A solvable model for homopolymers and self-similarity near critical point

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Abstract

We consider a model for the distribution of a long homopolymer with an attractive zero-range potential at the origin in $\mathbb{R}^3$ (polymer pinning and de-pinning). The distribution can be obtained as a limit of Gibbs distributions corresponding to properly normalized potentials concentrated in small neighborhoods of the origin as the size of the neighborhoods tends to zero. The distribution depends on the length T of the polymer and a parameter γ that corresponds, roughly speaking, to the difference between the inverse temperature in our model and the critical value of the inverse temperature.

At the critical point $\gamma_{cr} = 0$ the transition occurs from the globular phase (positive recurrent behavior of the polymer, $\gamma > 0$) to the extended phase (Brownian type behavior, $\gamma < 0$). The main result of the paper is a detailed analysis of the behavior of the polymer when γ is near γ_{cr}.

Our approach is based on analyzing the semigroups generated by the self-adjoint extensions $\mathcal{L}_\gamma$ of the Laplacian on $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ parametrized by γ, which are related to the distribution of the polymer. The main technical tool of the paper is the explicit formula for the resolvent of the operator $\mathcal{L}_\gamma$.

Key words: Gibbs measure, homopolymer, zero-range potential, phase transition, globular phase, diffusive phase.

2000 Mathematics Subject Classification Numbers: 60K37, 60K35, 82B26, 82B27, 82D60, 35K10.

1 Introduction

The study of polymer models has been a very active area of research in mathematical physics in recent years. Many of the physically relevant problems have been outlined in the

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paper of Lifschitz, Grosberg and Khokhlov [17]. To name just a few recent articles on the subject, with apologies to the many authors who have been omitted, we cite [2], [5], [4], [6], [7], [8]. In addition, there is an interesting exposition with many valuable references in [12]. The approach to the subject involves many essential ideas from statistical mechanics including strong connections to the developments in [11], [13] and [16].

Let us give a general qualitative description of the problem. We start with a path space $\Omega = C([0,T],\mathbb{R}^3)$, a Hamiltonian $H_T : \Omega \to (-\infty, 0]$ and a probability measure on Ω. A continuous function $\omega \in \Omega$ will be thought of as a realization of the polymer. The parameter $t \in [0,T]$ can be intuitively understood as the length along the polymer (although the functions $\omega = \omega(t)$ are not differentiable and the genuine notion of length can not be defined). One can consider models with the Hamiltonian given by

$$H_T(\omega) = -\int_0^T \int_0^T v(\omega(s_1) - \omega(s_2)) ds_1 ds_2,$$

where $v : \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$ is a local attractive potential (see [3], [14]). Our paper concerns a simpler "mean field" type model, where the polymer chain interacts with the external attractive potential (as in [17]). Namely, consider the Hamiltonian

$$H_T(\omega) = -\int_0^T v(\omega(s)) ds, \tag{1}$$

where v is a nonnegative compactly supported potential that is not identically equal to zero. The model considered here is that of a homopolymer since the potential is time-independent. Other models (of heteropolymers) which we do not consider in this paper allow for a potential that is a random stationary function of time (see [2] for example).

Let P_T^x be the Wiener measure on $\Omega = C([0,T],\mathbb{R}^3)$ shifted so that the trajectories start at x almost surely. This will be the reference measure corresponding to the infinite temperature, i.e., to the inverse temperature $\beta = 0$. For a value of inverse temperature $\beta \geq 0$, the polymer is distributed according to the Gibbs measure $\mathrm{P}_{\beta,T}^x$, whose density with respect to P_T^x is

$$\frac{d\mathrm{P}_{\beta,T}^x}{d\mathrm{P}_T^x}(\omega) = \frac{\exp(-\beta H_T(\omega))}{Z_\beta(T,x)}, \quad \omega \in C([0,T],\mathbb{R}^3). \tag{2}$$

The normalizing factor $Z_\beta(T,x) = \mathrm{E}_T^x e^{-\beta H_T}$ is called the partition function.

The phase transitions for the polymers are well-understood at the physical level (see [17]). For large β ($\beta > \beta_{cr}$ for some $\beta_{cr} > 0$) and any $t \in [0,T]$, with high probability with respect to the measure $\mathrm{P}_{\beta,T}^x$ the values of $\omega(t)$ are bounded by a constant that is independent of t. Thus the polymer is in the globular state. However, for $\beta < \beta_{cr}$ the typical shape of $\omega(t)$, $t \in [0,T]$, is that of a Brownian path in $\mathbb{R}^3$, and the polymer is said to be in the diffusive state. Rigorous results of this nature have also been proven in some generality (see [12], [2] for example). Of particular interest is the behavior of the polymer for the values of β that are near β_{cr}. Unfortunately, the detailed analysis of critical phenomena for general Hamiltonians is largely outside the range of techniques of modern mathematical physics. Most of the results on this type of problems concern the existence of a phase transition.

However, in our situation (homopolymer with a compactly supported non-negative potential), the complete analysis of the critical phenomena is possible. In [9] we studied the prevalent behavior of the polymer with respect to the measure $\mathrm{P}^x_{\beta,T}$ as $T \to +\infty$. In particular, we saw that for $d \geq 3$ there is $\beta_{cr} > 0$ such that:

(a) For $\beta > \beta_{cr}$, each constant $s > 0$ and each function $S(T)$ such that $S(T) \to +\infty$ and $T - S(T) \to +\infty$ as $T \to +\infty$, the distribution of $\omega(S(T)+t)$, $t \in [0, s]$, with respect to $\mathrm{P}^x_{\beta,T}$, converges to a recurrent Markov process on $[0, s]$.

(b) For $\beta < \beta_{cr}$, the distribution of $\omega(tT)/\sqrt{T}$, $t \in [0, 1]$, converges to the Brownian motion on the interval $[0, 1]$.

(c) For $\beta = \beta_{cr}$, the distribution of $\omega(tT)/\sqrt{T}$, $t \in [0, 1]$, converges to the distribution of a non-Gaussian Markov process on the interval $[0, 1]$.

Some of the earlier results for the discrete model based on the random walk and zero-range potential can be found in [10].

In the current paper we study the distributions of the continuous polymer in $\mathbb{R}^3$ which correspond to Schrodinger operators with zero-range potential. The potential is, in some sense, supported at the origin. Note, however, that the Hamiltonian H_T, the corresponding Gibbs measure and the Schrodinger operator L_β defined by (1), (2) and (3) are meaningless if v is a delta-function concentrated at the origin. Before we define these objects, let us note that in the case of a smooth potential, the finite-dimensional distributions of $\mathrm{P}^x_{\beta,T}$ can be expressed in terms of the fundamental solutions $p_\beta(t, x, y)$ of the parabolic equation

$$\frac{\partial u}{\partial t} = L_\beta u := \frac{1}{2}\Delta u + \beta v u. \tag{3}$$

Indeed, by (2) and the Feynman-Kac formula,

$$Z_\beta(T, x) = \int_{\mathbb{R}^3} p_\beta(T, x, y) dy,$$

and

$$\mathrm{P}^x_{\beta,T}(\omega(t_1) \in A_1, ..., \omega(t_k) \in A_k) = \tag{4}$$

$$Z_\beta^{-1}(T, x) \int_{A_1} ... \int_{A_k} \int_{\mathbb{R}^3} p_\beta(t_1, x, x_1)...p_\beta(t_k - t_{k-1}, x_{k-1}, x_k) p_\beta(T - t_k, x_k, y) dy dx_k...dx_1,$$

where $k \geq 1$, $0 \leq t_1 \leq ... \leq t_k \leq T$ and $A_1, ..., A_k$ are Borel sets in $\mathbb{R}^3$. Here we use the convention that $p_\beta(0, x, y) = \delta_x(y)$. The measure $\mathrm{P}^x_{\beta,T}$ corresponds to a non-homogeneous Markov process on $[0, T]$ (see [9]).

Now, instead of the operators L_β, we start with the Schrodinger operators with zero-range potential. They are defined (see [1]) as self-adjoint extension in $L^2(\mathbb{R}^3)$ of the operator

$$\Delta/2 : C_0^\infty(\mathbb{R}^3 \setminus \{0\}) \to C_0^\infty(\mathbb{R}^3 \setminus \{0\}).$$

There is a family of such extensions $\mathcal{L}_\gamma$ which depend on a parameter $\gamma \in \mathbb{R}$ that plays the same role as the difference between β and β_{cr} in the case of smooth potentials (see Section 2). Then we replace operator L_β in (3) by the operator $\mathcal{L}_\gamma$. This allows us to define the finite dimensional distributions and construct the Gibbs measure which corresponds to the zero-range potentials. Namely, let $\overline{p}_\gamma(t,x,y)$, $t \geq 0$, be the kernels of the operators in the semi-group generated by $\mathcal{L}_\gamma$. We can now define

$$\overline{Z}_\gamma(T,x) = \int_{\mathbb{R}^3} \overline{p}_\gamma(T,x,y)dy,$$

and use an analogue of (4) with $\overline{p}_\gamma$ instead of p_β and $\overline{Z}_\gamma(T,x)$ instead of $Z_\beta(T,x)$ in order to define the finite-dimensional distributions of the measure $\overline{\mathrm{P}}^x_{\gamma,T}$. (While neither $\overline{p}_\gamma(t_1,x,x_1)$ nor $\overline{Z}_\gamma(T,x)$ are defined when $x=0$, we can make sense of the expression $\overline{Z}_\gamma^{-1}(T,0)\overline{p}_\gamma(t_1,0,x_1)$ by taking the limit of $\overline{Z}_\gamma^{-1}(T,x)\overline{p}_\gamma(t_1,x,x_1)$ as $|x| \downarrow 0$.) It is not difficult to check that there is a measure on $C([0,T],\mathbb{R}^3)$ with such finite-dimensional distributions, and that this measure defines a non-homogeneous Markov process on $[0,T]$.

There is another way to obtain the measures $\overline{\mathrm{P}}^x_{\gamma,T}$ corresponding to zero-range potentials. One can construct $\overline{\mathrm{P}}^x_{\gamma,T}$ as the weak limit of measures corresponding to bounded compactly supported potentials as their supports shrink and the values of the potentials increase in a particular way. Namely, it turns out that $\mathcal{L}_\gamma$, $\gamma \in \mathbb{R}$, can be obtained as a strong resolvent limit, as $\varepsilon \downarrow 0$, of Schrodinger operators $\mathcal{L}^\varepsilon_\gamma = \Delta/2 + v^\varepsilon_\gamma$ with the potentials (see [1])

$$v^\varepsilon_\gamma = (\frac{\pi^2}{8\varepsilon^2} + \frac{\gamma}{\varepsilon})v(\frac{x}{\varepsilon}), \quad ||v||_{L^1(\mathbb{R}^3)} = \frac{4\pi}{3}. \tag{5}$$

It will be shown in this paper that the resolvent convergence of the Schrodinger operators implies the convergence of the corresponding Gibbs measures. Consider the Hamiltonian $H^\varepsilon_{\gamma,T}$ given by (1) with v^ε_γ instead of v, and the Gibbs measure $\mathrm{P}^{x,\varepsilon}_{\gamma,T}$ given by (2) with $\beta = 1$ and $H^\varepsilon_{\gamma,T}$ instead of H_T. For each $\gamma \in \mathbb{R}$ and $T > 0$ there are limits

$$\overline{\mathrm{P}}^x_{\gamma,T} = \lim_{\varepsilon \downarrow 0} \mathrm{P}^{x,\varepsilon}_{\gamma,T},$$

understood in the sense of weak convergence of measures on $C([0,T],\mathbb{R}^3)$ (see Section 2).

Unlike the case of a generic smooth potential, all the relevant analytic quantities can be found explicitly in the case of the zero-range potential polymer model. This allows us not only to obtain an analogue of the results (a)-(c) above, but also provide a detailed analysis of the behavior of the polymer distribution when γ tends to its critical value and, simultaneously, $T \to +\infty$.

We recall the main facts about the self-adjoint extensions and prove the convergence of $\mathrm{P}^{x,\varepsilon}_{\gamma,T}$ to $\overline{\mathrm{P}}^x_{\gamma,T}$ in Section 2. We refer the reader to [1] for a detailed treatment of the self-adjoint extensions of the Laplacian on $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. Here we only mention that for $d \geq 4$ the only closed self-adjoint extensions of the Laplacian on $C_0^\infty(\mathbb{R}^d \setminus \{0\})$ is the Laplacian on the entire space, and thus there is no analog of the distribution corresponding to a zero-range potential in $d \geq 4$. For $d = 1, 2$, the corresponding homopolymer models do not exhibit a phase transition, so we do not treat them here.

The goal of the current paper is to analyze the behavior of the polymer under the measure $\overline{\mathrm{P}}^x_{\gamma,T}$ for various values of γ and large T. Of particular interest is the behavior of the polymer when γ is near zero. We provide a detailed analysis of the behavior of $\overline{\mathrm{P}}^x_{\gamma(T),T}$ when $\gamma = \gamma(T) \to 0$ as $T \to +\infty$.

Two qualitatively different cases can be considered. The analysis of both cases relies on a self-similarity property of the measures $\overline{\mathrm{P}}^x_{\gamma,T}$. In the first case, $\gamma(T)$ is bounded and such that $\gamma(T)\sqrt{T} \to +\infty$ as $T \to +\infty$. We shall see that for each constant $s > 0$ and each function $S(T)$ such that $\gamma(T)\sqrt{S(T)} \to +\infty$ and $\gamma(T)\sqrt{T - S(T)} \to +\infty$ as $T \to +\infty$, the distribution of $\gamma(T)\omega(S(T) + (\gamma(T))^{-1}t)$, $t \in [0, s]$, with respect to $\overline{\mathrm{P}}^x_{\gamma(T),T}$, converges to a recurrent Markov process on $[0, s]$. The generator of the Markov process will be written out explicitly (see Section 3). In particular, the radial part is a diffusion process on the positive semi-axis with reflection at the origin. These results are also applicable in the situation when $\gamma > 0$ does not depend on T.

In the second case $\gamma(T)$ is such that $\gamma(T)\sqrt{T} \to \varkappa \in [-\infty, +\infty)$ as $T \to +\infty$. We will show that the distribution of $\omega(tT)/\sqrt{T}$, $t \in [0, 1]$, converges to the limiting measure on $C([0, 1], \mathbb{R}^3)$. The limiting measure can be identified as $\overline{\mathrm{P}}^0_{\varkappa,1}$ if $\varkappa > -\infty$ and the Wiener measure on $C([0, 1], \mathbb{R}^3)$ if $\varkappa = -\infty$. In particular, the distribution of $\omega(tT)/\sqrt{T}$, $t \in [0, 1]$, converges to the Wiener measure on $C([0, 1], \mathbb{R}^3)$ if $\gamma < 0$ does not depend on T. The limiting distribution for $\omega(T)/\sqrt{T}$ can be written out explicitly - it turns out to be compound Gaussian (see Section 4).

The paper is organized as follows. In Section 2 we recall some facts about the self-adjoint extensions of the Laplacian and introduce the associated family of processes with the zero-range potential. In Sections 3 and 4 we prove the main results concerning the cases when $\gamma(T)\sqrt{T} \to +\infty$ and $\gamma(T)\sqrt{T} \to \varkappa \in [-\infty, +\infty)$ as $T \to +\infty$, respectively. Finally, in Section 5 we prove the convergence of the processes with potentials v^ε_γ to those with a zero-range potential and prove some technical lemmas regarding the tightness of certain families of processes.

2 Measures corresponding to zero-range potentials

All the self-adjoint extensions of $\Delta/2$ are described in the following theorem, whose proof can be found in [1].

Theorem 2.1. *All the self-adjoint extensions of the Laplacian acting on $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ to an operator acting on $L^2(\mathbb{R}^3)$ form a one-parameter family $\mathcal{L}_\gamma$, $\gamma \in \mathbb{R}$. The spectrum of $\mathcal{L}_\gamma$ is given by*

$$\mathrm{spec}(\mathcal{L}_\gamma) = (-\infty, 0] \cup \left\{\frac{\gamma^2}{2}\right\}, \quad \gamma > 0,$$

$$\mathrm{spec}(\mathcal{L}_\gamma) = (-\infty, 0], \quad \gamma \leq 0.$$

The kernel of the resolvent of $\mathcal{L}_\gamma$ *is given by*

$$R_{\lambda,\gamma}(x,y) = \frac{e^{-\sqrt{2\lambda}|x-y|}}{\pi|x-y|} + \frac{1}{\sqrt{2\lambda}-\gamma}\frac{e^{-\sqrt{2\lambda}(|x|+|y|)}}{2\pi|x||y|}, \quad \lambda \notin \mathrm{spec}(\mathcal{L}_\gamma).$$

If $\gamma > 0$*, then* $\gamma^2/2$ *is a simple eigenvalue of* $\mathcal{L}_\gamma$ *with the eigenfunction*

$$\psi_\gamma(x) = \frac{\sqrt{\gamma}}{\sqrt{2\pi}}\frac{e^{-\gamma|x|}}{|x|}.$$

Since the spectrum of $\mathcal{L}_\gamma$ is bounded from above, the operators $\exp(t\mathcal{L}_\gamma)$, $t \geq 0$, are bounded in $L^2(\mathbb{R}^3)$. The kernel of $\exp(t\mathcal{L}_\gamma)$, $t > 0$, is given by

$$\overline{p}_\gamma(t,x,y) = \frac{1}{2\pi i}\int_{\Gamma(a)} e^{\lambda t} R_{\lambda,\gamma}(x,y)d\lambda = \frac{e^{-|x-y|^2/2t}}{(2\pi t)^{3/2}} + \frac{1}{4\pi^2 i}\int_{\Gamma(a)} \frac{e^{-\sqrt{2\lambda}(|x|+|y|)+\lambda t}}{(\sqrt{2\lambda}-\gamma)|x||y|}d\lambda, \quad (6)$$

where $x, y \neq 0$, $a > \gamma^2/2$ and $\Gamma(a)$ is the contour in the complex plane that is parallel to the imaginary axis and passes through a. Thus $\overline{p}_\gamma(t,x,y)$ can be interpreted as a formal fundamental solution of the equation

$$\frac{\partial u}{\partial t} = \mathcal{L}_\gamma u.$$

Let

$$\overline{Z}_\gamma(t,x) = \int_{\mathbb{R}^3} \overline{p}_\gamma(t,x,y)dy,$$

where $t > 0$, $x \neq 0$. Formula (6) implies that

$$\overline{Z}_\gamma(t,x) = 1 + \frac{1}{2\pi i}\int_{\Gamma(a)} e^{\lambda t}\frac{1}{\sqrt{2\lambda}-\gamma}\frac{e^{-\sqrt{2\lambda}|x|}}{\lambda|x|}d\lambda. \quad (7)$$

Define the measures $\overline{\mathrm{P}}^x_{\gamma,T}$, $x \in \mathbb{R}^3$, via their finite-dimensional distributions

$$\overline{\mathrm{P}}^x_{\gamma,T}(\omega(t_1) \in A_1, ..., \omega(t_k) \in A_k) = \quad (8)$$

$$\overline{Z}_\gamma^{-1}(T,x)\int_{A_1} ... \int_{A_k}\int_{\mathbb{R}^3} \overline{p}_\gamma(t_1,x,x_1)...\overline{p}_\gamma(t_k - t_{k-1}, x_{k-1}, x_k)\overline{p}_\gamma(T - t_k, x_k, y)dydx_k...dx_1,$$

where $k \geq 1$, $0 \leq t_1 \leq ... \leq t_k \leq T$ and $A_1, ..., A_k$ are Borel sets in $\mathbb{R}^3$. Here we use the conventions that $\overline{p}_\gamma(0,x,y) = \delta_x(y)$ and $\overline{Z}_\gamma^{-1}(T,0)\overline{p}_\gamma(t_1,0,x_1) = \lim_{|x|\downarrow 0}\overline{Z}_\gamma^{-1}(T,x)\overline{p}_\gamma(t_1,x,x_1)$. It is not difficult to show that there indeed is a measure on $C([0,T],\mathbb{R}^3)$ with the finite-dimensional distributions given by (8). We don't prove this here since the same conclusion follows from the proof of Theorem 2.3 below. Note the following self-similarity property.

Theorem 2.2. *For $a > 0$, let $f_a : C([0,T],\mathbb{R}^3) \to C([0,aT],\mathbb{R}^3)$ map a function ω to the function $f_a\omega$ via $(f_a\omega)(t) = \sqrt{a}\omega(t/a)$. Let $f_a\overline{\mathrm{P}}^x_{\gamma,T}$ be the push-forward of the measure $\overline{\mathrm{P}}^x_{\gamma,T}$ by this mapping. Then*

$$\overline{p}_\gamma(ta, x\sqrt{a}, y\sqrt{a}) = a^{-3/2}\overline{p}_{\gamma\sqrt{a}}(t,x,y), \qquad \overline{Z}_\gamma(ta, x\sqrt{a}) = \overline{Z}_{\gamma\sqrt{a}}(t,x) \tag{9}$$

and

$$f_a\overline{\mathrm{P}}^x_{\gamma,T} = \overline{\mathrm{P}}^{x\sqrt{a}}_{\gamma/\sqrt{a},Ta}. \tag{10}$$

Proof. The first statement follows after making the change of variables $\lambda' = \lambda/a$ in the integrals in the right hand sides of (6) and (7). The second statement now follows from (8). □

Let us discuss an alternative construction of the measure $\mathrm{P}^{x,\varepsilon}_{\gamma,T}$ that uses short-range potentials. Let v^ε_γ be defined by (5). Let the Hamiltonian $H^\varepsilon_{\gamma,T}$ be given by (1) with v^ε_γ instead of v, and the Gibbs measure $\mathrm{P}^{x,\varepsilon}_{\gamma,T}$ be given by (2) with $\beta = 1$ and $H^\varepsilon_{\gamma,T}$ instead of H_T. The following theorem will be proved in Section 5.

Theorem 2.3. *Let $x \in \mathbb{R}^3 \setminus 0$, $\gamma \in \mathbb{R}$ and $T > 0$. The measures $\mathrm{P}^{x,\varepsilon}_{\gamma,T}$ converge, in the sense of weak convergence of measures on $C([0,T],\mathbb{R}^3)$, as $\varepsilon \downarrow 0$, to the measure $\overline{\mathrm{P}}^x_{\gamma,T}$.*

From (8) it follows that the finite-dimensional distributions of $\overline{\mathrm{P}}^{x'}_{\gamma',T}$ converge to those of $\overline{\mathrm{P}}^0_{\gamma,T}$ as $x' \to 0$ and $\gamma' \to \gamma \in \mathbb{R}$. If $\gamma' \to -\infty$, then they converge to the finite-dimensional distributions of the Wiener measure on $C([0,T],\mathbb{R}^3)$. From Theorem 2.3 and Lemma 5.1 (from Section 5) it follows that the family $\{\overline{\mathrm{P}}^{x'}_{\gamma',T}, \gamma' \le c, |x'| \le 1\}$ is tight for each $c \in \mathbb{R}$, and therefore

$$\lim_{x'\to 0,\gamma'\to\gamma} \overline{\mathrm{P}}^{x'}_{\gamma',T} = \overline{\mathrm{P}}^0_{\gamma,T}, \qquad \lim_{x'\to 0,\gamma'\to -\infty} \overline{\mathrm{P}}^{x'}_{\gamma',T} = \mathrm{P}^0_T, \tag{11}$$

where P^0_T is the Wiener measure and the limits are understood in the sense of weak convergence of measures.

3 Distribution above the critical point

In this section we assume that $\gamma = \gamma(T)$ is bounded and such that $\gamma(T)\sqrt{T} \to +\infty$ as $T \to +\infty$. In particular, this covers the case when $\gamma > 0$ does not depend on T. We examine the behavior of the polymer paths with respect to $\overline{\mathrm{P}}^x_{\gamma(T),T}$ as $T \to +\infty$. First, we need the asymptotics of $\overline{p}_1(t,x,y)$ and $\overline{Z}_1(t,x)$ when $t \to +\infty$.

Lemma 3.1. *We have the following asymptotic expressions:*

$$\overline{p}_1(t,x,y) = \exp(t/2)(\psi_1(x)\psi_1(y) + q(t,x,y)), \tag{12}$$

where $\lim_{t\to+\infty}\sup_{|x|,|y|\ge k}|q(t,x,y)| = 0$ *for each* $k>0$*;*

$$\overline{Z}_1(t,x) = \exp(t/2)||\psi_1||_{L^1(\mathbb{R}^3)}(\psi_1(x)+Q(t,x)), \tag{13}$$

where $\lim_{t\to+\infty}\sup_{|x|\ge k}|Q(t,x)| = 0$ *for each* $k>0$*;*

$$\overline{Z}_1^{-1}(s,x)\overline{p}_1(t,x,y) = \exp((t-s)/2)||\psi_1||^{-1}_{L^1(\mathbb{R}^3)}(\psi_1(y)+\widetilde{q}(s,t,x,y)), \tag{14}$$

where $\lim_{s,t\to+\infty}\sup_{|x|\le k^{-1},|y|\ge k}|\widetilde{q}(s,t,x,y)| = 0$ *for each* $k>0$*.*

Proof. Formula (12) follows immediately from (6) if the integral over $\Gamma(a)$ in (6) is replaced by the integral over $\Gamma(b)$, $0<b<\gamma^2/2$, plus the residue at $\lambda=\gamma^2/2=1/2$. Formula (12) follows similarly from (7). □

Next, let us study the distribution of the end of the polymer with respect to the measure $\overline{\mathrm{P}}^x_{\gamma(T),T}$ as $T\to+\infty$.

Theorem 3.2. *If* $\gamma(T)$ *is bounded and such that* $\gamma(T)\sqrt{T}\to+\infty$ *as* $T\to+\infty$*, then the distribution of* $\gamma(T)\omega(T)$ *with respect to the measure* $\overline{\mathrm{P}}^x_{\gamma(T),T}$ *converges, weakly, as* $T\to+\infty$*, to the distribution with the density* $\psi_1/||\psi_1||_{L^1(\mathbb{R}^3)}$*.*

Proof. For fixed x, the density of $\omega(T)$ with respect to the Lebesgue measure is equal to

$$\overline{Z}^{-1}_{\gamma(T)}(T,x)\overline{p}_{\gamma(T)}(T,x,y).$$

Therefore, the density of $\gamma(T)\omega(T)$ is equal to

$$\gamma(T)^{-3}\overline{Z}^{-1}_{\gamma(T)}(T,x)\overline{p}_{\gamma(T)}(T,x,y/\gamma(T)) = \overline{Z}_1^{-1}(\gamma^2(T)T,x\gamma(T))\overline{p}_1(\gamma^2(T)T,x\gamma(T),y).$$

Here we used (9) with $\gamma=1$, $a=\gamma^2(T)$ and $t=T$. The latter expression is equal to

$$||\psi_1||^{-1}_{L^1(\mathbb{R}^3)}(\psi_1(y)+\widetilde{q}(\gamma^2(T)T,\gamma^2(T)T,x\gamma(T),y)), \tag{15}$$

where $\widetilde{q}$ is the same as in Lemma 3.1. When $T\to+\infty$, the expression in (15) converges to $\psi_1(x)/||\psi_1||_{L^1(\mathbb{R}^3)}$ uniformly in $|y|>k$ by Lemma 3.1. This justifies the weak convergence. □

Now let us examine the behavior of the polymer in a region separated both from zero and T. Let $S(T)$ be such that

$$\lim_{T\to+\infty}\gamma(T)\sqrt{S(T)} = \lim_{T\to+\infty}\gamma(T)\sqrt{T-S(T)} = +\infty. \tag{16}$$

Let $s>0$ be fixed. Consider the process $y^T(t)=\gamma(T)\omega(S(T)+t/\gamma^2(T))$, $0\le t\le s$. Let

$$r(t,x,y) = \frac{\overline{p}_1(t,x,y)\psi_1(y)}{\psi_1(x)}\exp(-t/2),\quad y\ne 0. \tag{17}$$

Observe that $\int_{\mathbb{R}^3} r(t,x,y)dy = 1$, and therefore r serves as the transition density for a Markov process. Also note that ψ_1^2 serves as an invariant density for the process.

Theorem 3.3. *If $\gamma(T)$ is bounded and such that $\gamma(T)\sqrt{T} \to +\infty$ as $T \to +\infty$, then the distribution of the process $y^T(t)$ with respect to the measure $\overline{\mathrm{P}}^x_{\gamma(T),T}$ converges as $T \to +\infty$, weakly in the space $C([0,s],\mathbb{R}^3)$, to the distribution of a stationary Markov process with transition density $r(t,x,y)$ and the invariant measure whose density is ψ_1^2.*

Proof. Let $0 \le t_1 < ... < t_n \le s$. The density of the random vector $(y^T(t_1),...,y^T(t_n))$ with respect to the Lebesgue measure on $\mathbb{R}^{3n}$ is equal to

$$\rho^T(x_1,...,x_n) =$$

$$\gamma(T)^{-3n}(\overline{Z}_{\gamma(T)}(T,x))^{-1}\overline{p}_{\gamma(T)}(S(T)+\frac{t_1}{\gamma^2(T)},x,\frac{x_1}{\gamma(T)})\overline{p}_{\gamma(T)}(\frac{t_2-t_1}{\gamma^2(T)},\frac{x_1}{\gamma(T)},\frac{x_2}{\gamma(T)})\ ...$$

$$...\ \overline{p}_{\gamma(T)}(\frac{t_n-t_{n-1}}{\gamma^2(T)},\frac{x_{n-1}}{\gamma(T)},\frac{x_n}{\gamma(T)})\overline{Z}_{\gamma(T)}(T-S(T)-\frac{t_n}{\gamma^2(T)},\frac{x_n}{\gamma(T)}).$$

Applying (9) with $\gamma = 1$, $a = \gamma^2(T)$ and $t = T$, we see that the right hand side of this equality is equal to

$$(\overline{Z}_1(\gamma^2(T)T,x\gamma(T)))^{-1}\overline{p}_1(\gamma^2(T)S(T)+t_1,x\gamma(T),x_1)\overline{p}_1(t_2-t_1,x_1,x_2)\ ...$$

$$...\ \overline{p}_1(t_n-t_{n-1},x_{n-1},x_n)\overline{Z}_1(\gamma^2(T)(T-S(T))-t_n,x_n).$$

We replace here all factors $\overline{p}_1$, except the first one, by r using (17). We replace the factors $(\overline{Z}_1(\gamma^2(T)T,x\gamma(T)))^{-1}\overline{p}_1(\gamma^2(T)S(T)+t_1,x\gamma(T),x_1)$ and $\overline{Z}_1(\gamma^2(T)(T-S(T))-t_n,x_n)$ by their asymptotic expansions given in Lemma 3.1. This leads to

$$\rho^T(x_1,...,x_n) = \psi_1^2(x_1)r(t_2-t_1,x_1,x_2)...r(t_n-t_{n-1},x_{n-1},x_n)+o(1), \quad T \to +\infty,$$

where the remainder tends to zero uniformly in $(x_1,...,x_n)$ with $\min(|x_1|,...,|x_n|) \ge k$. Since $k > 0$ can be chosen to be arbitrarily small, this justifies the convergence of the finite dimensional distributions of y^T to those of the Markov process. It remains to note that the family of measures induced by the processes y^T is tight, as follows from Lemma 5.2. □

Remark. Let us describe the generator of the limiting Markov process. Namely, let $C_0(\mathbb{R}^3)$ be the space of continuous functions on $\mathbb{R}^3$ with a finite limit at infinity. Consider the differential operator

$$Mu(x) = \frac{1}{2}\Delta u(x) + \frac{(\nabla\psi_1(x),\nabla u(x))}{\psi_1(x)} = \frac{1}{2}\Delta u(x) - (1+\frac{1}{|x|})\frac{\partial u(x)}{\partial r}, \quad x \ne 0,$$

$$Mu(0) = \lim_{|x|\downarrow 0}(\frac{1}{2}\Delta u(x) - (1+\frac{1}{|x|})\frac{\partial u(x)}{\partial r}).$$

Note that in the spherical coordinates M can be written as

$$Mu = \frac{1}{2}(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\widetilde{\Delta}u) - \frac{\partial u}{\partial r},$$

where $\widetilde{\Delta}$ is the Beltrami-Laplace operator on the sphere. For fixed x, the function $r(t,x,y)$ satisfies

$$\frac{\partial r(t,x,y)}{\partial t} = M^* r(t,x,y), \quad t > 0, \; y \in \mathbb{R}^3 \setminus 0,$$

where M^* is the formal adjoint of M. From here it easily follows ([15]) that the generator in the space $C_0(\mathbb{R}^3)$ of the Markov family with transition density $r(t,x,y)$ is given by the operator M with the domain

$$\mathcal{D} = \{u \in C^2(\mathbb{R}^3 \setminus \{0\}) \cap C_0(\mathbb{R}^3) : \; Mu \in C_0(\mathbb{R}^3), \; \lim_{r \downarrow 0} \frac{\int_{S^2} u(r,\varphi) d\mu(\varphi) - u(0)}{r} = 0\},$$

where μ is the Haar probability measure on S^2. The radial part of the limiting process is then a diffusion on the positive semi-axis with unit diffusion coefficient, unit drift towards the origin and reflection at the origin. If we denote the radial part of the limiting process by R_t, then the spherical part of the limiting process S_t on S^2 satisfies

$$dS_t = \frac{1}{R_t} dB_t, \quad R_t \neq 0. \tag{18}$$

Here B_t is the diffusion on the sphere whose generator is one half of the Beltrami-Laplace operator $\widetilde{\Delta}$. It is not difficult to see that if we denote the the first time when $R_t = 0$ by τ, then the sets of limit points of S_t as $t \uparrow \tau$ or $t \downarrow \tau$ coincide with the entire sphere S^2. For $t > 0$ the distribution of $S_{\tau+t}$ is uniform on the sphere.

Remark. If instead of assuming that $\lim_{T\to+\infty} \gamma(T)\sqrt{S(T)} = +\infty$ we assume that $S(T) = 0$, the result of Theorem 3.3 will hold with the only difference that the initial distribution for the limiting Markov process will now be concentrated at $x \lim_{T\to+\infty} \gamma(T)$, instead of being the invariant distribution, provided that the latter limit exists.

4 Distribution near and below the critical point

In this section we assume that $\gamma = \gamma(T)$ is such that $\gamma(T)\sqrt{T} \to \varkappa \in [-\infty, +\infty)$ as $T \to +\infty$. In particular, this covers the case when $\gamma < 0$ does not depend on T. We examine the behavior of the polymer paths with respect to $\overline{\mathrm{P}}^x_{\gamma(T),T}$ as $T \to +\infty$.

Consider the process $y^T(t) = \omega(tT)/\sqrt{T}$, $0 \leq t \leq 1$. Applying (10) with $a = 1/T$, we see that the distribution of the process y^T with respect to the measure $\overline{\mathrm{P}}^x_{\gamma(T),T}$ coincides with the distribution of the process $\omega(t)$, $t \in [0,1]$, with respect to the measure $\overline{\mathrm{P}}^{x/\sqrt{T}}_{\gamma(T)\sqrt{T},1}$. Since $x/\sqrt{T} \to 0$ and $\gamma(T)\sqrt{T} \to \varkappa$ as $T \to +\infty$, the following theorem is a consequence of (11).

Theorem 4.1. (Self-similarity near the critical point) *If $\gamma = \gamma(T)$ is such that $\gamma(T)\sqrt{T} \to \varkappa \in (-\infty, +\infty)$ as $T \to +\infty$, then the distribution of the process $y^T(t)$ with respect to the measure $\overline{\mathrm{P}}^x_{\gamma(T),T}$ converges as $T \to +\infty$ to the measure $\overline{\mathrm{P}}^0_{\varkappa,1}$.*

If $\gamma = \gamma(T)$ is such that $\gamma(T)\sqrt{T} \to -\infty$ as $T \to +\infty$, then the distribution of the process $y^T(t)$ with respect to the measure $\overline{\mathrm{P}}^x_{\gamma,T}$ converges as $T \to +\infty$ to the distribution of the 3-dimensional Brownian motion.

The finite-dimensional distributions of the limiting process can be written out using (6) and (8). In particular, when $\varkappa = 0$, the distribution of the limiting process is the same as the distribution of the corresponding process in the case of the smooth potential with $\beta = \beta_{cr}$ (see [9]).

Let us show that the limiting distribution of $y^T(T)$ is compound Gaussian.

Theorem 4.2. *The distribution of $\omega(1)$ (induced by the measure $\overline{\mathrm{P}}^0_{\varkappa,1}$) is compound Gaussian, i.e., its density is given by*

$$q_\varkappa(y) = \int_0^1 \frac{e^{-|y|^2/2\tau}}{(2\pi\tau)^{3/2}} v_\varkappa(\tau) d\tau,$$

where $v_\varkappa(\tau) > 0$, $\tau \in [0,1]$, $\varkappa \in \mathbb{R}$.

Proof. By (11), (7) and (8), the density of $\omega(1)$ with respect to the Lebesgue measure is equal to

$$q_\varkappa(y) = \lim_{|x|\downarrow 0} \frac{\overline{p}_\varkappa(1,x,y)}{\overline{Z}_\varkappa(1,x)} = \frac{1}{2\pi} \int_{\Gamma(a)} \frac{e^{-\sqrt{2\lambda}|y|+\lambda}}{(\sqrt{2\lambda}-\varkappa)|y|} d\lambda \left(\int_{\Gamma(a)} \frac{e^{\lambda} d\lambda}{(\sqrt{2\lambda}-\varkappa)\lambda} \right)^{-1},$$

where $a > \varkappa^2/2$. Note that

$$\frac{e^{-\sqrt{2\lambda}|y|}}{2\pi|y|} = \int_0^\infty \frac{e^{-|y|^2/2\tau}}{(2\pi\tau)^{3/2}} e^{-\lambda\tau} d\tau = \int_0^\infty \frac{d}{d\tau}\left(\frac{e^{-|y|^2/2\tau}}{(2\pi\tau)^{3/2}} \right) \frac{e^{-\lambda\tau}}{\lambda} d\tau,\ \ \lambda \in \Gamma(a).$$

We put this expression in the first integral in the previous formula. The integration by parts above was needed to guarantee the absolute convergence of the double integral and to change the order of integration. Below we will move the derivative back to the second factor. Thus

$$q_\varkappa(y) = \int_0^\infty \frac{d}{d\tau}\left(\frac{e^{-|y|^2/2\tau}}{(2\pi\tau)^{3/2}} \right) u_\varkappa(\tau) d\tau, \qquad u_\varkappa(\tau) = K \int_{\Gamma(a)} \frac{e^{\lambda(1-\tau)} d\lambda}{(\sqrt{2\lambda}-\varkappa)\lambda}, \tag{19}$$

where

$$K = \left(\int_{\Gamma(a)} \frac{e^{\lambda} d\lambda}{(\sqrt{2\lambda}-\varkappa)\lambda} \right)^{-1}.$$

If $\tau \geq 1$ and $\mathrm{Re}\lambda \geq a$, then the integrand in the second integral in (19) is analytic in λ and of order $O(|\lambda|^{-3/2})$ as $|\lambda| \to \infty$. Thus $u_\varkappa(\tau) = 0$ for $\tau \geq 1$, and

$$q_\varkappa(y) = \int_0^1 \frac{d}{d\tau}\left(\frac{e^{-|y|^2/2\tau}}{(2\pi\tau)^{3/2}} \right) u_\varkappa(\tau) d\tau = \int_0^1 \frac{e^{-|y|^2/2\tau}}{(2\pi\tau)^{3/2}} v_\varkappa(\tau) d\tau, \tag{20}$$

where

$$v_\varkappa(\tau) = -u'_\varkappa(\tau) = K\int_{\Gamma(a)} \frac{e^{\lambda(1-\tau)}d\lambda}{(\sqrt{2\lambda}-\varkappa)}, \qquad 0<\tau<1.$$

Since the integral over $\Gamma(a)$ is equal to the integral over a contour around negative λ-semiaxis (plus the contribution from the residue at $\varkappa^2/2$ if $\varkappa>0$), we have

$$\int_{\Gamma(a)} \frac{e^{\lambda(1-\tau)}d\lambda}{(\sqrt{2\lambda}-\varkappa)} = 2i\int_0^\infty \frac{\sqrt{2\sigma}e^{-\sigma(1-\tau)}d\sigma}{2\sigma+\varkappa^2} + \delta 2\pi i\varkappa e^{\varkappa^2(1-\tau)/2}, \qquad 0<\tau<1, \tag{21}$$

where $\delta=1$ if $\varkappa>0$, $\delta=0$ if $\varkappa\le 0$.

Let us evaluate K. We have

$$K^{-1} = \int_{\Gamma(a)} \frac{e^\lambda d\lambda}{(\sqrt{2\lambda}-\varkappa)\lambda} = \int_{\Gamma(a)} \frac{(\sqrt{2\lambda}+\varkappa)e^\lambda d\lambda}{(2\lambda-\varkappa^2)\lambda} = \int_{\Gamma(a)} \frac{\sqrt{2\lambda}e^\lambda d\lambda}{(2\lambda-\varkappa^2)\lambda} + \int_{\Gamma(a)} \frac{\varkappa e^\lambda d\lambda}{(2\lambda-\varkappa^2)\lambda}. \tag{22}$$

The first integral I_1 in the right-hand side can be written as the integral over the contour around the negative λ-semiaxis plus the contribution from the residue at $\varkappa^2/2$, i.e.

$$I_1 = -2\sqrt{2}i\int_0^\infty \frac{e^{-\sigma}d\sigma}{(2\sigma+\varkappa^2)\sqrt{\sigma}} + 2\pi i\frac{e^{\varkappa^2/2}}{|\varkappa|}.$$

We make the substitution $\sigma\to\sigma^2$ in the first term of the right-hand side and rewrite the second term using the identity

$$\frac{\pi}{|\varkappa|} = \int_0^\infty \frac{\sqrt{2}d\sigma}{\sigma^2+\varkappa^2/2}.$$

The second integral in the right-hand side of (22) can be expressed through the residues at $\lambda=0$ and $\lambda=\varkappa^2/2$. This implies, that

$$K^{-1} = 2\sqrt{2}i\int_0^\infty \frac{e^{\varkappa^2/2}-e^{-\sigma^2}}{\sigma^2+\varkappa^2/2}d\sigma + 2\pi i\frac{e^{\varkappa^2/2}-1}{\varkappa}.$$

This and (21) show that $v_\varkappa(\tau)>0$ for all $0\le\tau\le 1$, $-\infty<\varkappa<\infty$. This together with (20) justify the statement. □

5 Proof of Theorem 2.3. Tightness of certain families of processes

Proof of Theorem 2.3. As follows from (3) and (4), the finite-dimensional distributions of $\mathrm{P}^{x,\varepsilon}_{\gamma,T}$ are given by

$$\mathrm{P}^{x,\varepsilon}_{\gamma,T}(\omega(t_1)\in A_1,...,\omega(t_k)\in A_k) =$$

$$(\widetilde{Z}_\gamma^\varepsilon)^{-1}(T,x)\int_{A_1}...\int_{A_k}\int_{\mathbb{R}^3}\widetilde{p}_\gamma^\varepsilon(t_1,x,x_1)...\widetilde{p}_\gamma^\varepsilon(t_k-t_{k-1},x_{k-1},x_k)\widetilde{p}_\gamma^\varepsilon(T-t_k,x_k,y)dydx_k...dx_1,$$

where $\widetilde{p}_\gamma^\varepsilon(t,x,y)$ is the fundamental solution of the parabolic equation

$$\frac{\partial u}{\partial t}=\widetilde{L}_\gamma^\varepsilon u:=\frac{1}{2}\Delta u+v_\gamma^\varepsilon u$$

and

$$\widetilde{Z}_\gamma^\varepsilon(t,x)=\int_{\mathbb{R}^3}\widetilde{p}_\gamma^\varepsilon(t,x,y)dy.$$

Since $\overline{\mathrm{P}}^x_{\gamma,T}(\omega(t_i)=0\ \text{ for some } 1\le i\le k)=0$, in order to demonstrate the convergence of the finite-dimensional distributions, it is sufficient to prove that $\lim_{\varepsilon\downarrow 0}\widetilde{Z}_\gamma^\varepsilon(t,x)=\overline{Z}_\gamma(t,x)$ and that for continuous functions with compact supports $\varphi_1,...,\varphi_k$ such that $\mathrm{supp}(\varphi_i)\subset\mathbb{R}^3\setminus 0$, $1\le i\le k$, we have

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^3}...\int_{\mathbb{R}^3}\widetilde{p}_\gamma^\varepsilon(t_1,x,x_1)...\widetilde{p}_\gamma^\varepsilon(t_k-t_{k-1},x_{k-1},x_k)\varphi_1(x_1)...\varphi_k(x_k)dx_k...dx_1= \tag{23}$$

$$\int_{\mathbb{R}^3}...\int_{\mathbb{R}^3}\overline{p}_\gamma(t_1,x,x_1)...\overline{p}_\gamma(t_k-t_{k-1},x_{k-1},x_k)\varphi_1(x_1)...\varphi_k(x_k)dx_k...dx_1.$$

Let us start by showing that for each $k\in(0,1)$ and continuous function φ with compact support we have

$$\lim_{\varepsilon\downarrow 0}||v_\varepsilon-v||_{C(F\times[k,1/k])}=0, \tag{24}$$

where

$$v_\varepsilon=\int_{\mathbb{R}^3}\widetilde{p}_\gamma^\varepsilon(t,x,y)\varphi(y)dy,\quad v=\int_{\mathbb{R}^3}\overline{p}_\gamma(t,x,y)\varphi(y)dy,\quad F=\{x\in\mathbb{R}^3:|x|\in[k,1/k]\}.$$

Consider

$$u_{\lambda,\varepsilon}=(\mathcal{L}_\gamma^\varepsilon-\lambda)^{-1}\varphi,\quad u_\lambda=(\mathcal{L}_\gamma-\lambda)^{-1}\varphi,\quad \lambda\notin\mathrm{spec}(\mathcal{L}_\gamma).$$

It is shown in [1] that $\mathcal{L}_\gamma^\varepsilon$ converges to $\mathcal{L}_\gamma$ in the norm resolvent sense when $\lambda\notin\mathrm{spec}(\mathcal{L}_\gamma)$. Thus

$$||u_{\lambda,\varepsilon}-u_\lambda||_{L^2(\mathbb{R}^3)}\to 0\ \text{ as }\ \varepsilon\downarrow 0,\quad \lambda\notin\mathrm{spec}(\mathcal{L}_\gamma). \tag{25}$$

Since there is a neighborhood of F (that does not include the origin) where

$$(\frac{\Delta}{2}-\lambda)u_{\lambda,\varepsilon}=(\frac{\Delta}{2}-\lambda)u_\lambda=0,\quad 0<\varepsilon\le\varepsilon_0, \tag{26}$$

when ε_0 is small enough, from standard a priori estimates for elliptic equations and (25) it follows that

$$||u_{\lambda,\varepsilon}-u_\lambda||_{C(F)}\to 0\ \text{ as }\ \varepsilon\downarrow 0,\quad \lambda\notin\mathrm{spec}(\mathcal{L}_\gamma). \tag{27}$$

We have

$$v_\varepsilon = \frac{1}{2\pi i}\int_{\Gamma(a)} (\mathcal{L}^\varepsilon_\gamma - \lambda)^{-1}\varphi e^{\lambda t} d\lambda, \quad v = \frac{1}{2\pi i}\int_{\Gamma(a)} (\mathcal{L}_\gamma - \lambda)^{-1}\varphi e^{\lambda t} d\lambda. \tag{28}$$

Since the norm of the resolvent of an operator does not exceed the distance from the spectrum, we have

$$||u_{\lambda,\varepsilon}||_{L^2(\mathbb{R}^3)}, \quad ||u_\lambda||_{L^2(\mathbb{R}^3)} \le c|\lambda|^{-1}, \quad \lambda \in \Omega_a, \quad 0 < \varepsilon \le \varepsilon_0, \tag{29}$$

where Ω_a is the region in the complex λ-plane which is to the right of the two rays starting at $\lambda = a$ and forming $\pi/4$ angles with the the ray $[a, -\infty)$. From here it follows that the contour $\Gamma(a)$ in (28) can be replaced by $\Gamma'(a) = \partial\Omega_a$:

$$v_\varepsilon = \frac{1}{2\pi i}\int_{\Gamma'(a)} u_{\lambda,\varepsilon} e^{\lambda t} d\lambda, \quad v = \frac{1}{2\pi i}\int_{\Gamma'(a)} u_\lambda e^{\lambda t} d\lambda. \tag{30}$$

From (26) and (29) it follows that

$$||u_{\lambda,\varepsilon}||_{H^2(F)}, \quad ||u_\lambda||_{H^2(F)} \le c', \quad \lambda \in \Gamma'(a), \quad 0 < \varepsilon \le \varepsilon_0, \tag{31}$$

for some constant c', and the Sobolev embedding theorem implies that

$$||u_{\lambda,\varepsilon}||_{C(F)}, \quad ||u_\lambda||_{C(F)} \le c'', \quad \lambda \in \Gamma'(a), \quad 0 < \varepsilon \le \varepsilon_0. \tag{32}$$

This allows us to split the contour of integration in the integrals in (30) into two parts: a part where $|\lambda| >> 1$ and a bounded contour. The contribution to v_ε and v from the first part can be made arbitrarily small. The Lebesgue dominated convergence theorem implies that the difference of the integrals over the bounded part of the contour tends to zero. Thus we have (24).

It is easy to see that (24) implies (23). Let us now prove that $\lim_{\varepsilon\downarrow 0} \widetilde{Z}^\varepsilon_\gamma(t,x) = \overline{Z}_\gamma(t,x)$. Let ξ be an infinitely smooth function that is equal to zero for $|x| \le 1$ and equal to one for $|x| \ge 2$. Let

$$\widetilde{Z}^\varepsilon_\gamma = \widetilde{Z}^{\varepsilon,1}_\gamma + \widetilde{Z}^{\varepsilon,2}_\gamma, \quad \overline{Z}_\gamma = \overline{Z}^1_\gamma + \overline{Z}^2_\gamma,$$

where

$$\widetilde{Z}^{\varepsilon,1}_\gamma(t,x) = \int_{\mathbb{R}^3} \widetilde{p}^\varepsilon_\gamma(t,x,y)\xi(y)dy, \quad \widetilde{Z}^{\varepsilon,2}_\gamma(t,x) = \int_{\mathbb{R}^3} \widetilde{p}^\varepsilon_\gamma(t,x,y)(1-\xi(y))dy,$$

and $\overline{Z}^1_\gamma$, $\overline{Z}^2_\gamma$ are defined similarly. Then $\lim_{\varepsilon\downarrow 0} \widetilde{Z}^{\varepsilon,2}_\gamma(t,x) = \overline{Z}^2_\gamma(t,x)$ by (24) with $\varphi = 1-\xi$.

In order to estimate $|\widetilde{Z}^{\varepsilon,1}_\gamma(t,x) - \overline{Z}^1_\gamma(t,x)|$, we consider the following problem:

$$\frac{\partial u}{\partial t} - \mathcal{L}^\varepsilon_\gamma u = h, \quad u(0,x) = \xi(x), \tag{33}$$

where $h \in C_0^\infty(\mathbb{R}^3)$ depends on x only. The solution of this equation has the form

$$u = \widetilde{Z}_\gamma^{\varepsilon,1} - \frac{1}{2\pi i}\int_{\Gamma'(a)} \lambda^{-1}(\mathcal{L}_\gamma^\varepsilon - \lambda)^{-1} h e^{\lambda t} d\lambda.$$

Note that $u(t,x) = \xi(x)$ satisfies (33) with $h = -\Delta\xi/2$. Thus

$$\xi = \widetilde{Z}_\gamma^{\varepsilon,1} + \frac{1}{4\pi i}\int_{\Gamma'(a)} \lambda^{-1}(\mathcal{L}_\gamma^\varepsilon - \lambda)^{-1}(\Delta\xi) e^{\lambda t} d\lambda.$$

The same formula is valid with $\overline{Z}_\gamma^1$ instead of $\widetilde{Z}_\gamma^{\varepsilon,1}$ and $\mathcal{L}_\gamma$ instead of $\mathcal{L}_\gamma^\varepsilon$. Therefore,

$$|\widetilde{Z}_\gamma^{\varepsilon,1}(t,x) - \overline{Z}_\gamma^1(t,x)| = \frac{1}{4\pi}\left|\int_{\Gamma'(a)} \lambda^{-1}(\mathcal{L}_\gamma^\varepsilon - \lambda)^{-1}(\Delta\xi) e^{\lambda t} d\lambda - \int_{\Gamma'(a)} \lambda^{-1}(\mathcal{L}_\gamma - \lambda)^{-1}(\Delta\xi) e^{\lambda t} d\lambda\right|.$$

Note that $\Delta\xi$ has compact support, and therefore we can treat the right hand side in this formula in the same way as we treated the difference between v_ε and v thus obtaining that $\lim_{\varepsilon\downarrow 0} \widetilde{Z}_\gamma^{\varepsilon,1}(t,x) = \overline{Z}_\gamma^1(t,x)$.

It remains to note that the family of measures $\mathrm{P}_{\gamma,T}^{x,\varepsilon}$, $0 < \varepsilon \leq 1$, is tight, as follows from Lemma 5.1. □

Remark. It follows from the arguments above that for any $c, t > 0$, there is a constant C such that $\widetilde{Z}_\gamma^\varepsilon(t,x) \leq C$ provided that $\varepsilon \leq c$, $1/c \leq |x| \leq c$ and $\gamma \leq c$.

Next we prove two statements that are needed for the proofs of Theorems 2.3 and 3.3. Note that we already demonstrated the convergence of partition functions and finite-dimensional distributions in Theorem 2.3, and using these facts in the proofs of Lemmas 5.1 and 5.2 does not involve a circular argument.

Lemma 5.1. *Let v_γ^ε be given by (5), the corresponding Hamiltonian $H_{\gamma,T}^\varepsilon$ given by (1) with v_γ^ε instead of v, and the Gibbs measure $\mathrm{P}_{\gamma,T}^{x,\varepsilon}$ given by (2) with $\beta = 1$ and $H_{\gamma,T}^\varepsilon$ instead of H_T. For each $T > 0$ and $c \in \mathbb{R}$, the family of measures $\{\mathrm{P}_{\gamma,T}^{x,\varepsilon}, \varepsilon \leq c, \gamma \leq c, |x| \leq c\}$ is tight.*

Proof. To prove tightness it is enough to demonstrate that for each $\eta, \alpha > 0$ there is $0 < \delta < 1$ such that for all $u \in [0,T]$ we have

$$\mathrm{P}_{\gamma,T}^{x,\varepsilon}\left(\sup_{u \leq s \leq \min(u+\delta,T)} |\omega(s) - \omega(u)| > \alpha\right) \leq \delta\eta. \tag{34}$$

Let $\eta, \alpha > 0$ be fixed. If $\varepsilon_0 > 0$, the density of the measure $\mathrm{P}_{\gamma,T}^{x,\varepsilon}$ with respect to the measure induced by the Brownian motion starting at x is bounded from above uniformly in $\varepsilon_0 \leq \varepsilon \leq c$, $\gamma \leq c$, $|x| \leq c$. Thus we can find $0 < \delta < 1$ such that (34) holds for $\varepsilon_0 \leq \varepsilon \leq c$. Therefore, it is sufficient to prove (34) under the assumption that ε is sufficiently small so that $v_\gamma^\varepsilon(x) = 0$ for $|x| \geq \alpha/8$.

Consider the following events in $C([0,T],\mathbb{R}^3)$,

$$\widetilde{\mathcal{E}}_\delta^T = \{\omega : \sup_{u \le s \le \min(u+\delta,T)} |\omega(s)-\omega(u)| \ge \alpha\}, \quad \mathcal{E}_\delta^T = \{\omega : \sup_{s \le \min(\delta,T)} |\omega(s)-\omega(0)| \ge \alpha/8\}.$$

For a continuous function $\omega : [0,T] \to \mathbb{R}^3$, let

$$\tau = \min(T, \inf\{t \ge 0 : |\omega(t)| \ge \alpha/4\}).$$

Then

$$\mathrm{P}_{\gamma,T}^{x,\varepsilon}(\widetilde{\mathcal{E}}_\delta^T) = (\widetilde{Z}_\gamma^\varepsilon(T,x))^{-1}\mathrm{E}_T^x(\exp(\int_0^T v_\gamma^\varepsilon(\omega(t))dt)\chi_{\widetilde{\mathcal{E}}_\delta^T}) \le$$

$$(\widetilde{Z}_\gamma^\varepsilon(T,x))^{-1}\mathrm{E}_T^x\left(\exp(\int_0^\tau v_\gamma^\varepsilon(\omega(t))dt)\mathrm{E}_{T-\tau}^{\omega(\tau)}(\chi_{\mathcal{E}_\delta^\tau}\exp(\int_0^{T-\tau} v_\gamma^\varepsilon(\omega(t))dt))\right),$$

where E_T^x denotes the expectation with respect to the measure induced by the Brownian motion starting at the point x. Since

$$\mathrm{E}_T^x\exp(\int_0^\tau v_\gamma^\varepsilon(\omega(t))dt) \le \widetilde{Z}_\gamma^\varepsilon(T,x)$$

and

$$\mathrm{E}_{T-\tau}^{\omega(\tau)}(\chi_{\mathcal{E}_\delta^\tau}\exp(\int_0^{T-\tau} v_\gamma^\varepsilon(\omega(t))dt)) \le \sup_{\alpha/4 \le |x| \le c}\mathrm{E}_T^x(\chi_{\mathcal{E}_\delta^T}\exp(\int_0^T v_\gamma^\varepsilon(\omega(t))dt)),$$

we have

$$\mathrm{P}_{\gamma,T}^{x,\varepsilon}(\widetilde{\mathcal{E}}_\delta^T) \le \sup_{\alpha/4 \le |x| \le c}\mathrm{E}_T^x(\chi_{\mathcal{E}_\delta^T}\exp(\int_0^T v_\gamma^\varepsilon(\omega(t))dt)).$$

Let

$$\sigma = \min(\delta, \inf\{t \ge 0 : |\omega(t)-\omega(0)| = \alpha/8\}).$$

Then, since $v_\gamma^\varepsilon(x) = 0$ for $|x| \ge \alpha/8$,

$$\sup_{\alpha/4 \le |x| \le c}\mathrm{E}_T^x(\chi_{\mathcal{E}_\delta^T}\exp(\int_0^T v_\gamma^\varepsilon(\omega(t))dt)) \le \sup_{\alpha/4 \le |x| \le c}\mathrm{E}_T^x(\chi_{\mathcal{E}_\delta^T}\mathrm{E}_{T-\sigma}^{\omega(\sigma)}\exp(\int_0^{T-\sigma} v_\gamma^\varepsilon(\omega(t))dt)).$$

Note that

$$\mathrm{E}_{T-\sigma}^{\omega(\sigma)}\exp(\int_0^{T-\sigma} v_\gamma^\varepsilon(\omega(t))dt) \le \sup_{\alpha/8 \le |x| \le c+\alpha/8}\mathrm{E}_T^x\exp(\int_0^T v_\gamma^\varepsilon(\omega(t))dt) =$$

$$\sup_{\alpha/8 \le |x| \le c+\alpha/8}\widetilde{Z}_\gamma^\varepsilon(T,x) \le c(\alpha)$$

for some constant $c(\alpha)$, where the last inequality is due to the Remark following the proof of Theorem 2.3. It then remains to choose δ such that $\mathrm{E}_T^x(\chi_{\mathcal{E}_\delta^T}) \le \delta\eta/c(\alpha)$, and estimate (34) follows. □

Lemma 5.2. *Suppose that $\gamma(T)$ is bounded and such that $\gamma(T)\sqrt{T} \to +\infty$ as $T \to +\infty$. Suppose that $s > 0$, $S(T)$ satisfies (16) and T_0 is such that $S(T) + s/\gamma^2(T) \leq T$ for $T \geq T_0$. Let the process $y^T(t)$, $0 \leq t \leq s$, be defined on the probability space $C([0,T],\mathbb{R}^3)$ with the measure $\overline{\mathrm{P}}^x_{\gamma(T),T}$ by $y^T(t) = \gamma(T)\omega(S(T) + t/\gamma^2(T))$, $0 \leq t \leq s$. Then, for each $K > 0$, the family of measures on $C([0,s],\mathbb{R}^3)$ induced by the processes y^T with $T \geq T_0$ and $0 < |x| \leq K$ is tight.*

Proof. Due to self-similarity, it is sufficient to consider the case when $\gamma(T) = 1$. First observe that for any $\eta > 0$ there is $a > 0$ such that

$$\overline{\mathrm{P}}^x_{1,T}(|y^T(0)| > a) \leq \eta$$

for $T \geq T_0$ and $0 < |x| \leq K$. This easily follows from Lemma 3.1 in the same way that the convergence of finite-dimensional distributions was demonstrated in Theorem 3.3.

For a continuous function $\omega : [0,T] \to \mathbb{R}^3$, let

$$m^T(\omega,\delta) = \sup_{|t_1-t_2|\leq\delta,\ S(T)\leq t_1,t_2\leq S(T)+s} |\omega(t_1) - \omega(t_2)|.$$

To prove tightness, it is sufficient to show that for each $\alpha, \eta > 0$, there is $\delta > 0$ such that

$$\overline{\mathrm{P}}^x_{1,T}(m^T(\omega,\delta) > \alpha) \leq \eta \tag{35}$$

for $T \geq T_0$ and $0 < |x| \leq K$. By Theorem 2.3,

$$\overline{\mathrm{P}}^x_{1,T}(m^T(\omega,\delta) > \alpha) \leq \liminf_{\varepsilon\downarrow 0} \mathrm{P}^{x,\varepsilon}_{1,T}(m^T(\omega,\delta) > \alpha). \tag{36}$$

For a continuous function $\omega : [0,T] \to \mathbb{R}^3$, let

$$\tau = \min(S(T) + s, \inf\{t \geq S(T) : |\omega(t)| \geq \alpha/2\}).$$

For $0 \leq a \leq b \leq r$, let

$$\mathcal{E}^r_{a,b} = \{\omega \in C([0,r],\mathbb{R}^3) : \sup_{|t_1-t_2|\leq\delta,\ a\leq t_1,t_2\leq b} |\omega(t_1) - \omega(t_2)| > \alpha\}.$$

Then

$$\liminf_{\varepsilon\downarrow 0} \mathrm{P}^{x,\varepsilon}_{1,T}(m^T(\omega,\delta) > \alpha) = (\widetilde{Z}^\varepsilon_\gamma(T,x))^{-1}\mathrm{E}^x_T(\exp(\int_0^T v^\varepsilon_\gamma(\omega(t))dt)\chi_{\mathcal{E}^T_{S(T),S(T)+s}}) \leq$$

$$(\widetilde{Z}^\varepsilon_\gamma(T,x))^{-1}\mathrm{E}^x_T\left(\exp(\int_0^\tau v^\varepsilon_\gamma(\omega(t))dt)\mathrm{E}^{\omega(\tau)}_{T-\tau}(\exp(\int_0^{T-\tau} v^\varepsilon_\gamma(\omega(t))dt)\chi_{\mathcal{E}^{T-\tau}_{0,S(T)+s-\tau}})\right) \leq \tag{37}$$

$$(\widetilde{Z}^\varepsilon_\gamma(T,x))^{-1}(\widetilde{Z}^\varepsilon_\gamma(S(T)+s,x)) \sup_{|x|\geq\alpha/2} \mathrm{E}^x_{T-S(T)}(\exp(\int_0^{T-S(T)} v^\varepsilon_\gamma(\omega(t))dt)\chi_{\mathcal{E}^{T-S(T)}_{0,s}}).$$

Since $v^\varepsilon_\gamma(x) = 0$ when $x \geq \alpha/4$, provided that ε is small enough, we have by the Markov property

$$\sup_{|x|\geq \alpha/2} \mathrm{E}^x_{T-S(T)}(\exp(\int_0^{T-S(T)} v^\varepsilon_\gamma(\omega(t))dt)\chi_{\mathcal{E}^{T-S(T)}_{0,s}}) \leq \tag{38}$$

$$\sup_{|x|\geq\alpha/4} \widetilde{Z}^\varepsilon_\gamma(T-S(T),x)\mathrm{P}^0_s(\{\omega : \sup_{|t_1-t_2|\leq\delta,\ 0\leq t_1,t_2\leq s} |\omega(t_1)-\omega(t_2)| > \alpha\}).$$

Recall from the proof of Theorem 2.3 that $\lim_{\varepsilon\downarrow 0} \widetilde{Z}^\varepsilon_\gamma(t,x) = \overline{Z}_\gamma(t,x)$. Therefore, combining (36), (37) and (38) we see that

$$\overline{\mathrm{P}}^x_{1,T}(m^T(\omega,\delta) > \alpha) \leq (\overline{Z}_\gamma(T,x))^{-1}(\overline{Z}_\gamma(S(T)+s,x))\times$$

$$\sup_{|x|\geq\alpha/4} \overline{Z}_\gamma(T-S(T),x)\mathrm{P}^0_s(\{\omega : \sup_{|t_1-t_2|\leq\delta,\ 0\leq t_1,t_2\leq s} |\omega(t_1)-\omega(t_2)| > \alpha\}).$$

Formula (35) now follows from Lemma 3.1. □

Acknowledgements: While working on this article, M. Cranston was supported by NSF grant DMS-0706198, L. Koralov was supported by NSF grant DMS-0706974, S. Molchanov and B. Vainberg were supported by NSF grant DMS-0706928.

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