Math 405 Take-home exam

Spring, 2016 due: **Tuesday, May 10 at 12:30pm** Laskowski

Ground rules:

- This exam is open book, open notes; but you are expected to do this exam by yourself and without access to electronic sources (e.g., no internet searches, no Wolfram Alpha, etc.) This is on the **Honor system**.
- Solving a problem and writing it up are two separate enterprises! Once you solve a question, spend some time preparing a nice, cogent writeup of the argument. For problems with a computational component, all steps must be shown. You may want to write solutions to the six problems on different pages.
- Each of the six problems is worth 25 points.

1. Let Let $A = \begin{pmatrix} 0 & 4 & 4 \\ 1 & 1 & 0 \\ -2 & 3 & 4 \end{pmatrix}$.
   (a) Find a matrix B in Jordan normal form that is similar to A.
   (b) Find an invertible matrix S such that $B = S^{-1}AS$.
2. Let $T : V \to V$ be a linear transformation with characteristic polynomial $h(x) = (x-2)^2(x-3)^6$ and minimal polynomial $m(x) = (x-2)(x-3)^3$. List all possible sets of elementary divisors of T, and for each such set, give a matrix in Jordan canonical form having that set of elementary divisors.
3. Let V be a finite dimensional vector space over a field F, and let $T : V \to V$ be a linear transformation. Say that T *maps bases to bases* if, for every basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ of V, the set $\{T(v_1), \ldots, T(v_n)\}$ is also a basis of V.

   Prove that T is invertible if and only if T maps bases to bases.

4. Recall that a linear transformation $T : V \to V$ is *idempotent* if $T^2 = T$.

   (a) Prove that if $T : V \to V$ is idempotent, then $V = U \oplus W$, where $U = \{v \in V : T(v) = v\}$ and $W = \{v \in V : T(v) = 0\}$.

   (b) Find all linear transformations $T : V \to V$ that are both idempotent and invertible.

5. Let $f, g \in F[x]$ be relatively prime polynomials, let $T : V \to V$ be a linear transformation, and let $w \in V$ be a non-zero vector. Prove that if $f(T)(w) = 0$, then $g(T)(w) \neq 0$.

6. Suppose V is an n-dimensional vector space, $T : V \to V$ is a linear transformation with characteristic polynomial

$$h(x) = (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$$

   with $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ distinct. Let $S : V \to V$ be any linear transformation that commutes with T, i.e., $ST = TS$. Prove that S is diagonalizable.