Most(?) theories have Borel complete reducts

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September 20, 2021

Abstract

We prove that many seemingly simple theories have Borel complete reducts. Specifically, if a countable theory has uncountably many complete $1$-types, then it has a Borel complete reduct. Similarly, if $Th(M)$ is not small, then $M^{eq}$ has a Borel complete reduct, and if a theory $T$ is not $\omega$-stable, then the elementary diagram of some countable model of $T$ has a Borel complete reduct.

1 Introduction

In their seminal paper [1], Friedman and Stanley define and develop a notion of Borel reducibility among classes of structures with universe $\omega$ in a fixed, countable language $L$ that are Borel and invariant under permutations of $\omega$. It is well known (see e.g., [3] or [2]) that such classes are of the form $\text{Mod}(\Phi)$, the set of models of $\Phi$ whose universe is precisely $\omega$ for some sentence $\Phi \in L_{\omega,1}$, but here we concentrate on first-order, countable theories $T$. For countable theories $T, S$ in possibly different language, a Borel reduction is a Borel function $f : \text{Mod}(T) \to \text{Mod}(S)$ that satisfies $M \cong N$ if and only if $f(M) \cong f(N)$. One says that $T$ is Borel reducible to $S$ if there is a Borel reduction $f : \text{Mod}(T) \to \text{Mod}(S)$. As Borel reducibility is transitive, this induces a quasi-order on the class of all countable theories, where we say $T$ and $S$ are Borel equivalent if there are Borel reductions in both directions. In [1], Friedman and Stanley show that among Borel invariant classes (hence among countable first-order theories) there is a maximal class with respect to $\leq_B$.

*Both authors partially supported by NSF grant DMS-1855789.
We say $\Phi$ is \textit{Borel complete} if it is in this maximal class. Examples include the theories of graphs, linear orders, groups, and fields.

The intuition is that Borel complexity of a theory $T$ is related to the complexity of invariants that describe the isomorphism types of countable models of $T$. Given an $L$-structure $M$, one naturally thinks of the reducts $M_0$ of $M$ to be ‘simpler objects’ hence the invariants for a reduct ‘should’ be no more complicated than for the original $M$, but we will see that this intuition is incorrect. As a paradigm, let $T$ be the theory of ‘independent unary predicates’ i.e., $T = Th(2^\omega, U_n)$, where each $U_n$ is a unary predicate interpreted as $U_n = \{ \eta \in 2^\omega : \eta(n) = 1 \}$. The countable models of $T$ are rather easy to describe. The isomorphism type of a model is specified by which countable, dense subset of ‘branches’ is realized, and how many elements realize each of those branches. However, with Theorem 3.2, we will see that $T$ has a Borel complete reduct.

To be precise about reducts, we have the following definition.

\textbf{Definition 1.1.} Given an $L$-structure $M$, a \textit{reduct} $M'$ of $M$ is an $L'$-structure with the same universe as $M$, and for which the interpretation every atomic $L'$-formula $\alpha(x_1, \ldots, x_k)$ is an $L$-definable subset of $M^k$ (without parameters). An $L'$-theory $T'$ is a \textit{reduct of an $L$-theory} $T$ if $T' = Th(M')$ for some reduct $M'$ of some model $M$ of $T$.

In the above definition, it would be equivalent to require that the interpretation in $M'$ of every $L'$-formula $\theta(x_1, \ldots, x_k)$ is a 0-definable subset of $M^k$.

\section{An engine for Borel completeness results}

This section is devoted to proving Borel completeness for a specific family of theories. All of the theories $T_h$, are in the same language $L = \{ E_n : n \in \omega \}$ and are indexed by strictly increasing functions $h : \omega \rightarrow \omega \setminus \{ 0 \}$. For a specific choice of $h$, the theory $T_h$ asserts that

- Each $E_n$ is an equivalence relation with exactly $h(n)$ classes; and
- The $E_n$’s cross-cut, i.e., for all nonempty, finite $F \subseteq \omega$, $E_F(x, y) := \bigwedge_{n \in F} E_n(x, y)$ is an equivalence relation with precisely $\prod_{n \in F} h(n)$ classes.

It is well known that each of these theories $T_h$ is complete and admits elimination of quantifiers. Thus, in any model of $T_h$, there is a unique 1-type. However, the strong type structure is complicated.\footnote{Recall that in any structure $M$, two elements $a, b$ have the same \textit{strong type}, $stp(a) = stp(b)$, if $M \models E(a, b)$ for every 0-definable equivalence relation. Because of the quantifier elimination, in any model $M \models T_h$, $stp(a) = stp(b)$ if and only if $M \models E_n(a, b)$ for every $n \in \omega$.} So much so, that the whole of this section is devoted to the proof of:
**Theorem 2.1.** For any strictly increasing \( h : \omega \to \omega \setminus \{0\} \), \( T_h \) is Borel complete.

**Proof.** Fix a strictly increasing function \( h : \omega \to \omega \setminus \{0\} \). We begin by describing representatives \( \mathcal{B} \) of the strong types and a group \( G \) that acts faithfully and transitively on \( \mathcal{B} \). As notation, for each \( n \), let \( [h(n)] \) denote the \( h(n) \)-element set \( \{1, \ldots, h(n)\} \) and let \( \text{Sym}([h(n)]) \) be the (finite) group of permutations of \( [h(n)] \). Let

\[
\mathcal{B} = \{ f : \omega \to \omega : f(n) \in [h(n)] \text{ for all } n \in \omega \}
\]

and let \( G = \Pi_{n \in \omega} \text{Sym}([h(n)]) \) be the direct product. As notation, for each \( n \in \omega \), let \( \pi_n : G \to \text{Sym}([h(n)]) \) be the natural projection map. Note that \( G \) acts coordinate-wise on \( \mathcal{B} \) by: For \( g \in G \) and \( f \in \mathcal{B} \), \( g \cdot f \) is the element of \( \mathcal{B} \) satisfying \( g \cdot f(n) = \pi_n(g)(f(n)) \).

Define an equivalence relation \( \sim \) on \( \mathcal{B} \) by:

\[
f \sim f' \text{ if and only if } \{ n \in \omega : f(n) \neq f'(n) \} \text{ is finite.}
\]

For \( f \in \mathcal{B} \), let \([f]\) denote the \( \sim \)-class of \( f \) and, abusing notation somewhat, for \( W \subseteq \mathcal{B} \)

\[
[W] := \bigcup\{ [f] : f \in W \}.
\]

Observe that for every \( g \in G \), the permutation of \( \mathcal{B} \) induced by the action of \( g \) maps \( \sim \)-classes onto \( \sim \)-classes, i.e., \( G \) also acts transitively on \( \mathcal{B}/\sim \).

We first identify a countable family of \( \sim \)-classes that are ‘sufficiently indiscernible’. Our first lemma is where we use the fact that the function \( h \) defining \( T_h \) is strictly increasing.

**Lemma 2.2.** There is a countable set \( Y = \{ f_i : i \in \omega \} \subseteq \mathcal{B} \) such that whenever \( i \neq j \), \( \{ n \in \omega : f_i(n) = f_j(n) \} \) is finite.

**Proof.** We recursively construct \( Y \) in \( \omega \) steps. Suppose \( \{ f_i : i < k \} \) have been chosen. Choose an integer \( N \) large enough so that \( h(N) > k \) (hence \( h(n) > k \) for all \( n \geq N \)). Now, construct \( f_k \in \mathcal{B} \) to satisfy \( f_k(n) \neq f_i(n) \) for all \( n \geq N \) and all \( i < k \).

Fix an enumeration \( \langle f_i : i \in \omega \rangle \) of \( Y \) for the whole of the argument. The ‘indiscernibility’ of \( Y \) alluded to above is formalized by the following definition and lemma.

**Definition 2.3.** Given a permutation \( \sigma \in \text{Sym}(\omega) \), a group element \( g \in G \) respects \( \sigma \) if \( g \cdot [f_i] = [f_{\sigma(i)}] \) for every \( i \in \omega \).

**Lemma 2.4.** For every permutation \( \sigma \in \text{Sym}(\omega) \), there is some \( g \in G \) respecting \( \sigma \).
Proof. Note that since \( h \) is increasing, \( h(n) \geq n \) for every \( n \in \omega \). Fix a permutation \( \sigma \in \text{Sym}(\omega) \) and we will define some \( g \in G \) respecting \( \sigma \) coordinate-wise. Using Lemma 2.2, choose a sequence
\[
0 = N_0 < N_1 < N_2 < \ldots
\]
of integers such that for all \( i \in \omega \), both \( f_i(n) \neq f_j(n) \) and \( f_{\sigma(i)}(n) \neq f_{\sigma(j)}(n) \) hold for all \( n \geq N_i \) and all \( j < i \).

Since \( \{N_i\} \) are increasing, it follows that for each \( i \in \omega \) and all \( n \geq N_i \), the subsets \( \{f_j(n) : j \leq i\} \) and \( \{f_{\sigma(j)}(n) : j \leq i\} \) of \( [h(n)] \) each have precisely \((i + 1)\) elements. Thus, for each \( i < \omega \) and for each \( n \geq N_i \), there is a permutation \( \delta_n \in \text{Sym}([h(n)]) \) satisfying
\[
\bigwedge_{j \leq i} \delta_n(f_j(n)) = f_{\sigma(j)}(n)
\]
[Simply begin defining \( \delta_n \) to meet these constraints, and then complete \( \delta_n \) to a permutation of \( [h(n)] \) arbitrarily.] Using this, define \( g := \langle \delta_n : n \in \omega \rangle \), where each \( \delta_n \in \text{Sym}([h(n)]) \) is constructed as above. To see that \( g \) respects \( \sigma \), note that for every \( i \in \omega \), \( (g \cdot f_i)(n) = f_{\sigma(i)}(n) \) for all \( n \geq N_i \), so \( (g \cdot f_i) \sim f_{\sigma(i)} \). \( \square \)

Definition 2.5. For distinct integers \( i \neq j \), let \( d_{i,j} \in B \) be defined by:

\[
d_{i,j}(n) := \begin{cases} f_i(n) & \text{if } n \text{ even;} \\ f_j(n) & \text{if } n \text{ odd.} \end{cases}
\]

Let \( Z := \{d_{i,j} : i \neq j\} \).

Note that \( d_{i,j} \not\in f_k \) for all distinct \( i, j \) and all \( k \in \omega \), hence \( \{f_i : i \in \omega \} \) and \( \{d_{i,j} : i \neq j\} \) are disjoint.

Lemma 2.6. For all \( \sigma \in \text{Sym}(\omega) \), if \( g \in G \) respects \( \sigma \), then \( g \cdot [d_{i,j}] = [d_{\sigma(i),\sigma(j)}] \) for all \( i \neq j \).

Proof. Choose \( \sigma \in \text{Sym}(\omega) \), \( g \) respecting \( \sigma \), and \( i \neq j \). Choose \( N \) such that \( (g \cdot [f_i])(n) = [f_{\sigma(i)}](n) \) and \( (g \cdot [f_j])(n) = [f_{\sigma(j)}](n) \) for every \( n \geq N \). Since \( d_{i,j}(n) = f_i(n) \) for \( n \geq N \) even,
\[
(g \cdot d_{i,j})(n) = \pi_n(g)(d_{i,j}(n)) = \pi_n(g)(f_i(n)) = (g \cdot f_i)(n) = f_{\sigma(i)}(n)
\]
Dually, \( (g \cdot d_{i,j})(n) = f_{\sigma(j)}(n) \) when \( n \geq N \) is odd, so \( (g \cdot d_{i,j}) \sim d_{\sigma(i),\sigma(j)} \). \( \square \)
With the combinatorial preliminaries out of the way, we now prove that \( T_h \) is Borel complete. We form a highly homogeneous model \( M^* \models T_h \) and thereafter, all models we consider will be countable, elementary substructures of \( M^* \). Let \( A = \{a_f : f \in B\} \) and \( B = \{b_f : f \in B\} \) be disjoint sets and let \( M^* \) be the \( L \)-structure with universe \( A \cup B \) and each \( E_n \) interpreted by the rules:

- For all \( f \in B \) and \( n \in \omega \), \( E_n(a_f,b_f) \); and
- For all \( f, f' \in B \) and \( n \in \omega \), \( E_n(a_f,a_{f'}) \iff f(n) = f'(n) \).

with the other instances of \( E_n \) following by symmetry and transitivity. For any finite \( F \subseteq \omega \), \( \{f \upharpoonright F : f \in B\} \) has exactly \( \Pi_{n \in F} h(n) \) elements, hence \( E_F(x,y) := \bigwedge_{n \in F} E_n(x,y) \) has \( \Pi_{n \in F} h(n) \) classes in \( M^* \). Thus, the \( \{E_n : n \in \omega\} \) cross cut and \( M^* \models T_h \).

Let \( E_\infty(x,y) \) denote the (type definable) equivalence relation \( \bigwedge_{n \in \omega} E_n(x,y) \). Then, in \( M^* \), \( E_\infty \) partitions \( M^* \) into 2-element classes \( \{a_f,b_f\} \), indexed by \( f \in B \). Note also that every \( g \in G \) induces an \( L \)-automorphism \( g^* \in \text{Aut}(M^*) \) by

\[
g^*(x) := \begin{cases} a_{(g \cdot f)} & \text{if } x = a_f \text{ for some } f \in B \\ b_{(g \cdot f)} & \text{if } x = b_f \text{ for some } f \in B \end{cases}
\]

Recall the set \( Y = \{f_i : i \in \omega\} \) from Lemma 2.2, so \( [Y] = \{[f_i] : i \in \omega\} \). Let \( M_0 \subseteq M^* \) be the substructure with universe \( \{a_f : f \in [Y]\} \). As \( T_h \) admits elimination of quantifiers and as \([Y]\) is dense in \( B \), \( M_0 \preceq M^* \). Moreover, every substructure \( M \) of \( M^* \) with universe containing \( M_0 \) will also be an elementary substructure of \( M^* \), hence a model of \( T_h \).

To show that \( \text{Mod}(T_h) \) is Borel complete, we define a Borel mapping from \{irreflexive graphs \( G = (\omega,R) \)\} to \( \text{Mod}(T_h) \) as follows: Given \( G \), let \( Z(R) := \{d_{i,j} \in Z : G \models R(i,j)\} \), so \([Z(R)] = \bigcup \{[d_{i,j}] : d_{i,j} \in Z(R)\} \). Let \( M_G \preceq M^* \) be the substructure with universe \( M_0 \cup \{a_d,b_d : d \in [Z(R)]\} \)

That the map \( G \mapsto M_G \) is Borel is routine, given that \( Y \) and \( Z \) are fixed throughout.

Note that in \( M_G \), every \( E_\infty \)-class has has either one or two elements. Specifically, for each \( d \in [Z(R)] \), the \( E_\infty \)-class \([a_d]_\infty = \{a_d,b_d\} \), while the \( E_\infty \)-class \([a_f]_\infty = \{a_f\} \) for every \( f \in [Y] \).

We must show that for any two graphs \( G = (\omega,R) \) and \( H = (\omega,S) \), \( G \) and \( H \) are isomorphic if and only if the \( L \)-structures \( M_G \) and \( M_H \) are isomorphic.

To verify this, first choose a graph isomorphism \( \sigma : (\omega,R) \rightarrow (\omega,S) \). Then \( \sigma \in \text{Sym}(\omega) \) and, for distinct integers \( i \neq j \), \( d_{i,j} \in Z(R) \) if and only if \( d_{\sigma(i),\sigma(j)} \in Z(S) \). Apply Lemma 2.4 to get \( g \in G \) respecting \( \sigma \) and let \( g^* \in \text{Aut}(M^*) \) be the \( L \)-automorphism.
induced by $g$. By Lemma 2.6 and Definition 2.3, it is easily checked that the restriction of $g^*$ to $M_G$ is an $L$-isomorphism between $M_G$ and $M_H$.

Conversely, assume that $\Psi : M_G \to M_H$ is an $L$-isomorphism. Clearly, $\Psi$ maps $E_\infty$-classes in $M_G$ to $E_\infty$-classes in $M_H$. In particular, $\Psi$ permutes the 1-element $E_\infty$-classes $\{\{a_f\} : f \in [Y]\}$ of both $M_G$ and $M_H$, and maps the 2-element $E_\infty$-classes $\{\{a_{d_i}, b_{d_i}\} : d \in [Z(S)]\}$ of $M_G$ onto the 2-element $E_\infty$-classes $\{\{a_{d_i}, b_{d_i}\} : d \in [Z(S)]\}$ of $M_H$. That is, $\Psi$ induces a bijection $F : [Y \sqcup Z(R)] \to [Y \sqcup Z(S)]$ that permutes $[Y]$.

As well, by the interpretations of the $E_n$'s, for $f, f' \in [Y \sqcup Z(R)]$ and $n \in \omega$,

$$f(n) = f'(n) \text{ if and only if } F(f)(n) = F(f')(n).$$

From this it follows that $F$ maps $\sim$-classes onto $\sim$-classes. As $F$ permutes $[Y]$ and as $[Y] = \bigcup\{[f_i] : i \in \omega\}$, $F$ induces a permutation $\sigma \in Sym(\omega)$ given by $\sigma(i)$ is the unique $i^* \in \omega$ such that $F([f_i]) = [f_{i^*}]$.

We claim that this $\sigma$ induces a graph isomorphism between $G = (\omega, R)$ and $H = (\omega, S)$. Indeed, choose any $(i,j) \in R$. Thus, $d_{i,j} \in Z(R)$. As $F$ is $\sim$-preserving, choose $N$ large enough so that $F(f_i)(n) = F(f_{\sigma(i)})(n)$ and $F(f_j)(n) = F(f_{\sigma(j)})(n)$ for every $n \geq N$. By definition of $d_{i,j}$, $d_{i,j}(n) = f_i(n)$ for $n \geq N$ even, so $F(d_{i,j})(n) = F(f_i)(n) = f_{\sigma(i)}(n)$ for such $n$. Dually, for $n \geq N$ odd, $F(d_{i,j})(n) = F(f_j)(n) = f_{\sigma(j)}(n)$. Hence, $F(d_{i,j}) \sim d_{\sigma(i),\sigma(j)} \in [Z(S)]$. Thus, $(\sigma(i), \sigma(j)) \in S$. The converse direction is symmetric (i.e., use $\Psi^{-1}$ in place of $\Psi$ and run the same argument).

**Remark 2.7.** If we relax the assumption that $h : \omega \to \omega \setminus \{0\}$ is strictly increasing, there are two cases. If $h$ is unbounded, then the proof given above can easily be modified to show that the associated $T_h$ is also Borel complete. Conversely, with Theorem 6.2 of [6] the authors prove that if $h : \omega \to \omega \setminus \{0\}$ is bounded, then $T_h$ is not Borel complete. The salient distinction between the two cases is that when $h$ is bounded, the associated group $G$ has bounded exponent. However, even in the bounded case $T_h$ has a Borel complete reduct by Lemma 3.1 below.

### 3 Applications to reducts

We begin with one easy lemma that, when considering reducts, obviates the need for the number of classes to be strictly increasing.

**Lemma 3.1.** Let $L = \{E_n : n \in \omega\}$ and let $f : \omega \to \omega \setminus \{0, 1\}$ be any function. Then every model $M$ of $T_f$, the complete theory asserting that each $E_n$ is an equivalence relation with $f(n)$ classes, and that the $\{E_n\}$ cross-cut, has a Borel complete reduct.
Proof. Given any function \( f : \omega \to \omega \setminus \{0, 1\} \), choose a partition \( \omega = \bigsqcup \{F_n : n \in \omega\} \) into non-empty finite sets for which \( \Pi_{k \in F_n} f(k) < \Pi_{k \in F_m} f(k) \) whenever \( n < m < \omega \). For each \( n \), let \( h(n) := \Pi_{k \in F_n} f(k) \) and let \( E^*_n(x, y) := \bigwedge_{k \in F_n} E_k(x, y) \). Then, as \( h \) is strictly increasing and \( \{E^*_n\} \) is a cross-cutting set of equivalence relations with each \( E^*_n \) having \( h(n) \) classes.

Now let \( M \models T_f \) be arbitrary and let \( L' = \{E^*_n : n \in \omega\} \). As each \( E^*_n \) described above is 0-definable in \( M \), there is an \( L' \)-reduct \( M' \) of \( M \). It follows from Theorem 2.1 that \( T' = Th(M') \) is Borel complete, so \( T_f \) has a Borel complete reduct. \( \square \)

**Theorem 3.2.** Suppose \( T \) is a complete theory in a countable language with uncountably many 1-types. Then every model \( M \) of \( T \) has a Borel complete reduct.

**Proof.** Let \( M \models T \) be arbitrary. As usual, by the Cantor-Bendixon analysis of the compact, Hausdorff Stone space \( S_1(T) \) of complete 1-types, choose a set \( \{\varphi_\eta(x) : \eta \in 2^{<\omega}\} \) of 0-definable formulas, indexed by the tree \( (2^{<\omega}, \leq) \) ordered by initial segment, satisfying:

1. \( M \models \exists x \varphi_\eta(x) \) for each \( \eta \in 2^{<\omega} \);
2. For \( \nu \leq \eta \), \( M \models \forall x (\varphi_\eta(x) \to \varphi_\nu(x)) \);
3. For each \( n \in \omega \), \( \{\varphi_\eta(x) : \eta \in 2^n\} \) are pairwise contradictory.

By increasing these formulas slightly, we can additionally require

4. For each \( n \in \omega \), \( M \models \forall x (\bigvee_{\eta \in 2^n} \varphi_\eta(x)) \).

Given such a tree of formulas, for each \( n \in \omega \), define

\[
\delta^0_n(x) := \bigwedge_{\eta \in 2^n} [\varphi_\eta(x) \to \varphi^0_n(x)] \quad \text{and} \quad \delta^1_n(x) := \bigwedge_{\eta \in 2^n} [\varphi_\eta(x) \to \varphi^1_n(x)]
\]

Because of (4) above, \( M \models \forall x (\delta^0_n(x) \lor \delta^1_n(x)) \) for each \( n \). Also, for each \( n \), let

\[
E_n(x, y) := [\delta^0_n(x) \leftrightarrow \delta^0_n(y)]
\]

From the above, each \( E_n \) is a 0-definable equivalence relation with precisely two classes.

**Claim.** The equivalence relations \( \{E_n : n \in \omega\} \) are cross-cutting.

**Proof.** It suffices to prove that for every \( m > 0 \), the equivalence relation \( E^*_m(x, y) := \bigwedge_{n<m} E_n(x, y) \) has \( 2^m \) classes. So fix \( m \) and choose a subset \( A_m = \{a_\eta : \eta \in 2^m\} \subseteq M \) forming a set of representatives for the formulas \( \{\varphi_\eta(x) : \eta \in 2^m\} \). It suffices to show that \( M \models \neg E^*_m(a_\eta, a_\nu) \) whenever \( \eta \neq \nu \) are from \( 2^m \). But this is clear. Fix distinct \( \eta \neq \nu \) and choose any \( k < m \) such that \( \eta(k) \neq \nu(k) \). Then \( M \models \neg E_k(a_\eta, a_\nu) \), hence \( M \models \neg E^*_m(a_\eta, a_\nu) \). \( \square \)
Thus, taking the 0-definable relations \( \{ E_n \} \), \( M \) has a reduct that is a model of \( T_f \) (where \( f \) is the constant function 2). As reducts of reducts are reducts, it follows from Lemma 3.1 and Theorem 2.1 that \( M \) has a Borel complete reduct.

We highlight how unexpected Theorem 3.2 is with two examples. First, the theory of ‘Independent unary predicates’ mentioned in the Introduction has a Borel complete reduct.

Next, we explore the assumption that a countable, complete theory \( T \) is not small, i.e., for some \( k \) there are uncountably many \( k \)-types. We conjecture that some model of \( T \) has a Borel complete reduct. If \( k = 1 \), then by Theorem 3.2, every model of \( T \) has a Borel complete reduct. If \( k > 1 \) is least, then it is easily seen that there is some complete \((k-1)\) type \( p(x_1, \ldots, x_{k-1}) \) with uncountably many complete \( q(x_1, \ldots, x_k) \) extending \( p \). Thus, if \( M \) is any model of \( T \) realizing \( p \), say by \( \bar{a} = (a_1, \ldots, a_{k-1}) \), the expansion \((M, a_1, \ldots, a_{k-1})\) has a Borel complete reduct, also by Theorem 3.2. Similarly, we have the following result.

**Corollary 3.3.** Suppose \( T \) is a complete theory in a countable language that is not small. Then for any model \( M \) of \( T \), \( M^{eq} \) has a Borel complete reduct.

**Proof.** Let \( M \) be any model of \( T \) and choose \( k \) least such that \( T \) has uncountably many complete \( k \)-types consistent with it. In the language \( L^{eq} \), there is a sort \( U_k \) and a definable bijection \( f : M^k \rightarrow U_k \). Hence \( Th(M^{eq}) \) has uncountably many 1-types consistent with it, each extending \( U_k \). Thus, \( M^{eq} \) has a Borel complete reduct by Theorem 3.2.

Finally, recall that a countable, complete theory is not \( \omega \)-stable if, for some countable model \( M \) of \( T \), the Stone space \( S_1(M) \) is uncountable. From this, we immediately obtain our final corollary.

**Corollary 3.4.** If a countable, complete \( T \) is not \( \omega \)-stable, then for some countable model \( M \) of \( T \), the elementary diagram of \( M \) in the language \( L(M) = L \cup \{ c_m : m \in M \} \) has a Borel complete reduct.

**Proof.** Choose a countable \( M \) so that \( S_1(M) \) is uncountable. Then, in the language \( L(M) \), the theory of the expanded structure \( M^1 \) in the language \( L(M) \) has uncountably many 1-types, hence it has a Borel complete reduct by Theorem 3.2.

The results above are by no means characterizations. Indeed, there are many Borel complete \( \omega \)-stable theories. In [5], the first author and Shelah prove that any \( \omega \)-stable theory that has eni-DOP or is eni-deep is not only Borel complete, but also \( \lambda \)-Borel complete for all \( \lambda \).\(^2\) As well, there are \( \omega \)-stable theories with only countably many countable models.

\(^2\)Definitions of eni-DOP and eni-deep are given in Definitions 2.3 and 6.2, respectively, of [5], and the definition of \( \lambda \)-Borel complete is recalled in Section 4 of this paper.
that have Borel complete reducts. To illustrate this, we introduce three interrelated theories. The first, \( T_0 \) in the language \( L_0 = \{ U, V, W, R \} \) is the paradigmatic DOP theory. \( T_0 \) asserts that:

- \( U, V, W \) partition the universe;
- \( R \subseteq U \times V \times W \);
- \( T_0 \models \forall x \forall y \exists z R(x, y, z) \); [more formally, for each \( n \), \( T_0 \models \forall x \forall y \exists^n z R(x, y, z) \);]
- \( T_0 \models \forall x \forall y \forall y' \forall z [R(x, y, z) \land R(x', y', z) \rightarrow (x = x' \land y = y')] \).

\( T_0 \) is both \( \omega \)-stable and \( \omega \)-categorical and its unique countable model is rather tame. The complexity of \( T_0 \) is only witnessed with uncountable models, where one can code arbitrary bipartite graphs in an uncountable model \( M \) by choosing the cardinalities of the sets \( R(a, b, M) \) among \( (a, b) \in U \times V \) to be either \( \aleph_0 \) or \( |M| \).

To get bad behavior of countable models, we expand \( T_0 \) to an \( L^* = L_0 \cup \{ f_n : n \in \omega \} \)-theory \( T \supseteq T_0 \) that additionally asserts:

- Each \( f_n : U \times V \rightarrow W \);
- \( \forall x \forall y R(x, y, f_n(x, y)) \) for each \( n \); and
- for distinct \( n \neq m \), \( \forall x \forall y (f_n(x, y) \neq f_m(x, y)) \).

This \( T \) is \( \omega \)-stable with eni-DOP and hence is Borel complete by Theorem 4.12 of [5]. However, \( T \) has an expansion \( T^* \) in a language \( L^* := L \cup \{ c, d, g, h \} \) whose models are much better behaved. Let \( T^* \) additionally assert:

- \( U(c) \land V(d) \);
- \( g : U \rightarrow V \) is a bijection with \( g(c) = d \);
- Letting \( W^* := \{ z : R(c, d, z) \} \), \( h : U \times V \times W^* \rightarrow W \) is an injective map that is the identity on \( W^* \) and, for each \( (x, y) \in U \times V \), maps \( W^* \) onto \( \{ z \in W : R(x, y, z) \} \); and moreover
  - \( h \) commutes with each \( f_n \), i.e., \( \forall x \forall y (h(x, y, f_n(c, d)) = f_n(x, y)) \).

Then \( T^* \) is \( \omega \)-stable and two-dimensional (the dimensions being \( |U| \) and \( |W^* \setminus \{ f_n(c, d) : n \in \omega \} | \) ), hence \( T^* \) has only countably many countable models. However, \( T^* \) visibly has a Borel complete reduct, namely \( T \).
4 Observations about the theories $T_h$

In addition to their utility in proving Borel complete reducts, the theories $T_h$ in Section 2 illustrate some novel behaviors. First off, model theoretically, these theories are extremely simple. More precisely, each theory $T_h$ is weakly minimal with the geometry of every strong type trivial (such theories are known as mutually algebraic in [4]). Additionally, the theories $T_h$ are the simplest known examples of theories that are Borel complete, but not $\lambda$-Borel complete for all cardinals $\lambda$. For any infinite cardinal, $\lambda$-Borel completeness was introduced in [5]. Instead of looking at $L$-structures with universe $\omega$, we consider $X^\lambda_L$, the set of $L$-structures with universe $\lambda$. We topologize $X^\lambda_L$ analogously; namely a basis consists of all sets $U\varphi(\alpha_1,\ldots,\alpha_n) := \{M \in X^\lambda_L : M \models \varphi(\alpha_1,\ldots,\alpha_n)\}$ for all $L$-formulas $\varphi(x_1,\ldots,x_n)$ and all $(\alpha_1,\ldots,\alpha_n) \in \lambda^n$. Define a subset of $X^\lambda_L$ to be $\lambda$-Borel if it is the smallest $\lambda^+$-algebra containing the basic open sets, and call a function $f : X^\lambda_{L_1} \to X^\lambda_{L_2}$ to be $\lambda$-Borel if the inverse image of every basic open set is $\lambda$-Borel. For $T,S$ theories in languages $L_1,L_2$, respectively we say that $\text{Mod}_\lambda(T)$ is $\lambda$-Borel reducible to $\text{Mod}_\lambda(S)$ if there is a $\lambda$-Borel $f : \text{Mod}_\lambda(T) \to \text{Mod}_\lambda(S)$ preserving back-and-forth equivalence in both directions (i.e., $M \equiv_{\infty,\omega} N \iff f(M) \equiv_{\infty,\omega} f(N)$).

As back-and-forth equivalence is the same as isomorphism for countable structures, $\lambda$-Borel reducibility when $\lambda = \omega$ is identical to Borel reducibility. As before, for any infinite $\lambda$, there is a maximal class under $\lambda$-Borel reducibility, and we say a theory is $\lambda$-Borel complete if it is in this maximal class. All of the ‘classical’ Borel complete theories, e.g., graphs, linear orders, groups, and fields, are $\lambda$-Borel complete for all $\lambda$. However, the theories $T_h$ are not.

Lemma 4.1. If $T$ is mutually algebraic in a countable language, then there are at most $\beth_2$ pairwise $\equiv_{\infty,\omega}$-inequivalent models (of any size).

Proof. We show that every model $M$ has an $(\infty,\omega)$-elementary substructure of size $2^{|\text{Dom}(M)|}$, which suffices. So, fix $M$ and choose an arbitrary countable $M_0 \trianglelefteq M$. By Proposition 4.4 of [4], $M \setminus M_0$ can be decomposed into countable components, and any permutation of isomorphic components induces an automorphism of $M$ fixing $M_0$ pointwise. As there are at most $2^{|\text{Dom}(M)|}$ non-isomorphic components over $M_0$, choose a substructure $N \subseteq M$ containing $M_0$ and, for each isomorphism type of a component, $N$ contains either all of copies in $M$ (if there are only finitely many) or else precisely $\aleph_0$ copies if $M$ contains infinitely many copies. It is easily checked that $N \preceq_{\infty,\omega} M$.

Corollary 4.2. No mutually algebraic theory $T$ in a countable language is $\lambda$-Borel complete for $\lambda \geq \beth_2$. In particular, $T_h$ is Borel complete, but not $\lambda$-Borel complete for large $\lambda$. 

Proof. Fix $\lambda \geq \beth_2$. It is readily checked that there is a family of $2^\lambda$ graphs that are pairwise not back and forth equivalent. As there are fewer than $2^\lambda \equiv \beth_\infty \omega$-classes of models of $T$, there cannot be a $\lambda$-Borel reduction of graphs into $\text{Mod}_\lambda(T)$. \qed

In [8], another example of a Borel complete theory that is not $\lambda$-Borel complete for all $\lambda$ is given (it is dubbed $TK$ there) but the $T_h$ examples are cleaner. In order to understand this behavior, in [8] we call a theory $T$ *grounded* if every potential canonical Scott sentence $\sigma$ of a model of $T$ (i.e., in some forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$, $\sigma$ is a canonical Scott sentence of some model, then $\sigma$ is a canonical Scott sentence of a model in $\mathbb{V}$). Proposition 5.1 of [8] proves that every theory of refining equivalence relations is grounded. By contrast, we have

**Proposition 4.3.** If $T$ is Borel complete with a cardinal bound on the number of $\equiv_{\beth_\infty \omega}$-classes of models, then $T$ is not grounded. In particular, $T_h$ is not grounded.

Proof. Let $\kappa$ denote the number of $\equiv_{\beth_\infty \omega}$-classes of models of $T$. If $T$ were grounded, then $\kappa$ would also bound the number of potential canonical Scott sentences. As the class of graphs has a proper class of potential canonical Scott sentences, it would follow from Theorem 3.10 of [8] that $T$ could not be Borel complete. \qed

**References**


