Midterm Exam

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Name: ______________________

1. (10 points each) Let S be the subspace of $\mathbb{R}^4$ generated by the three vectors

$$\{\langle 1,2,0,2\rangle, \langle 2,0,1,1\rangle, \langle 1,-2,1,-1\rangle\}$$

(a) Find a basis for S.

ANS: Row reduce, or note by inspection that the three vectors are linearly dependent (second row=sum of first and third). So, e.g., any two of these vectors form a basis. [As would any two non-zero vectors after row reducing.]

(b) Find an orthonormal basis for S with respect to the usual inner product on $\mathbb{R}^4$ (i.e., dot product).

ANS: Start with $\{v_1, v_2\} = \{\langle 1,2,0,2\rangle, \langle 2,0,1,1\rangle\}$. Let $u_1 := \frac{v_1}{||v_1||} = (1/3)\langle 1,2,0,2\rangle$. Let $w_2 = v_2 - proj_{u_1} v_2 = (v_2, u_1)v_2 = (1/9)\langle 14,-8,9,1\rangle$ and let $u_2 = \frac{w_2}{||w_2||} = (1/27\sqrt{38})\langle 14,-8,9,1\rangle$.

2. **True/False** (10 points each) For each statement, give a brief justification as to why it is True or False.

_______________ a) $A = \{f : f(0) = 1\}$ is a subspace of $\mathcal{F}(\mathbb{R})$, the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$.

FALSE – e.g., the zero function is not in A.

_______________ b) $B = \{f : f(1) = 0\}$ is a subspace of $\mathcal{F}(\mathbb{R})$, the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$.

TRUE – If $f, g \in B$, then $(f+g)(1) = f(1) + g(1) = 0$, so $(f+g) \in B$. Also, if $\alpha \in \mathbb{R}$, then $(\alpha f)(1) = \alpha f(1) = 0$. Thus, B is a subspace.

_______________ c) There is a set of 7 linearly independent vectors in $M_3(\mathbb{R})$, the vector space of all 3×3 matrices with coefficients in $\mathbb{R}$.

TRUE – $M_3(\mathbb{R})$ has dimension 9, so e.g., any 7 elements of a basis will be linearly independent.

_______________ d) The polynomial x^2 is a linear combination of $\{2, 3x, x(x-1)\}$ in $P_2(\mathbb{R})$, the vector space of polynomials of degree at most 2.

TRUE – $P_2(\mathbb{R})$ has dimension 3, and it is readily checked that the three functions are linearly independent. Thus, every element of $P_2(\mathbb{R})$ is a linear combination of these three.

_______________ e) The solution set of $5x_1 + 2x_2 - x_4 = 3$ is a 3-dimensional linear manifold of $\mathbb{R}^4$.

TRUE – The corresponding homogeneous equation $5x_1 + 2x_2 - x_4 = 0$ describes a subspace S of dimension 3. By superposition, the solution set of the given equation has the form $p + S$, where p is any particular solution. This is what it means to be a 3-dimensional linear manifold.

3. (10 points each) Let $P_2(\mathbb{R})$ denote the vector space of polynomials of degree at most 2. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by:

$$T(a_0 + a_1x + a_2x^2) = 3a_2x + a_1$$

(a) Prove that T is a linear transformation.

ANS: Directly check that for every $p, q \in P_2(\mathbb{R})$, $T(p+q) = T(p) + T(q)$ and that $T(\alpha p) = \alpha T(p)$ for $\alpha \in \mathbb{R}$.

(b) Find the matrix of T with respect to the standard basis $\mathcal{B} = \{1, x, x^2\}$.

ANS: $T(1) = 0$, $T(x) = 1$ and $T(x^2) = 3x$. So $\Phi_{\mathcal{B}}$ is a 3×3 matrix

$$\Phi_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) Prove that $\mathcal{C} = \{5, 2x-1, x(x-1)\}$ is a basis for $P_2(\mathbb{R})$.

ANS: Say $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_0(5) + \lambda_1(2x-1) + \lambda_2(x^2 - x)$ is the zero vector, i.e., the identically zero function. Then, by looking at the x^2 terms, $\lambda_2 = 0$. Once we have this, then the x terms give $\lambda_1 = 0$, and thus $\lambda_0 = 0$ by the constant terms. So $\mathcal{C}$ is linearly independent. As $\dim(P_2(\mathbb{R})) = 3$, $\mathcal{C}$ is a basis.

(d) Find the change of basis matrix $M_{\mathcal{B}}^{\mathcal{C}}$ that transforms $\mathcal{C}$-coefficients of a vector to $\mathcal{B}$-coefficients of the same vector.

Since $5 = 5(1) + 0x + 0x^2$, $2x-1 = (-1)(1) + 2x + 0x^2$, and $x(x-1) = 0(1) + (-1)x + 1x^2$,

$$M_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} 5 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

4. (20 points) Let $T : V \to V$ be a linear transformation and let T^2 denote the composition $T \circ T$ i.e., $T^2(v) = T(T(v))$. Prove that:

$$Null(T^2) = Null(T) \qquad \text{if and only if} \qquad T(V) \cap Null(T) = \{0\}.$$

PROOF: First, assume $Null(T^2) = Null(T)$. Choose any vector $w \in T(V) \cap Null(T)$. Since $w \in T(V)$, choose $u \in V$ such that $T(u) = w$. Since $w \in Null(T)$, $T(T(u)) = T(w) = 0$. That is, $u \in Null(T^2)$. Since $Null(T) = Null(T^2)$, $u \in Null(T)$. Thus, $w = T(u) = 0$. Thus, $T(V) \cap Null(T) = \{0\}$.

Conversely, assume $T(V) \cap Null(T) = \{0\}$. We must show $Null(T^2) = Null(T)$. One direction is obvious – For any linear transformation T, $T(0) = 0$. Thus, $Null(T) \subseteq Null(T^2)$ for any linear transformation $T : V \to V$. For the other direction, choose any $u \in Null(T^2)$. Let $w = T(u)$. Visibly, $w \in T(V)$. But also, since $u \in Null(T^2)$, $T(w) = T(T(u)) = 0$. That is, $w \in Null(T)$. By hypothesis, $w = 0$. That is, $T(u) = 0$, so $u \in Null(T)$.

5. (a) (10 points) Suppose that $T : V \to W$ is a linear transformation that is 1-1. Prove that if $\{v_1, \ldots, v_n\} \subseteq V$ is linearly independent, then $\{T(v_1), \ldots, T(v_n)\} \subseteq W$ is linearly independent as well.

PROOF: Choose any $\lambda_1 \ldots, \lambda_n$ such that $\lambda_1 T(v_1) + \ldots \lambda_n T(v_n) = 0$. As T is a linear transformation,

$$T(\lambda_1 v_1 + \ldots \lambda_n v_n) = \lambda_1 T(v_1) + \ldots \lambda_n T(v_n) = 0$$

Since T is 1-1, this implies that $\lambda_1 v_1 + \ldots \lambda_n v_n = 0$. But, since $\{v_1, \ldots, v_n\}$ are linearly independent, it follows that $\lambda_1 = \ldots = \lambda_n = 0$. Thus, $\{T(v_1), \ldots, T(v_n)\} \subseteq W$ is linearly independent.

(b) (10 points) Let V and W be two finite dimensional vector spaces over the same field F. Prove that there is a 1-1 linear transformation $T : V \to W$ if and only if $\dim(V) \leq \dim(W)$. [Hint: You may want to use (a) above.]

PROOF: First, assume there is a 1-1 linear transformation $T : V \to W$. Let $n = \dim(V)$ and $m = \dim(W)$. Let $\{v_1, \ldots, v_n\}$ be a basis for V. By (a), $\{T(v_1), \ldots, T(v_n)\}$ is a linearly independent set of size n. As we know there cannot be a linearly independent set in W of size more than $\dim(W)$, it follows that $n \leq \dim(W)$.

Conversely, assume that $\dim(V) \leq \dim(W)$. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ be bases for V and W, respectively. Define a $T : V \to W$ to be the unique linear transformation satisfying $T(v_i) = w_i$ for $1 \leq i \leq n$. As $Null(T) = \{0\}$, T is 1-1. [In fact, T is an isomorphism between V and the subspace $S(w_1, \ldots, w_n)$ of W.]