let $L = \Delta + V\frac{\partial}{\partial s}$ be the differential expression in the left-hand side of (28), and let $L^* = \Delta - V\frac{\partial}{\partial s}$ be the conjugate expression (see (30)). Since $L^*\pi = 0$ and v_1 satisfies (28), we have

$$\int_{\Omega'} (Lv_1)\pi dx - \int_{\Omega'} v_1 L^* \pi dx = V_{\text{eff}} \int_{\Omega'} \pi dx = V_{\text{eff}}.$$

This relation can be re-written using the Green formula and applying the divergence theorem to the vector field $\overrightarrow{F} = (v_1\pi, 0, 0)$. This leads to

$$V_{\text{eff}} = \int_{\partial\Omega'} [(\Delta v_1)\pi - v_1\Delta\pi + V\frac{\partial(v_1\pi)}{\partial s}]dx = \int_{\partial\Omega'} (\frac{\partial v_1}{\partial n}\pi - v_1\frac{\partial \pi}{\partial n} + Vv_1\pi n_1)dS.$$

The integrand on the right hand side vanishes on S^l since π satisfies the boundary condition (30) on S^l and $\frac{\partial v_1}{\partial n} = 0$ on S^l. Thus

$$V_{\text{eff}} = \int_{S_0 \bigcup S_1} (\frac{\partial v_1}{\partial n}\pi - v_1\frac{\partial \pi}{\partial n} + Vv_1\pi n_1)dS. \tag{33}$$

We substitute here $v_1 = s + \psi_1$. The integral with ψ_1 is zero due to the periodicity of the functions ψ_1 and π. Thus (33) holds with v_1 replaced by s. If we also take into account that

$$\int_{S_0 \bigcup S_1} \frac{\partial s}{\partial n}\pi dS = 0,$$

then we obtain the first equality (31) with $S = S_1$. Then this equality holds with any S since V_{eff} is invariant with respect to the shift $s \to s + a$. Let us provide another way to show that the second term in (31) does not depend on the choice of S. Let $\Omega_{a,b} = \Omega \bigcap \{a < s < b\}$. Then

$$\begin{aligned} 0 = \int_{\Omega_{a,b}} L^*\pi dx = \int_{\partial\Omega_{a,b}} (\pi'_n - V\pi n_1)dS = \int_{S_a \bigcup S_b} (\pi'_n - V\pi n_1)dS \\ = \int_{S_b} (\pi'_s - V\pi)dS - \int_{S_a} (\pi'_s - V\pi)dS. \end{aligned}$$

In order to prove the second equality (31), we write the first one with $S = S_a$ and integrate with respect to a over the interval $(0, 1)$. The last equality (31) follows from the divergence theorem.

The proof of (32) is similar. We have

$$\int_{\Omega'}(V_{\text{eff}}v_1+\sigma^2)\pi dx=\int_{\Omega'}(Lv_2)\pi dx-\int_{\Omega'}v_2L^*\pi dx=\int_{S_0\bigcup S_1}(\frac{\partial v_2}{\partial n}\pi-v_2\frac{\partial \pi}{\partial n}+Vv_2\pi n_1)dS.$$

We plug here $v_2=\frac{s^2}{2}+s\psi_1+\psi_2$. The integral with ψ_2 vanishes due to the periodicity of ψ_2 and π. Hence v_2 can be replaced by $\frac{s^2}{2}+s\psi_1$. We reduce the integral over S_0 to an integral over S_1 using the substitution $s\to s-1$. Taking into account the periodicity of ψ_1 and π we obtain

$$\int_{\Omega'}(V_{\text{eff}}v_1+\sigma^2)\pi dx=\int_{S_1}[(1+\frac{\partial\psi_1}{\partial s})\pi-(\frac{1}{2}+\psi_1)\frac{\partial\pi}{\partial s}+V(\frac{1}{2}+\psi_1)\pi]dS$$
$$=\int_{S_1}[\frac{\partial v_1}{\partial s}\pi-v_1\frac{\partial\pi}{\partial s}+Vv_1\pi]dS-\frac{1}{2}V_{\text{eff}}.$$

The last relation is the consequence of the first equality (31) and the equality $v_1=s+\psi_1$. Hence,

$$\sigma^2=\int_{S_0}[\frac{\partial v_1}{\partial s}\pi-v_1\frac{\partial\pi}{\partial s}+Vv_1\pi]dS-V_{\text{eff}}\int_{\Omega'}v_1\pi dx-\frac{1}{2}V_{\text{eff}}. \tag{34}$$

It remains to show that

$$\int_{\Omega'}|\nabla v_1|^2\pi dx=\int_{S_0}[\frac{\partial v_1}{\partial s}\pi-v_1\frac{\partial\pi}{\partial s}+Vv_1\pi]dS-V_{\text{eff}}\int_{\Omega'}v_1\pi dx-\frac{1}{2}V_{\text{eff}}. \tag{35}$$

We note that $L(v_1^2)=2v_1Lv_1+2|\nabla v_1|^2=2V_{\text{eff}}v_1+2|\nabla v_1|^2$. Thus

$$\int_{\Omega'}|\nabla v_1|^2\pi dx=\frac{1}{2}\int_{\Omega'}L(v_1^2)\pi dx-V_{\text{eff}}\int_{\Omega'}v_1\pi dx$$
$$=\int_{S_0\bigcup S_1}[v_1\frac{\partial v_1}{\partial n}\pi-\frac{1}{2}v_1^2(\frac{\partial\pi}{\partial n}-V\pi n_1)]dS-V_{\text{eff}}\int_{\Omega'}v_1\pi dx.$$

We took into account here that the integrand of the first term on the right vanishes on S^l. The last inequality implies (35) if the integral over S_1 in the equality above is rewritten in terms of the integral over S_0 using the first formula (31) and periodicity of the functions π and v_1-s. The proof of the theorem is complete. Now let us justify the remark.

Assume that $V=0$. From the divergence theorem and periodicity of ψ_1 it follows that

$$\int_{\Omega'}\frac{\partial\psi_1}{\partial s}dx=\int_{\partial\Omega'}\psi_1n_1dS=\int_{S^l}\psi_1n_1dS.$$

Since $n_1=\frac{\partial s}{\partial n}$ and the latter function on S^l is equal to $-\frac{\partial\psi_1}{\partial n}$ (due to (28)), we have

$$\int_{\Omega'}\frac{\partial\psi_1}{\partial s}dx=\int_{\partial\Omega'}\psi_1n_1dS=-\int_{S^l}\psi_1\frac{\partial\psi_1}{\partial n}dS=-\int_{\partial\Omega'}\psi_1\frac{\partial\psi_1}{\partial n}dS.$$

Now we recall that $\Delta\psi_1 = \Delta(v_1 - s) = 0$. Hence the latter formula together with the Green formula imply

$$\int_{\Omega'} \frac{\partial \psi_1}{\partial s} dx = -\int_{\Omega'} |\nabla \psi_1|^2 dx. \tag{36}$$

Furthermore, $\pi \equiv 1/|\Omega'|$ since $V = 0$. From $v_1 = \psi_1 + s$, (32), and (36) it follows that

$$\sigma^2 |\Omega'| = \int_{\Omega'} |\nabla v_1|^2 dx = \int_{\Omega'} (1 + 2\frac{\partial \psi_1}{\partial s} + |\nabla \psi_1|^2) dx = \int_{\Omega'} (1 - |\nabla \psi_1|^2) dx$$

□

Theorem 2.6. *The following relation holds:* $V_{\text{eff}} = \sigma_0^2 V + O(V^2)$ *as* $V \to 0$ *where* $\sigma_0^2 = \sigma^2|_{V=0} > 0$ *is the effective diffusivity at* $V = 0$. *In particular, if* $|V|$ *is small enough, then* $V_{\text{eff}} \neq 0$ *when* $V \neq 0$ *and* V_{eff} *is a monotone function of* V.

Proof. From (30) it follows that π is an analytic function of V. Then (31) implies that V_{eff} is analytic in V, and (28) implies that v_1 and ψ_1 depend on V analytically. We expand $\pi = \frac{1}{|\Omega'|} + V\pi_1 + O(V^2)$ using the Taylor approximation at $V = 0$. Then (30) leads to the following problem for π_1:

$$\Delta \pi_1 = 0, \quad x \in \Omega'; \quad \frac{\partial \pi_1}{\partial n}|_{S^l} = \frac{n_1}{|\Omega'|}; \quad \pi_1|_{S_1} = \pi_1|_{S_0}, \; (\pi_1)'_s|_{S_1} = (\pi_1)'_s|_{S_0}.$$

Since $n_1 = \frac{\partial s}{\partial n}$, only the factor $-1/|\Omega'|$ in the boundary condition makes the problem for π_1 different from the problem for $\psi_1^0 = \psi_1|_{V=0}$. The latter problem can be obtained from (28) if we put there $V = V_{\text{eff}} = 0$. Hence $\pi_1 = -\frac{\psi_1^0}{|\Omega'|}$, i.e., $\pi = \frac{1}{|\Omega'|} - \frac{V\psi_1^0}{|\Omega'|} + O(V^2)$. We put this into the second of relations (31) and obtain that

$$V_{\text{eff}} = V(1 + \frac{\int_{\Omega'} (\psi_1^0)'_s dx}{|\Omega'|}) + O(V^2) = V(1 - \frac{\int_{\Omega'} |\nabla \psi_1^0|^2 dx}{|\Omega'|}) + O(V^2).$$

The last relation follows from (36). It remains only to use the Remark after Theorem 2.5.

□

We still need to prove inequality (19), which was used in the arguments above.

Lemma 2.7. *There are positive constants* a *and* b *such that the inequality*

$$|G(1, x, y)| \leq a e^{-b(x_1 - y_1)^2}$$

holds for the Green function $G(t, x, y)$ *of problem (1).*

Proof. The statement is very similar to the upper Aronson estimate at a fixed time for the Green function of a parabolic problem. The difference from the classical estimate is that here we have reflection on the boundary of the tube, rather than a problem in the entire space. For this reason, we sketch an independent proof.

For each y, denote by D the domain that consists of three cells of periodicity of Ω: the one which contains y, and two adjacent cells. Denote by $S^{(1)}, S^{(2)}$ the vertical parts of the boundary of D that are defined by the equations $x_1 = [y] - 1, x_1 = [y] + 2$, respectively. Let $\widetilde{G}(t, x, y)$ be the Green function of the problem

$$\frac{\partial \widetilde{G}}{\partial t} = \Delta \widetilde{G} + V \frac{\partial \widetilde{G}}{\partial s}, \ x \in D; \ \ \frac{\partial \widetilde{G}}{\partial n}|_{\partial D \bigcap \partial \Omega} = 0; \ \ \widetilde{G}(0, x, y) = \delta_y(x), \quad x \in D, \tag{37}$$

supplemented by the standard periodicity condition on $S^{(1)}, S^{(2)}$. The Green function $\widetilde{G}$ exists and is infinitely smooth when $t \neq 0, x \neq y$ [12]. Let $\zeta^y(t, x)$ be a cut-off function that is equal to one in a small neighborhood of the point $t = 0, x = y$ and is equal to zero when $t > 1/2$ or $|x - y| > 1/2$. Obviously, one can choose ζ^y in such a way that $\frac{\partial \zeta^y}{\partial n}|_{\partial D \bigcap \partial \Omega} = 0$.

Now, instead of estimating the fundamental solution, we can estimate $u^y(t, x) = G(t, x, y) - \zeta^y(t, x)\widetilde{G}(t, x, y)$, which satisfies

$$\frac{\partial u^y}{\partial t} = \Delta u^y + V \frac{\partial u^y}{\partial s} + h_1(t, x, y), \ x \in \Omega; \ \ \frac{\partial u^y}{\partial n}|_{\partial \Omega} = 0; \ \ u^y(0, x) = h_2(x, y), \quad x \in \Omega,$$

where h_1, h_2 are smooth, uniformly in t, x, y bounded functions supported in the 1/2-neighborhoods of the points $t = 0, x = y$ and $x = y$, respectively.

Consider the diffusion process X_t^x defined via

$$dX_t^x = \sqrt{2} dW_t + V dt + \chi_{\partial \Omega}(X_t^x) n(X_t^x) d\xi_t^x, \qquad X_0^x = x, \tag{38}$$

where ξ_t^x is the local time of X_t^x on the boundary of Ω. By the Feynman-Kac formula, the solution u^y can be represented as

$$u^y(1, x) = \int_{1/2}^{1} \mathrm{E} h_1(t, X_t^x) dt + \mathrm{E} h_2(X_1^x).$$

By the uniform boundedness of h_1 and h_2, we conclude that

$$u^y(1, x) \leq C \sup_{0 \leq t \leq 1} \mathrm{P}(X_t^x \in B_y) \leq \mathrm{P}(X_t^x \text{ reaches } B_y \text{ before time } 1),$$

where B_y is the unit ball centered at $x = y$. From the large deviation principle for diffusion processes with reflection on the boundary (see [1]), it follows that there are positive constants c_1 and c_2 such that

$$\sup_{x \in \Omega} \mathrm{P}(\sup_{t \in [0, \varepsilon]} |X_t^x - x| \geq 1) \leq c_1 e^{-c_2/\varepsilon} \tag{39}$$

for each $\varepsilon > 0$. In order for the process X_t^x to reach the unit neighborhood of y, it needs to cross at least $k := [|x_1 - y_1| - 1]$ regions, each bounded by a pair of vertical hyperplanes separated by unit distance. Since the total time available is equal to one, at least $[k/2]$ of the regions need to be crossed in time that is smaller than $2/k$. By (39) and the Markov property of the process, the probability that $[k/2]$ given regions are crossed in time smaller than $2/k$ is estimated from above by $(c_1 e^{-kc_2/2})^{[k/2]}$. Therefore,

$$\mathrm{P}(X_t^x \text{ reaches } B_y \text{ before time } 1) \leq \frac{k!}{[k/2]!(k-[k/2])!}(c_1 e^{-kc_2/2})^{[k/2]},$$

where the combinatorial factor is due to the fact that a priori we don't know which of the regions will be crossed fast. The right hand side is estimated from above by $c_3 e^{-c_4 k^2}$ with positive c_3 and c_4. Finally, the desired estimate on $u^y(1,x)$ follows once we recall that $k = [|x_1 - y_1| - 1]$. □

3 Periodic tubes with nearly separated dead zones.

Consider problem (1) with $V = 0$ (i.e., without drift) in a periodic domain $\Omega = \Omega(\varepsilon) \subset \mathbb{R}^d$ that has the form of an ε-independent periodic tube Ω_0 with a periodic system of cavities connected to the tube Ω_0 by narrow channels (see Fig. 3). Denote one cell of periodicity of the domain $\Omega(\varepsilon)$ by $\Omega'(\varepsilon)$ and one cell of periodicity of Ω_0 by Ω_0'.

We do not impose restrictions on the shape of the cavities and channels besides the periodicity condition, smoothness of the boundary $\partial\Omega(\varepsilon)$ and the assumption that the channel enters the cell of periodicity Ω_0' of the main tube in an ε neighborhood of some point $x_0 \in \partial\Omega_0'$. The latter means that by omitting an ε-neighborhood of x_0 from $\Omega'(\varepsilon)$, one disconnects Ω_0' from the other part of $\Omega'(\varepsilon)$. For simplicity we will assume that there is only one cavity and one channel on each period, but it is not difficult to extend all the arguments to the case of several cavities per period or multiple channels connecting the cavity with the tube. Let us stress that no regularity in ε of the cavity and channels is assumed. For example, their volume may grow, decay, or oscillate as $\varepsilon \to 0$. We also do not need to make a distinction between the cavity and the channel.

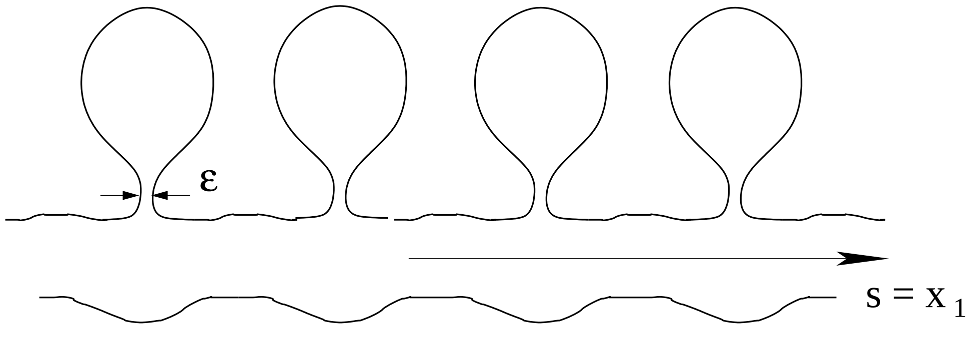

Figure 3: Periodic tube with nearly separated cavities (dead zones).

We assume that the cell Ω_0' is defined by the restriction $0 < s = x_1 < 1$. We denote by $S_0,\ S_1$ the parts of the boundary of Ω_0' that belong to the planes through the points

$s = 0$ and $s = 1$, respectively. Let $S^l = \partial\Omega_0' \backslash [S_0 \bigcup S_1]$ be the lateral part of the boundary $\partial\Omega_0'$ of the cell corresponding to the main tube.

In order to describe the effective diffusivity σ^2 for the problem in $\Omega(\varepsilon)$ and the effective diffusivity σ_0^2 for the problem in the main tube Ω_0 (without the cavities and channels), we introduce the auxiliary problem

$$\Delta u = 0, \quad x \in \Omega_0; \quad u = s + \psi, \quad \text{where } \psi \text{ is 1-periodic in } s; \quad u_n'|_{S^l} = 0. \tag{40}$$

We will call function u the harmonic coordinate (for the domain Ω_0). This function was introduced in the previous section where it was denoted by v_1 (see (28)).

Theorem 3.1. *Let $V = 0$. Then the following formulas are valid for the effective diffusivity σ_0^2 in the case of a periodic tube Ω_0 and for the effective diffusivity $\sigma_0^2(\varepsilon)$ in the case of the tube with cavities:*

$$\sigma_0^2 = \frac{\int_{\Omega_0'} |\nabla u|^2 dx}{|\Omega_0'|}, \tag{41}$$

$$\sigma^2(\varepsilon) = \frac{|\Omega_0'|\sigma_0^2}{|\Omega'(\varepsilon)|} + O(\varepsilon^{\frac{d}{2}}), \quad \varepsilon \to 0. \tag{42}$$

Remark. When Ω_0 is a cylinder, formula (42) can be found in [5], [10], under certain assumptions on the connecting channels.

Proof of Theorem 3.1. In the case of domain Ω_0, we will use notations u, ψ for functions v_1, ψ_1, defined in (27), (28) (compare (28) and (40)), and we preserve the original notations v_1, ψ_1 when the problem in $\Omega(\varepsilon)$ is considered.

We note that π is a constant when $V = 0$. From the normalization condition ($\int_{\Omega'} \pi dx = 1$) it follows that $\pi = 1/|\Omega_0'|$ for the problem in Ω_0 and $\pi = 1/|\Omega'(\varepsilon)|$ for the problem in $\Omega(\varepsilon)$. Hence, formula (41) follows from (32). It only remains to prove (42). In fact, (32) was derived without any specific assumptions on the periodic domain Ω, and it is valid for both domains Ω_0 and $\Omega(\varepsilon)$. Thus

$$\sigma^2(\varepsilon) = \frac{\int_{\Omega'(\varepsilon)} |\nabla v_1|^2 dx}{|\Omega'(\varepsilon)|}. \tag{43}$$

We are going to compare the functions u and v_1 and derive (42) from (43) and (41).

Let $x_0 \in S^l$ be the point on the lateral side of the cell Ω_0' where the channel enters the main tube. Harmonic coordinates u and v_1 (for tubes Ω_0 and $\Omega(\varepsilon)$, respectively) are defined up to arbitrary additive constants. We fix these constants assuming that $u(x_0) = v_1(x_0) = 0$.

We fix a function $\alpha = \alpha(\varepsilon, x) \in C^\infty(\Omega_0')$ such that $\alpha = 0$ when $|x - x_0| < 2\varepsilon$, $\alpha = 1$ when $|x - x_0| > 3\varepsilon$ and $\frac{\partial\alpha}{\partial n} = 0$ on S^l. Moreover, we assume that $|\nabla\alpha| \leq c_1/\varepsilon$ for some ε-independent c_1. We consider function $w = \alpha(x)u(x)$ and extend it by zero in the channel and the cavity. Then $w \in C^\infty(\Omega'(\varepsilon))$.

Let us estimate $v_1 - w$. Obviously, this difference satisfies the following relations in $\Omega'(\varepsilon)$:

$$\Delta(v_1 - w) = f := 2\nabla\alpha\nabla u + u\Delta\alpha, \quad x \in \Omega'(\varepsilon); \quad \frac{\partial(v_1 - w)}{\partial n} = 0, \quad x \in S^l(\varepsilon), \tag{44}$$

and satisfies the periodicity condition on S_0, S_1. We multiply both sides of the equation above by $v_1 - w$, integrate over $\Omega'(\varepsilon)$ (the right hand side is supported in Ω'_0), and apply the Green formula to the left-hand side and to the second term on the right. This implies that

$$\|\nabla(v_1 - w)\|^2_{L_2(\Omega'(\varepsilon))} = \int_{\Omega'_0} [2(v_1 - w)\nabla\alpha\nabla u + (v_1 - w)u\Delta\alpha]dx =$$

$$\int_{\Omega'_0} [2(v_1 - w)\nabla\alpha\nabla u - \nabla((v_1 - w)u)\nabla\alpha]dx = \int_{\Omega'_0} [(v_1 - w)\nabla\alpha\nabla u - u\nabla(v_1 - w)\nabla\alpha]dx$$

$$= \int_{\Omega'_0} (v_1 - w)\nabla(\alpha - 1)\nabla u - \int_{\Omega'_0} u\nabla(v_1 - w)\nabla\alpha dx. \tag{45}$$

Next, we use the following Green formula based on (40) and the fact that $\alpha - 1$ vanishes in a neighborhood of $S_0 \bigcup S_1$:

$$0 = \int_{\Omega'_0} (v_1 - w)(\alpha - 1)\Delta u dx = -\int_{\Omega'_0} \nabla[(v_1 - w)(\alpha - 1)]\nabla u dx.$$

Thus, the first term in the right-hand side of (45) can be replaced by

$$-\int_{\Omega'_0} (\alpha - 1)\nabla(v_1 - w)\nabla u dx.$$

After that, (45) implies the following estimate:

$$\|\nabla(v_1 - w)\|_{L_2(\Omega'(\varepsilon))} \le \|(\alpha - 1)\nabla u\|_{L_2(\Omega'_0)} + \|u\nabla\alpha\|_{L_2(\Omega'_0)}.$$

Function u is smooth and ε-independent. Obviously, $\|\alpha - 1\|_{L_2(\Omega'_0)} = O(\varepsilon^{d/2}),\ \varepsilon \to 0$. A similar estimate is valid for $u\nabla\alpha$ since $|u| < C\varepsilon$ on the support of $\nabla\alpha$ (recall that $u(x_0) = 0$). Hence

$$\|\nabla(v_1 - w)\|_{L_2(\Omega'(\varepsilon))} = O(\varepsilon^{\frac{d}{2}}), \quad \varepsilon \to 0.$$

Thus one can make the following changes in formula (43) with the accuracy of $O(\varepsilon^{d/2})$: replace v_1 by $w = \alpha u$, replace integration over $\Omega'(\varepsilon)$ by the integration over Ω'_0, and then drop α (since it was shown above that $\|\nabla[(\alpha - 1)u]\| = O(\varepsilon^{d/2})$). In other words, one can replace the numerator in (43) by the numerator from (41) plus $O(\varepsilon^{d/2})$. Then it remains only to use (41). □

References

[1] R.F. Anderson, S. Orey, Small random perturbation of dynamical systems with reflecting boundary, Nagoya Math. J., Vol 60 (1976) pp. 189-216.

[2] N. S. Bakhvalov, G. P. Panasenko, Homogenization: averaging processes in pe- riodic media. Mathematical problems in mechanics of composite materials, "Nauka, Moscow, 1984; English transl., Math. Appl. (Soviet Ser.), vol. 36, Kluwer Acad. Publ. Group, Dordrecht, 1989.

[3] A. Bensoussan, J.-L. Lions, G. Papanicolaou, Asymptotic analysis for periodic structures, Stud. Math. Appl., vol. 5, North-Holland Publishing Co., Amsterdam-N.Y., 1978.

[4] A. Berezhkovskii, M. Pustovoit, S. Besrukov, Channel-facilitated membrane transport: Average lifetime in the channel, J. Chemical Phys., 119 (2003), 3943.

[5] A. Berezhkovskii, L. Dagdug, Analytic treatment of biased diffusion in tubes with periodic dead ends, J. Chemical Phys., 134 (2011), 124109.

[6] R. Bunoiu, G. Cardone, T. Suslina, Spectral approach to homogenization of an elliptic operator periodic in some directions. Math. Methods Appl. Sci. 34 (2011), no. 9, 10751096.

[7] C. Conca, M. Vanninathan, Homogenization of periodic structures via Bloch decomposition, Siam J. Appl. Math. 57, No. 6 (1997) pp. 1639-1659.

[8] C. Conca, R. Orive, M. Vanninathan, Bloch approximation in homogenization on bounded domains, Asymptot. Anal. 41 (2005), No. 1, 71-91.

[9] C. Conca, J. San Martn, L. Smaranda, M. Vanninathan, Burnett coefficients and laminates. Appl. Anal. 91 (2012), no. 6, 11551176.

[10] L. Dagdug, A. Berezhkovskii, Y. Makhnovskii, V. Zitserman, Transient diffusion in the tube with dead ends, J. Chemical Phys., 127 (2007), 224712.

[11] F. den Hollander, S. Molchanov, O. Zeitouni, Random Media at Saint-Flour. Reprints of lectures from the Annual Saint-Flour Probability Summer School held in Saint-Flour. Probability at Saint-Flour. Springer, Heidelberg, 2012.

[12] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, Inc, 1964.

[13] M. Freidlin, W. Hu, On diffusion in narrow random channels. J. Stat. Phys. 152 (2013), no. 1, 136158.

[14] G. Grubb, Functional Calculus of Pseudodifferential Boundary Problems, Birkhauser, Boston, 1996.

[15] S.A. Nazarov, I.L. Pankratova, A.L. Piatnitski, Homogenization of the spectral problem for periodic elliptic operators with sign-changing density function. Arch. Ration. Mech. Anal. 200 (2011), no. 3, 747–788.

[16] P. Reimann, Brownian motors: noisy transport far from equilibrium, Physical reports, 361 (2002), 57-265.

[17] E. Sanchez-Palencia, Nonhomogeneous Media and Vibration Theory. Lecture Notes in Physics, vol. 127. Springer: Berlin, New York, 1980.

[18] T. Suslina, Homogenization of the Neumann problem for elliptic systems with periodic coefficients. SIAM J. Math. Anal. 45 (2013), no. 6, 34533493.

[19] V. V. Zhikov, S. M. Kozlov, O. A. Olejnik, Homogenization of differential operators, “Nauka, Moscow, 1993; English transl., Springer-Verlag, Berlin, 1994.

[20] R. Zwanzig, Effective diffusion coefficient for a Brownian particle in a 2-D periodic channel, Physica, 117(A) (1983), 277-280.