

Characterizing the existence of a Borel complete expansion

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Abstract

We develop general machinery to cast the class of potential canonical Scott sentences of an infinitary sentence Φ as a class of structures in a related language. From this, we show that Φ has a Borel complete expansion if and only if S_∞ divides $\text{Aut}(M)$ for some countable model $M \models \Phi$. From this, we prove that for theories T_h asserting that $\{E_n\}$ is a countable family of cross cutting equivalence relations with $h(n)$ classes, if $h(n)$ is uniformly bounded then T_h is not Borel complete, providing a converse to Theorem 2.1 of [12].

1 Introduction

Back and forth systems are an invaluable tool in understanding the descriptive set theoretic complexity of a class of models. Indeed, within the realm of countable structures, back and forth equivalence is the same as isomorphism. As every structure can be made countable in a forcing extension $\mathbb{V}[G]$ of the set-theoretic universe \mathbb{V} , two arbitrary structures in the same vocabulary are back and forth equivalent if and only if they are potentially isomorphic, i.e., they can become isomorphic in some forcing extension.

Classically, given a countable structure M , there is a ‘preferred back and forth system’ $\mathcal{F}_\infty := \{(\bar{a}, \bar{b}) \in M^{2n} : (M, \bar{a}) \equiv_{\infty, \omega} (M, \bar{b})\}$ and the data of (M, \mathcal{F}_∞) is coded into a *Scott sentence* $\varphi_M \in L_{\omega_1, \omega}$, whose countable models N are precisely those isomorphic to

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M . This assignment can be done canonically, giving a canonical object $\text{css}(M)$. Similarly, a structure M of any size has an associated canonical Scott sentence $\text{css}(M)$, which is a sentence of $L_{\infty, \omega}$ that describes the back-and-forth class of M . Extending on this correspondence, given a sentence $\Phi \in L_{\omega_1, \omega}$, we can look at $\text{CSS}(\Phi)_{\text{sat}} := \{\text{css}(M) : M \models \Phi\}$, which may be a proper class, but is bijective with the class $\text{Mod}(\Phi) / \equiv_{\infty, \omega}$. However, for applications, a possibly larger class $\text{CSS}(\Phi)_{\text{ptl}}$ is useful. This class consists of sentences $\psi \in \mathbb{V}$, $\psi \in L_{\infty, \omega}$, such that in every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$, if $(\psi \in L_{\omega_1, \omega})^{\mathbb{V}[G]}$, then $\psi = \text{css}(N)$ for some $N \models \Phi$ in $\mathbb{V}[G]$. In [16] sentences in $\text{CSS}(\Phi)_{\text{ptl}}$ are the *potential canonical Scott sentences* of models of Φ .

The number of potential canonical Scott sentences is relevant to the complexity of $(\text{Mod}_{\omega}(\Phi), \cong)$, the set of models of Φ with universe ω , modulo isomorphism. In [16] it is proved that if there is a Borel reduction $(\text{Mod}_{\omega}(\Phi), \cong) \leq_B (\text{Mod}_{\omega}(\Psi), \cong)$, then $|\text{CSS}(\Phi)_{\text{ptl}}| \leq |\text{CSS}(\Psi)_{\text{ptl}}|$ (where we interpret these ‘cardinalities’ to be ∞ if they are proper classes). This is quite useful, although in the above formulation the elements of $\text{CSS}(\Phi)_{\text{ptl}}$ are simply sentences of $L_{\infty, \omega}$ and are hard to manipulate.

There are two parts to this paper. In the first, we obtain a natural bijection between $\text{CSS}(\Phi)_{\text{ptl}}$ and a class of structures in an associated language L^b . In Section 2 we begin by introducing ‘sharp’ back and forth systems and structures and describe a canonical *flattening* operation that associates an L^b -structure $\mathbb{B} = (M, \mathcal{F})^b$ to each sharp structure (M, \mathcal{F}) . We show that the flattening operation transforms (infinitary) L -formulas φ into L^b -formulas φ^b in a natural way. Using this, we distinguish the class of *Hausdorff* flat structures that correspond to flattenings of the ‘preferred’ sharp structures $(M, \mathcal{F}_{\infty})$. In Section 3 we give a list $\text{Ax}^b(L)$ of infinitary L^b -sentences that axiomatize these flattenings. We show that for a countable flat structure \mathbb{B} , we can recover a sharp structure (M, \mathcal{F}) whose flattening is isomorphic to \mathbb{B} . We show that the class of Hausdorff flat structures is absolute between forcing extensions and we define a canonical set of representatives Hdf^* of the L^b -isomorphism types of Hausdorff flat structures. We see that this class Hdf^* is naturally bijective with $\text{CSS}(L)_{\text{ptl}}$, so we propose to take Hdf^* as an improved representation of potential canonical Scott sentences. All of this relativizes to models of a given Φ , so $\text{Hdf}^*(\Phi)$ is naturally bijective with $\text{CSS}(\Phi)_{\text{ptl}}$.

The second part of the paper exploits the fact that the class $\text{Flat}(\Phi)$ of flat models of Φ^b is axiomatized by the sentence $\Phi^b \wedge \text{Ax}^b(L) \in (L^b)_{\omega_1, \omega}$. Non-Hausdorff flat models correspond to flattenings of sharp structures (M, \mathcal{F}) where \mathcal{F} is something other than \mathcal{F}_{∞} , which in turn codes the structure of an expansion M^* of M . In Section 4 we make this correspondence precise and then in Section 5, by applying Morley’s theorem to the $(L^b)_{\omega_1, \omega}$ -sentence $\Phi^b \wedge \text{Ax}^b(L)$, we see that the existence of a Borel complete expansion of a model of Φ is equivalent to Hjorth’s notion of ‘ S_{∞} divides $\text{Aut}(M)$ ’ for some countable

$M \models \Phi$.¹ This connection is indirect, in that we prove that each of these properties is equivalent to $\Phi^b \wedge \text{Ax}^b(L)$ having arbitrarily large models, which in turn is equivalent to this infinitary theory having Ehrenfeucht-Mostowski models.

Theorem 1.1. *The following are equivalent for a sentence $\Phi \in L_{\omega_1, \omega}$.*

1. Φ has a Borel complete expansion (i.e., there is some countable $L' \supseteq L$ and $\psi \in (L')_{\omega_1, \omega}$ such that $\psi \vdash \varphi$ and ψ is Borel complete);
2. $\Phi^b \wedge \text{Ax}^b(L)$ has arbitrarily large models;
3. $\Phi^b \wedge \text{Ax}^b(L)$ admits Ehrenfeucht-Mostowski models;
4. S_∞ divides $\text{Aut}(M)$ for some countable $M \models \Phi$.

We glean two consequences from Theorem 1.1. First, we get a distinction between the Borel complexity of first order theories and sentences of $L_{\omega_1, \omega}$. In particular, it follows that every first order theory with an infinite model has a Borel complete expansion, whereas there are sentences of $L_{\omega_1, \omega}$ (even complete ones) that do not. Second, we use Theorem 1.1 to prove that the theory of cross-cutting equivalence relations T_h in the language $\{E_n : n \in \omega\}$ with a uniform, finite bound on the number of E_n -classes is not Borel complete. This is a converse to Theorem 2.1 of [12], where it is proved that the analogous theory with unboundedly many classes is Borel complete.

1.1 Preliminaries

Following [16], let HC denote the set of hereditarily countable sets. We begin by defining a class of formulas whose restrictions to HC are well behaved.

Definition 1.2. Suppose $\varphi(x)$ is any formula of set theory, possibly with a hidden parameter from HC .

- $\varphi(HC) = \{a \in HC : (HC, \in) \models \varphi(a)\}$ and if $\mathbb{V}[G]$ is a forcing extension of \mathbb{V} , then $\varphi(HC)^{\mathbb{V}[G]} = \{a \in HC^{\mathbb{V}[G]} : \mathbb{V}[G] \models 'a \in \varphi(HC)'\}$.
- $\varphi(x)$ is *HC-forcing invariant* if, for every twice-iterated forcing extension $\mathbb{V}[G][G']$,

$$\varphi(HC)^{\mathbb{V}[G][G']} \cap HC^{\mathbb{V}[G]} = \varphi(HC)^{\mathbb{V}[G]}$$

¹In [5], Hjorth proved that $\text{Aut}(M)$ satisfies the Topological Vaught Conjecture on analytic sets if and only if S_∞ divides $\text{Aut}(M)$.

The reader is cautioned that when computing $\varphi(\text{HC})^{\mathbb{V}[G]}$, the quantifiers of φ range over $\text{HC}^{\mathbb{V}[G]}$ as opposed to the whole of $\mathbb{V}[G]$. The reason for the double iteration of forcing is to make the notion of a formula $\varphi(x)$ being HC-invariant absolute between forcing extensions. Visibly, the class of HC-forcing invariant formulas is closed under boolean combinations, and it follows from Lévy Absoluteness that every Σ_1 formula of set theory is HC-invariant.

The principal idea we wish to exploit is that every set is potentially in HC. That is, for any set $A \in \mathbb{V}$, there is a forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ such that $A \in \text{HC}^{\mathbb{V}[G]}$. With this in mind, for any HC-invariant formula $\varphi(x)$, let

$$\varphi_{\text{ptl}} = \{A \in \mathbb{V} : \mathbb{V}[G] \models \varphi(A) \text{ for some (equiv. every) } \mathbb{V}[G] \supseteq \mathbb{V} \text{ with } A \in \text{HC}^{\mathbb{V}[G]}\}$$

To ease notation, call a sequence of sets $X = (X^{\mathbb{V}[G]} : \mathbb{V}[G] \supseteq \mathbb{V})$ *strongly definable* if there is an HC-invariant $\varphi(x)$ defining each $X^{\mathbb{V}[G]}$ in $\mathbb{V}[G]$. For X strongly definable, we write X_{ptl} for φ_{ptl} , where $\varphi(x)$ is any HC-invariant formula defining X . We call X *short* if X_{ptl} is a set, as opposed to a proper class. For a short X , we let $\|X\| := |X_{\text{ptl}}|$, and we write $\|X\| = \infty$ when X is not short.

If a strongly definable X is defined by a formula $\varphi(x)$ that is absolute between any forcing extensions, then using the fact that every set $A \in \mathbb{V}$ is in $\text{HC}^{\mathbb{V}[G]}$ for some forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$, X_{ptl} is all sets $A \in \mathbb{V}$ such that $\varphi(A)$ holds, i.e., $A \in X^{\mathbb{V}[G]}$ for some forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$. We abuse notation slightly and refer to this class as X .

As examples, for any countable language L and any $\Phi \in L_{\omega_1, \omega}$, both $\text{Mod}_{\text{HC}}(L) = \{M \in \text{HC} : M \text{ is an } L\text{-structure}\}$ and $\text{Mod}_{\text{HC}}(\Phi) = \{M \in \text{Mod}_{\text{HC}}(L) : M \models \Phi\}$ are strongly definable. Here, as being an L -structure and satisfaction are absolute, we write $\text{Mod}(L)$ and $\text{Mod}(\Phi)$ in place of the more cumbersome $(\text{Mod}_{\text{HC}}(L))_{\text{ptl}}$ or $(\text{Mod}_{\text{HC}}(\Phi))_{\text{ptl}}$.

Working with strongly definable sets, we say a notion holds *persistently* if the relation holds in every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$. As examples, we say $f : X \rightarrow Y$ persistently if, for every $\mathbb{V}[G] \supseteq \mathbb{V}$, $f^{\mathbb{V}[G]}$ is a function with domain $X^{\mathbb{V}[G]}$, taking elements in $Y^{\mathbb{V}[G]}$. *Persistently injective* and *persistently bijective* functions are defined similarly. We record the following facts, which appear as Lemmas 2.5 and 2.14 of [16].

Lemma 1.3. *Suppose f, A, B are strongly definable with $A, B \subseteq \text{HC}$ and $f : A \rightarrow B$ is persistently a function. Then f_{ptl} is a class functional from A_{ptl} to B_{ptl} .*

If, in addition, $f : A \rightarrow B$ is persistently injective (resp. bijective) then f_{ptl} is injective (resp. bijective).

We also include one new notion that was not discussed in [16]. For a persistent function $f : A \rightarrow B$ as in Lemma 1.3, we say f has a *persistent cross section* if there is a strongly definable $g : B \rightarrow A$ such that, persistently, $(\forall b \in B) f \circ g(b) = b$.

Lemma 1.4. *Suppose f, A, B are strongly definable with $A, B \subseteq HC$ and $f : A \rightarrow B$ is persistently a function. If $g : B \rightarrow A$ is a persistent cross section of f , then $f_{\text{ptl}} : A_{\text{ptl}} \rightarrow B_{\text{ptl}}$ is surjective.*

Proof. Given $b \in B_{\text{ptl}}$, put $a := g_{\text{ptl}}(b)$. Then $f_{\text{ptl}}(a) = b$, so f_{ptl} is surjective. \square

Much of Descriptive Set Theory, or at least the study of Borel reducibility revolves around analyzing the complexity of equivalence relations, and hence quotients. Call a pair (X, E) a *strongly definable quotient* if both X and E are strongly definable and E is persistently an equivalence relation on X . We say (X, E) is *HC-reducible* to (Y, F) ,

$$(X, E) \leq_{HC} (Y, F)$$

if both (X, E) and (Y, F) are strongly definable quotients and there is a strongly definable $f \subseteq X \times Y$ such that, persistently,

- The E -saturation of $\text{dom}(f)$ is X . That is, for every $x \in X$, there is an $x' \in X$ and $y' \in Y$ where xEx' holds and $(x', y') \in f$.
- f induces a well-defined injection on equivalence classes. That is, if (x, y) and (x', y') are in f , then xEx' holds if and only if yFy' does.

As examples, if X, E, Y, B are Borel subsets of Polish spaces then any Borel reduction is an *HC-reduction*, as is any absolutely Δ_2^1 -reduction.

A *representation* of a strongly definable quotient (X, E) is a pair f, Z of strongly definable families such that $f : X \rightarrow Z$ is persistently surjective and persistently,

$$(\forall a, b \in X)[E(a, b) \leftrightarrow f(a) = f(b)]$$

It is easily checked that if f, Z and f', Z' are two representations of the same (X, E) , then $\|Z\| = \|Z'\|$, so we define $\|(X, E)\|$ to be this common value, which may be ∞ . We call (X, E) *short* if $\|(X, E)\| < \infty$. The following fact is essentially Proposition 2.18 of [16].

Fact 1.5. *Suppose (X, E) and (Y, F) are both represented, strongly definable quotients. If $(X, E) \leq_{HC} (Y, F)$, then $\|(X, E)\| \leq \|(Y, F)\|$.*

Continuing with our example above, for a countable language L , let $\text{Mod}_\omega(L)$ be the Polish space of L -structures with universe ω . By a theorem of Lopez-Escobar, [13], the Borel subsets of $\text{Mod}_\omega(L)$, invariant under the action of S_∞ , are of the form $\text{Mod}_\omega(\Phi)$ for some $\Phi \in L_{\omega_1, \omega}$. In [3], Friedman and Stanley introduced the concept of Borel reducibility between isomorphism classes of this form. For infinitary Φ, Ψ , possibly in different languages, $(\text{Mod}_\omega(\Phi), \cong)$ is *Borel reducible* to $(\text{Mod}_\omega(\Psi), \cong)$ if there is a Borel $f :$

$\text{Mod}_\omega(\Phi) \rightarrow \text{Mod}_\omega(\Psi)$ satisfying $M \cong N$ iff $f(M) \cong f(N)$ for all $M, N \in \text{Mod}_\omega(\Phi)$. They showed that among invariant Borel subsets, there is a maximal class, dubbed *Borel complete*, with respect to Borel reducibility.

Placing this into our context, there is a representation

$$css : (\text{Mod}_{HC}(L), \cong) \rightarrow CSS(L)$$

where, for each L -structure $M \in HC$, $css(M)$ is the *canonical Scott sentence* of M , which is a sentence of $L_{\omega_1, \omega}$ (see Definition 3.1 of [16] for a formal definition of this map). The salient property of $css(M)$ is described by

Fact 1.6. *For any countable language L and any $M, N \in \text{Mod}_{HC}(L)$, $N \models css(M)$ if and only if $N \cong M$.*

That the set $CSS(L)$ is strongly definable and that css is a persistent HC-invariant surjection follows from Karp's Completeness Theorem for sentences of $L_{\omega_1, \omega}$ (see e.g., Theorem 3 of [8] and Lemma 3.3 of [16]).

For any $\Phi \in L_{\omega_1, \omega}$, the restriction of the css representation gives a representation $CSS(\Phi)$ of $(\text{Mod}_{HC}(\Phi), \cong)$. Thus, $CSS(\Phi)_{\text{ptl}}$, the class of *potential canonical Scott sentences* is the class of all $\psi \in \mathbb{V}$, where $\psi \in L_{\infty, \omega}$, but for some/every $\mathbb{V}[G] \supseteq \mathbb{V}$ where $(\psi \in L_{\omega_1, \omega})^{\mathbb{V}[G]}$, we have $(\psi \in CSS(\Phi))^{\mathbb{V}[G]}$.

Note that $CSS(\Phi)$ is also a representation of $(\text{Mod}_\omega(\Phi), \cong)$, the isomorphism classes of models of Φ with universe ω . We write $\|\Phi\|$ in place of $\|(\text{Mod}_{HC}(\Phi), \cong)\| = |CSS(\Phi)_{\text{ptl}}|$ (which may be ∞). Clearly, if $(\text{Mod}_\omega(\Phi), \cong)$ is Borel complete, then Φ is not short, i.e., $\|\Phi\| = \infty$.

Note, however, that for some Φ , the class $CSS(\Phi)_{\text{ptl}}$ can be strictly larger than $CSS(\Phi)_{\text{sat}}$, the latter being the class of all $css(M)$ for $M \in \text{Mod}(\Phi)$. When these two classes are not equal, we say that Φ is *not grounded*. One of the central goals of Section 3 is to get better control of the class of potential canonical Scott sentences of Φ when Φ is not grounded.

2 Back and forth systems, sharp expansions and their flattenings

Fix, for the whole of this section, a countable language L . We begin with the well known definition of a back and forth system, but then we add additional adjectives.

Definition 2.1. Given an arbitrary L -structure M , a (finitary) *back-and-forth system* on M is a non-empty set \mathcal{F} of pairs (\bar{a}, \bar{b}) of finite tuples from M with $\text{lg}(\bar{a}) = \text{lg}(\bar{b})$ satisfying:

1. For $(\bar{a}, \bar{b}) \in \mathcal{F}$, the map $\bar{a} \mapsto \bar{b}$ is q.f.- L -elementary (i.e., for any atomic L -formula $\alpha(\bar{x})$, $M \models \alpha(\bar{a}) \leftrightarrow \alpha(\bar{b})$); and
2. For all $(\bar{a}, \bar{b}) \in \mathcal{F}$ and singleton $c \in M$, there is $d \in M$ such that $(\bar{a}c, \bar{b}, d) \in \mathcal{F}$; and dually for every $d' \in M$ there is $c' \in M$ such that $(\bar{a}c', \bar{b}d') \in \mathcal{F}$.

For an L -structure M of any size, we define the *subsequence maps* as follows. For each $n \in \omega$, all $k \leq n$ and all injective $f : k \rightarrow n$, f induces a mapping from $M^n \rightarrow M^k$ defined as: for every $\bar{a} = (a_0, \dots, a_{n-1}) \in M^n$, $\bar{a} \upharpoonright_f$ is the subsequence of \bar{a} $(a_{f(0)}, \dots, a_{f(k-1)})$. As a special case, when $k = n$, $f \in \text{Sym}(\{0, \dots, n-1\})$ is a permutation and $\bar{a} \upharpoonright_f$ is the associated permutation of \bar{a} .

Definition 2.2. Let \mathcal{F} be any back and forth system on an L -structure M .

- We say \mathcal{F} is *downward closed* if, for all $k \leq n$ and all injective $f : k \rightarrow n$, whenever $(\bar{a}, \bar{b}) \in \mathcal{F} \cap M^{2n}$ $(\bar{a} \upharpoonright_f, \bar{b} \upharpoonright_f) \in \mathcal{F}$ as well.
- \mathcal{F} is a *sharp* back and forth system on M if \mathcal{F} is both downward closed and, for every $k \in \omega$, $\mathcal{F} \cap M^{2k}$ is an equivalence relation E_k on M^k for every k .

It is readily verified that if \mathcal{F} is a back-and-forth system on M , then so is its *downward closure* $dc(\mathcal{F}) = \{(\bar{a} \upharpoonright_f, \bar{b} \upharpoonright_f) : (\bar{a}, \bar{b}) \in \mathcal{F}, f : k \rightarrow n \text{ injective}\}$. As well, if for every $k \in \omega$ we let E_k be the symmetric and transitive closure of $(dc(\mathcal{F}) \cap M^{2k}) \cup \{(\bar{a}, \bar{a}) : \bar{a} \in M^k\}$, then the sequence of equivalence relations $\{E_k : k \in \omega\}$ is the smallest sharp back-and-forth system containing \mathcal{F} . For the remainder of this paper, we will only consider sharp back-and-forth systems.

We formalize the operations described by expanding the language.

Definition 2.3. For a given countable language L , let $L^\sharp = L \cup \{E_k : k \in \omega\}$, where each E_k is a $2k$ -ary relation symbol. A *sharp structure* (M, \mathcal{F}) is an L^\sharp expansion of an L -structure M by interpreting each E_k as $\mathcal{F} \cap M^{2k}$ for some distinguished sharp back-and-forth system \mathcal{F} . Let $\text{Ax}^\sharp(L)$ axiomatize sharp structures.

Regardless of the size of M , there is a unique largest sharp back and forth system on M , which we denote by \mathcal{F}_∞ and we let M^\sharp denote the canonical sharp expansion (M, \mathcal{F}_∞) . To describe \mathcal{F}_∞ we recall Karp's notion of back-and-forth equivalence of structures.

Definition 2.4. Fix an L -structure M . For tuples \bar{a}, \bar{b} of the same length, we say (M, \bar{a}) and (M, \bar{b}) are *back-and-forth equivalent* if there is some back-and-system \mathcal{F} containing (\bar{a}, \bar{b}) . We denote this as $(M, \bar{a}) \equiv_{\infty, \omega} (M, \bar{b})$. This introduces an equivalence relation on each n and we say $\text{tp}_\infty(\bar{a}) = \text{tp}_\infty(\bar{b})$ if (M, \bar{a}) and (M, \bar{b}) are back-and-forth equivalent.

It is straightforward to check that $\mathcal{F}_\infty := \{(\bar{a}, \bar{b}) : \text{tp}_\infty(\bar{a}) = \text{tp}_\infty(\bar{b})\}$ is a sharp back-and-forth system, and in fact, \mathcal{F}_∞ contains every sharp back-and-forth system on M .

Another source of examples is found via expansions. Suppose $L^* \supseteq L$ and M^* is an expansion of M . Define $\mathcal{F}^* := \{(\bar{a}, \bar{b}) : (M^*, \bar{a}) \equiv_{\infty, \omega} (M^*, \bar{b})\}$. Then (M, \mathcal{F}^*) is also a sharp expansion of M and $\mathcal{F}^* \subseteq \mathcal{F}_\infty$. In fact, every sharp expansion (M, \mathcal{F}) corresponds to an expansion M^* of M , formed by adding a new k -ary predicate for every E_k -class in \mathcal{F} . Then \mathcal{F} is the largest sharp back and forth system on M^* . The complexity of this correspondence is the topic of Section 4.

2.1 Flattenings of sharp structures

Here we introduce the central notion of this paper.

Given a sharp structure (M, \mathcal{F}) in the language $L^\sharp = L \cup \{E_n : n \in \omega\}$, it is natural to form a ‘restricted eq-expansion’ by adding new sorts for each of the $2n$ -ary equivalence relations E_n . That is, for each n , add a new unary predicate symbol $U_n(z)$ and an n -ary function symbol π_k and let $(M, \mathcal{F})^{eq}$ be formed by interpreting each of the predicates U_n as being names for the E_n -classes, and $\pi_n : M^n \rightarrow U_n$ is the canonical projection, i.e., $\pi_n(\bar{a}) = \bar{a}/E_n$.

Note that the subsequence maps $\bar{a} \mapsto \bar{a} \upharpoonright_f$ are definable in this language, so we also get induced (unary!) projection maps $P_{k,n}^f : U_n \rightarrow U_k$ indexed by injections $f : k \rightarrow n$, and these commute with the projection maps π_n , i.e., if $f : k \leq n$ is any injection, then for any $\bar{a} \in M^n$, $P_{k,n}^f \pi_n(\bar{a}) = \pi_k(\bar{a} \upharpoonright_f)$.

We now define the *flattening* of (M, \mathcal{F}) , which naively can be described as the result of passing to the *eq*-structure described above, and then ‘forgetting the home sort.’ More formally,

Definition 2.5. For any language L , let $\text{QF}_L = \bigcup \{\text{QF}_n : n \in \omega\}$, where QF_n is the set of all quantifier-free formulas $\alpha_n(x_0, \dots, x_{n-1})$, such that every free variable is among $\{x_0, \dots, x_{n-1}\}$. [Every quantifier-free formula α appears in infinitely many QF_n .] Let

$$L^b := \{U_n : n \in \omega\} \cup \{\alpha_n^b(z) : \alpha_n(x_0, \dots, x_{n-1}) \in \text{QF}_L\} \cup \{P_{k,n}^f : f : k \rightarrow n \text{ is an injection}\}$$

where each U_n and α_n^b are unary predicates, and $P_{k,n}^f : U_n \rightarrow U_k$ is a unary function symbol.

Given any sharp structure (M, \mathcal{F}) , its *flattening* is the L^b -structure $\mathbb{B} = (M, \mathcal{F})^b$ with universe $\bigcup_{n \in \omega} M^n/E_n$, where each $U_n^{\mathbb{B}} = M^n/E_n$, $\alpha_n^b(\mathbb{B}) = \{\bar{a}/E_n \in U_n : M \models \alpha_n(\bar{a})\}$, and $P_{k,n}^f : U_n^{\mathbb{B}} \rightarrow U_k^{\mathbb{B}}$ given by $P_{k,n}^f(\bar{a}/E_n) = \bar{a} \upharpoonright_f / E_k$.

Note that the definitions of $\alpha_n^{\mathbb{B}}$ and $P_{k,n}^f$ are well defined because \mathcal{F} is a sharp back-and-forth system on M . In particular, if $E_n(\bar{a}, \bar{b})$ holds, then $qftp_M(\bar{a}) = qftp_M(\bar{b})$.

The correspondence $\alpha \mapsto \alpha^b$ extends naturally to arbitrary L -formulas, and even to $L_{\infty, \omega}$ -formulas.

Definition 2.6. For every formula $\varphi(x_0, \dots, x_{n-1})$ of $L_{\infty, \omega}$, we recursively define an $(L^b)_{\infty, \omega}$ -formula $\varphi^b(z)$ as follows:

- For $\alpha_n(x_0, \dots, x_{n-1}) \in \text{QF}_L$, let α_n^b be the distinguished L^b predicate symbol;
- $(\bigwedge_{i \in I} \varphi_i)^b := \bigwedge_{i \in I} \varphi_i^b$;
- $(\neg \varphi)^b := \neg(\varphi^b)$; and
- $(\exists y \theta(x_0, \dots, x_{n-1}, y))^b := \exists w (U_{n+1}(w) \wedge \theta^b(w) \wedge P_{n, n+1}^{id}(w) = z)$.

Let $\text{Fm}^b = \{\varphi^b(z) : \varphi \in L_{\infty, \omega}\}$, which is a subset of $(L^b)_{\infty, \omega}$.

For a flat structure \mathbb{B} and $\mathbf{a} \in \mathbb{B}$, $\text{b-tp}_{\mathbb{B}}(\mathbf{a}) := \{\varphi^b(z) \in \text{Fm}^b : \mathbb{B} \models \varphi^b(\mathbf{a})\}$.

Note that every infinitary L^b -formula in Fm^b has a single free variable. The following Lemma is straightforward.

Lemma 2.7. *For all n and all $\varphi(x_0, \dots, x_{n-1}) \in L_{\infty, \omega}$, for all L -structures M and for all sharp back-and-forth systems $\mathcal{F} = \{E_n : n \in \omega\}$ on M we have: (letting \mathbb{B} denote the flattening $(M, \mathcal{F})^b$ for all $\bar{a} \in M^n$)*

$$M \models \varphi(\bar{a}) \quad \text{if and only if} \quad \mathbb{B} \models \varphi^b(\bar{a}/E_n)$$

Proof. We prove this by induction on the complexity of φ . For quantifier-free L -formulas, this correspondence is built into the definition of a flattening, and it is routine to see this correspondence is preserved under (infinitary) boolean combinations. To see that it is preserved under quantification, assume the Lemma holds for $\theta(\bar{x}, y)$ with $\text{lg}(\bar{x}) = n$. Fix any (M, \mathcal{F}) and choose any $\bar{a} \in M^n$. Let \mathbb{B} denote the flattening and let $\mathbf{a} = \bar{a}/E_n$. On one hand, if $M \models \exists y \theta(\bar{a}, y)$, choose $b \in M$ such that $M \models \theta(\bar{a}, b)$ and let $\mathbf{b} = \bar{a}b/E_{n+1} \in U_{n+1}$. As our inductive hypothesis gives $\mathbb{B} \models \theta^b(\mathbf{b})$, we conclude $\mathbb{B} \models \varphi^b(\mathbf{a})$. Conversely, choose $\mathbf{b} \in U_{n+1}$ witnessing $\mathbb{B} \models \varphi^b(\mathbf{a})$, i.e., $\mathbb{B} \models \theta^b(\mathbf{b}) \wedge P_{n, n+1}^{id}(\mathbf{b}) = \mathbf{a}$. As $\mathbb{B} \models \theta^b(\mathbf{b})$, by induction there is some $\bar{b} \in M^n$ and $b^* \in M$ such that $\bar{b}b^*/E_{n+1} = \mathbf{b}$ and $M \models \theta(\bar{b}, b^*)$. It follows from the interpretation of $P_{n, n+1}^{id}$ in \mathbb{B} that $\bar{b}/E_n = \mathbf{a}$, which is also equal to \bar{a}/E_n . Thus, $(\bar{a}, \bar{b}) \in \mathcal{F}$. As \mathcal{F} is a back-and-forth system, there is $a^* \in M$ so that $(\bar{a}a^*, \bar{b}b^*) \in \mathcal{F}$, so $\bar{a}a^*/E_{n+1} = \bar{b}b^*/E_{n+1} = \mathbf{b}$. By the inductive hypothesis applied to $\theta(\bar{x}, y)$, we conclude $M \models \theta(\bar{a}, a^*)$, i.e., $M \models \varphi(\bar{a})$, as required. \square \square

Definition 2.8. An L^b -structure \mathbb{B} is *Hausdorff* if Fm^b separates points, i.e., distinct elements from \mathbb{B} have distinct b -types. That is, $b\text{-tp}_{\mathbb{B}}(\mathbf{a}) = b\text{-tp}_{\mathbb{B}}(\mathbf{b})$ implies $\mathbf{a} = \mathbf{b}$.

Lemma 2.9. For any L -structure M , the flattening of the canonical expansion $M^\sharp = (M, \mathcal{F}_\infty)$ is Hausdorff. Conversely, if (M, \mathcal{F}) is any sharp expansion of M whose flattening is Hausdorff, then $\mathcal{F} = \mathcal{F}_\infty$.

Proof. Let \mathbb{B} be the flattening of the canonical M^\sharp . To see that \mathbb{B} is Hausdorff, choose any $\mathbf{a} \neq \mathbf{b}$ from \mathbb{B} . If \mathbf{a}, \mathbf{b} are in different U_n 's they are clearly separated, so assume $\mathbb{B} \models U_n(\mathbf{a}) \wedge U_n(\mathbf{b})$. By the definition of the flattening, choose $\bar{a}, \bar{b} \in M^n$ such that $\bar{a}/E_n = \mathbf{a}$ and $\bar{b}/E_n = \mathbf{b}$. As $\mathbf{a} \neq \mathbf{b}$, $(\bar{a}, \bar{b}) \notin \mathcal{F}_\infty$, so $\text{tp}_\infty(\bar{a}) \neq \text{tp}_\infty(\bar{b})$. Choose any infinitary $\varphi(\bar{x}) \in \text{tp}_\infty(\bar{a}) \setminus \text{tp}_\infty(\bar{b})$. Then $\mathbb{B} \models \varphi^b(\mathbf{a}) \wedge \neg\varphi^b(\mathbf{b})$, so \mathbb{B} is Hausdorff.

For the converse, recall that for any back-and-forth system \mathcal{F} on M , if $(\bar{a}, \bar{b}) \in \mathcal{F}$, then $\text{tp}_\infty(\bar{a}) = \text{tp}_\infty(\bar{b})$, hence $\mathcal{F} \subseteq \mathcal{F}_\infty$. Now assume \mathcal{F} is any sharp back-and-forth system that is not equal to \mathcal{F}_∞ , and we show that its flattening \mathbb{B} cannot be Hausdorff. Since $\mathcal{F}_\infty \subseteq \mathcal{F}$, there is some n and $\bar{a}, \bar{b} \in M^n$ with $(\bar{a}, \bar{b}) \in \mathcal{F}_\infty \setminus \mathcal{F}$, i.e., $\text{tp}_\infty(\bar{a}) = \text{tp}_\infty(\bar{b})$, but $\mathbf{a} \neq \mathbf{b}$, where $E_n = \mathcal{F} \cap M^{2n}$, $\mathbf{a} = \bar{a}/E_n$ and $\mathbf{b} = \bar{b}/E_n$. Thus, $\mathbf{a}, \mathbf{b} \in \mathbb{B}$ and for every $\varphi(\bar{x}) \in L_{\infty, \omega}$, $\mathbb{B} \models \varphi^b(\mathbf{a}) \leftrightarrow \varphi^b(\mathbf{b})$. So \mathbf{a} and \mathbf{b} witness that \mathbb{B} is not Hausdorff. \square

3 Flat structures, the inverse map, and potential canonical Scott sentences

3.1 Axiomatizing flat structures

We formalize the notion of a L^b -structure \mathbb{B} . We will see that every flattening of a sharp expansion is a flat structure, and conversely, every *countable* \mathbb{B} is the flattening of some sharp expansion of some countable L -structure. Throughout, we have a fixed (countable) language L , along with the associated language L^b .

As notation, for any L^b -structure \mathbb{B} and any $\mathbf{b} \in U_n$, let

$$\Delta_{\mathbf{b}} := \{\text{quantifier-free } \beta(x_0, \dots, x_{n-1}) : \mathbb{B} \models \beta^b(\mathbf{b})\}$$

In particular, $\Delta_{\mathbf{b}}$ is a set of L -formulas, all of whose free variables are among $\{x_0, \dots, x_{n-1}\}$. The definition below lists several conditions that hold of any flattening of a sharp structure (M, \mathcal{F}) .

Definition 3.1. Let $\text{Ax}^b(L)$ denote the following infinitary L^b axioms, and call an L^b -structure \mathbb{B} a *flat structure* if $\mathbb{B} \models \text{Ax}^b(L)$.

1. Structural axioms:

- (a) The unary predicates $\{U_n : n \in \omega\}$ partition the universe.
- (b) For all $k \leq n$ and injective $f : k \rightarrow n$, $P_{k,n}^f$ is a (unary) function: $U_n \rightarrow U_k$.
- (c) For all $n \geq 1$, $P_{n,n}^{id} : U_n \rightarrow U_n$ (where $id : n \rightarrow n$ is the identity) is the identity function.
- (d) For all $n \geq 1$ and all $\mathbf{b} \in U_n$, $\Delta_{\mathbf{b}}$ is a complete, quantifier-free type in the variables (x_0, \dots, x_{n-1}) (i.e., there is some L -structure M and some $\bar{a} \in M^n$ such that $M \models \beta(\bar{a})$ for all $\beta \in \Delta_{\mathbf{b}}$ and, for every quantifier-free $\beta(x_0, \dots, x_{n-1})$, exactly one of $\beta, \neg\beta$ is contained in $\Delta_{\mathbf{b}}$.)

2. Relational axioms:

- (a) For all injective $f : k \rightarrow n$ and $g : n \rightarrow m$, and all $\mathbf{a} \in U_m$, $P^{g \circ f}(\bar{a}) = P_{k,n}^f(P_{n,m}^g(\mathbf{a}))$.
- (b) For all injective $f : k \rightarrow n$, all $\mathbf{a} \in U_n$, and all quantifier-free $\alpha_k(x_0, \dots, x_{k-1})$,

$$\alpha_k(x_0, \dots, x_{k-1}) \in \Delta_{P_{k,n}^f(\mathbf{a})} \quad \text{if and only if} \quad \alpha_k(x_{f(0)}, \dots, x_{f(k-1)}) \in \Delta_{\mathbf{a}}$$

3. Equality Axiom: For all $1 \leq k \leq n$, all injective $f, g : k \rightarrow n$ and all $\mathbf{a} \in U_n$, if $(\bigwedge_{i \in k} x_{f(i)} = x_{g(i)}) \in \Delta_{\mathbf{a}}$, then $P_{k,n}^f(\mathbf{a}) = P_{k,n}^g(\mathbf{a})$.

As notation: If $\mathbf{a} \in U_k$ and $\mathbf{b} \in U_n$, $\mathbf{a} \leq \mathbf{b}$ denotes $\mathbf{a} = P_{k,n}^{id}(\mathbf{b})$ (so $k \leq n$).

- 4. Amalgamation: For all $k \leq n, m$, if $\mathbf{a} \in U_k$, $\mathbf{b} \in U_n$, $\mathbf{c} \in U_m$, $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a} \leq \mathbf{c}$, then there is $\mathbf{d} \in U_{n+m-k}$ such that $\mathbf{b} \leq \mathbf{d}$ and $P^v(\mathbf{d}) = \mathbf{c}$, where $v : m \rightarrow n + (m - k)$ is defined by $v(i) = i$ for all $i < k$ and $v(i) = n + (i - k)$ for all $k \leq i < m$.
- 5. Duplication: $\mathbb{B} \models \sigma^b$, where σ is the L -sentence $\forall x_0 \exists x_1 (x_0 = x_1)$, i.e., for every $\mathbf{a} \in U_1$, there is $\mathbf{b} \in U_2$ such that $P^{f_0}(\mathbf{b}) = P^{f_1}(\mathbf{b}) = \mathbf{a}$ for the two functions $f_0, f_1 : 1 \rightarrow 2$ and $(x_0 = x_1) \in \Delta_{\mathbf{b}}$.
- 6. Constants: For all constant symbols $c \in L$, $\mathbb{B} \models \sigma_c^b$, where σ_c is the L -sentence $\exists x_0 (x_0 = c)$.
- 7. Functions: For all k -ary function symbols $f \in L$, $\mathbb{B} \models \sigma_f^b$, where σ_f is the L -sentence $\forall x_0 \dots \forall x_{k-1} \exists x_k (x_k = f(x_0, \dots, x_{k-1}))$.

We remark that without additional conditions, the element \mathbf{d} obtained by Amalgamation is not uniquely described.

3.2 Generalized subsequences

In anticipation of Proposition 3.7, where we recover an L -structure M from a flat structure \mathbb{B} , as formulas must also be defined on tuples that may have duplicate values, we must show that elements of \mathbb{B} encode these as well. We begin with a one-step lemma, whose proof follows immediately from (AP) and Duplication.

Lemma 3.2. *Let \mathbb{B} be any flat structure. Suppose $m \geq 1$, $j \in m$, and $\mathbf{a} \in U_m$. There is $\mathbf{c} \in U_{m+1}$ such that $\mathbf{a} \leq \mathbf{c}$ and $(x_j = x_m) \in \Delta_{\mathbf{c}}$.*

By iterating Lemma 3.2 n times, we get the following lemma.

Lemma 3.3. *Suppose $f^* : n \rightarrow m$ is any function (not necessarily injective). For any flat structure \mathbb{B} and any $\mathbf{a} \in U_m$, there is a unique $\mathbf{c} \in U_{m+n}$ such that $\mathbf{a} \leq \mathbf{c}$ and $(\bigwedge_{i \in n} x_{f^*(i)} = x_{m+i}) \in \Delta_{\mathbf{c}}$.*

Proof. The existence of such a \mathbf{c} follows by iterating Lemma 3.2 n times. As for uniqueness, suppose $\mathbf{c}_1, \mathbf{c}_2 \in U_{m+n}$ both satisfy these conditions. Let $\mathbf{d} \in U_{m+2n}$ be an amalgamation of \mathbf{c}_1 and \mathbf{c}_2 over \mathbf{a} . In particular, $\mathbf{c}_1 \leq \mathbf{d}$, and $\mathbf{c}_2 = P_{m+n, m+2n}^g$, where g is defined as $g(i) = i$ for all $i < m$ and $g(i) = i + n$ if $m \leq i < m + n$. Because of the Equality Axiom, to conclude that $\mathbf{c}_1 = \mathbf{c}_2$, it suffices to show that $(x_{id(i)} = x_{g(i)}) \in \Delta_{\mathbf{d}}$ for all $i < m + n$. The first part is easy. For $i < m$, $g(i) = i$ since $\mathbf{a} \leq \mathbf{c}_2$, hence $(x_i = x_{g(i)}) \in \Delta_{\mathbf{d}}$ trivially. For $m \leq i < m + n$, let $j = i - m$. Since $\mathbf{a} \leq \mathbf{c}_1$ and $\mathbf{a} \leq \mathbf{c}_2$, $(x_{f^*(j)} = x_j)$ is in both $\Delta_{\mathbf{c}_1}$ and $\Delta_{\mathbf{c}_2}$. Since $\mathbf{c}_1 \leq \mathbf{d}$, we have $(x_{f^*(j)} = x_j) \in \Delta_{\mathbf{d}}$ as well. But $P_{m+n, m+2n}^g(\mathbf{d}) = \mathbf{c}_2$ implies $(x_{g(f^*(j))} = x_{g(i)}) \in \Delta_{\mathbf{d}}$. Because $f^*(j) < m$, $g(f^*(j)) = f^*(j)$, so $(x_{f^*(j)} = x_{g(i)}) \in \Delta_{\mathbf{d}}$. As $\Delta_{\mathbf{d}}$ is a complete, q.f. type, we conclude $(x_i = x_{g(i)}) \in \Delta_{\mathbf{d}}$. Thus, as noted above, we conclude $\mathbf{c}_1 = \mathbf{c}_2$ by the Equality Axiom. \square

In light of this uniqueness result, we make the following definitions. For the remainder of this subsection, we assume \mathbb{B} is a fixed flat structure, and we work inside \mathbb{B} .

Definition 3.4. Suppose $f^* : n \rightarrow m$ is any function and $\mathbf{a} \in U_m$.

The f^* -blowup of \mathbf{a} is the (unique) $\mathbf{c} \in U_{m+n}$ satisfying $\mathbf{a} \leq \mathbf{c}$ and $(\bigwedge_{i \in n} x_{f^*(i)} = x_{m+i}) \in \mathbf{c}$.

For any integer ℓ , the (injective) ℓ -shift function $v_\ell : n \rightarrow n + \ell$ is defined by $v_\ell(i) = \ell + i$.

The generalized projection $P^{f^*}(\mathbf{a})$ is the element $\mathbf{b} \in U_n$ satisfying $\mathbf{b} = P_{n, m+n}^{v_m}(\mathbf{c})$, where \mathbf{c} is the f^* -blowup of \mathbf{a} and $v_m : n \rightarrow m + n$ is the m -shift function.

We show that these generalized projections satisfy statements analogous to the axioms for (hardwired) projections P^f (for injective functions f). We begin with a Generalized Equality criterion.

Lemma 3.5. *Suppose $f^*, g^* : n \rightarrow m$ and $\mathbf{a} \in U_m$ satisfies $(\bigwedge_{i \in n} x_{f^*(i)} = x_{g^*(i)}) \in \Delta_{\mathbf{a}}$. Then $P^{f^*}(\mathbf{a}) = P^{g^*}(\mathbf{a})$.*

Proof. Let $\mathbf{c} \geq \mathbf{a}$ denote the f^* -blowup of \mathbf{a} . As $f^*(i) = g^*(i)$ for each $i \in n$, it follows from Lemma 3.3 that \mathbf{c} is also the g^* -blowup of \mathbf{a} . Thus, by definition, $P^{f^*}(\mathbf{a}) = P_{n, m+n}^v(\mathbf{c}) = P^{g^*}(\mathbf{a})$. \square

To show that Generalized Composition holds, we introduce some more notation. Given a sequence $\langle \mathbf{a}_0, \dots, \mathbf{a}_{\ell-1} \rangle \in \mathbb{B}^\ell$ with $\mathbf{a}_i \in U_{k_i}$, let $s_i = \sum_{j < i} k_j$ and let $s = s_{\ell-1}$. For each $i < \ell$, let $f_i : k_i \rightarrow s$ be the s_{i-1} -shift function defined by $f_i(j) = s_{i-1} + j$ (where we take $s_{-1} = 0$). A *joint embedding* of $\langle \mathbf{a}_0, \dots, \mathbf{a}_{\ell-1} \rangle$ is any $\mathbf{d} \in U_s$ such that $P^{f_i}(\mathbf{d}) = \mathbf{a}_i$ for every $i \leq \ell$. \mathbf{d} is not uniquely determined.

It is easily checked that for any sequence $\langle \mathbf{a}_0, \dots, \mathbf{a}_{\ell-1} \rangle$ and any joint embedding $\mathbf{d} \in U_s$, $\mathbf{a}_0 \leq \mathbf{d}$ and, letting $v_{k_0} : (s - k_0) \rightarrow s$ be the k_0 -shift function, the ‘truncation’ $P^{v_{k_0}}(\mathbf{d})$ is a joint embedding of $\langle \mathbf{a}_1, \dots, \mathbf{a}_{\ell-1} \rangle$. Given this, Generalized Composition follows easily.

Lemma 3.6. *Suppose $f^* : k \rightarrow n$ and $g^* : n \rightarrow m$ are arbitrary functions. For any $\mathbf{a} \in U_m$ we have $P^{g^* f^*}(\mathbf{a}) = P^{f^*}(P^{g^*}(\mathbf{a}))$.*

Proof. Let $\mathbf{b} \in U_{m+n}$ be the g^* -blowup of \mathbf{a} , let $\mathbf{c} \in U_{m+k}$ be the $g^* f^*$ -blowup of \mathbf{a} , and, as $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a} \leq \mathbf{c}$, let \mathbf{d} be an amalgamation of \mathbf{b} and \mathbf{c} over \mathbf{a} . [In fact, \mathbf{d} is uniquely determined, but we don’t need this.] Then \mathbf{d} is a joint embedding of $\langle \mathbf{a}, P^{g^*}(\mathbf{a}), P^{g^* f^*}(\mathbf{a}) \rangle$. Let $\mathbf{e} = P^{v_m}(\mathbf{d})$ be the truncation of \mathbf{d} ; so by the preceding remarks, \mathbf{e} is a joint embedding of $\langle P^{g^*}(\mathbf{a}), P^{g^* f^*}(\mathbf{a}) \rangle$. We claim that \mathbf{e} is the f^* -blowup of $P^{g^*}(\mathbf{a})$. This observation completes the proof, since the f^* -blowup of $P^{g^*}(\mathbf{a})$ is a joint embedding of $\langle P^{g^*}(\mathbf{a}), P^{f^*}(P^{g^*}(\mathbf{a})) \rangle$; and hence, by comparing truncations, we get $P^{g^* f^*}(\mathbf{a}) = P^{f^*}(P^{g^*}(\mathbf{a}))$.

To prove the claim, since \mathbf{e} is a joint embedding, $P^{g^*}(\mathbf{a}) \leq \mathbf{e}$. It remains to show that $(x_{f^*(i)} = x_{n+i}) \in \Delta_{\mathbf{e}}$ for all $i < k$. Fix $i < k$ and let $j = f^*(i) \in n$. Now, in \mathbf{b} , $(x_{g^*(j)} = x_{m+j}) \in \Delta_{\mathbf{b}}$, hence in $\Delta_{\mathbf{d}}$, since $\mathbf{b} \leq \mathbf{d}$. On the other hand, in \mathbf{c} , $(x_{g^* f^*(i)} = x_{m+i}) \in \Delta_{\mathbf{c}}$, hence $(x_{g^* f^*(i)} = x_{m+n+i}) \in \Delta_{\mathbf{d}}$ from the amalgamation. As $\Delta_{\mathbf{d}}$ is a complete type, these two statements imply $(x_{m+j} = x_{m+n+i}) \in \Delta_{\mathbf{d}}$. As $\mathbf{e} = P^{v_m}(\mathbf{d})$ and recalling $j = f^*(i)$, we have $(x_{f^*(i)} = x_{n+i}) \in \Delta_{\mathbf{e}}$, as required. \square

3.3 Reconstructing (M, \mathcal{F}) from a countable \mathbb{B}

Throughout this section we fix a countable language L and a countable, flat structure \mathbb{B} in the language L^b . The goal of this section is to prove the following proposition.

Proposition 3.7. *For every countable, flat structure \mathbb{B} , there is a countable L -structure M and a sharp back-and-forth system \mathcal{F} such that $(M, \mathcal{F})^b \cong \mathbb{B}$. Moreover, the sharp structure (M, \mathcal{F}) is unique up to L^\sharp -isomorphism.*

Towards the proof of Proposition 3.7, we introduce a new notion.

Definition 3.8. A cofinal sequence in \mathbb{B} $\langle \mathbf{a}_n : n \in \omega \rangle$ satisfies: For all $n \in \omega$

1. $\mathbf{a}_n \in U_n$ and $\mathbf{a}_n \leq \mathbf{a}_{n+1}$; and
2. For all m and $k \leq n, \ell$, for all $\mathbf{b} \in U_k$ and $\mathbf{c} \in U_\ell$, if $\mathbf{b} = P_{k,n}^f(\mathbf{a}_n)$ and $\mathbf{b} = P_{k,\ell}^g(\mathbf{c})$, then there is an $m \geq n$ and an injective $g' : \ell \rightarrow m$ so that $g' \circ g = id \circ f$ and $P^{g'}(\mathbf{a}_m) = \mathbf{c}$.

That is, a cofinal sequence represents a kind of Fraïssé sequence of elements from \mathbb{B} . The following lemma is easily achieved by proper bookkeeping.

Lemma 3.9. *Every countable flat structure \mathbb{B} has a cofinal sequence. In fact, any finite sequence $\langle \mathbf{a}_i : i < n \rangle$ satisfying $\mathbf{a}_i \in U_i$ and $\mathbf{a}_i \leq \mathbf{a}_{i+1}$ for each i can be extended to a cofinal sequence.*

Now fix a countable flat structure \mathbb{B} in the language L^b and fix a cofinal sequence $\langle \mathbf{a}_n : n \in \omega \rangle$. We will construct a countable L -structure M and a sharp back-and-forth system $\mathcal{F} = \{E_n : n \in \omega\}$ such that $(M, \mathcal{F})^b$ is L^b -isomorphic to \mathbb{B} as follows:

- First, as $\mathbf{a}_n \leq \mathbf{a}_{n+1}$, we have $\Delta_{\mathbf{a}_n} \subseteq \Delta_{\mathbf{a}_{n+1}}$, and as each $\Delta_{\mathbf{a}_n}$ describes a complete, quantifier-free type in the variables (x_0, \dots, x_{n-1}) , $q(x_0, x_1, \dots) := \bigcup_{n \in \omega} \Delta_{\mathbf{a}_n}$ describes a complete, quantifier-free type in ω variables $X = \{x_i : i \in \omega\}$.
- Next, define an equivalence relation \sim on X as: $x_i \sim x_j$ iff $(x_i = x_j) \in q$. As q is a consistent type, \sim indeed is an equivalence relation.
- Define an L -structure M to have universe X / \sim . For any quantifier-free L -formula α with n free variables and any $(i_0, \dots, i_{n-1}) \in \omega^n$, say $M \models \alpha([x_{i_0}], \dots, [x_{i_{n-1}}])$ if and only if $\alpha(x_{i_0}, \dots, x_{i_{n-1}}) \in q$. It is readily checked that this is well defined.
- Additionally, for each $n \in \omega$, define a *covering map* $cov_n : M^n \rightarrow U_n$ as: For any $\bar{a} = (a_0, \dots, a_{n-1}) \in M^n$ (so each a_j is a \sim -class of elements from X), let $cov_n(\bar{a}) = P^h(\bar{a}_m)$ for any/all sufficiently large m and any function $h : n \rightarrow m$ satisfying $x_{h(j)} \in a_j$ for all $j < n$. Again, it is easily checked that this definition does not depend on our choice of m or h .

- Finally, for each $n \in \omega$, define $E_n(\bar{x}, \bar{y})$ on M^{2n} as $E_n(\bar{a}, \bar{b}) \Leftrightarrow \text{cov}_n(\bar{a}) = \text{cov}_n(\bar{b})$ and let $\mathcal{F} := \bigcup_{n \in \omega} E_n$.

Much of the verification that \mathcal{F} is a sharp back-and-forth system and that $(M, \mathcal{F})^b \cong \mathbb{B}$ is codified with the following fundamental lemma, which will be used in later sections as well.

Lemma 3.10. *Suppose $n \in \omega$, $\bar{a} \in M^n$, $\text{cov}_n(\bar{a}) = \mathbf{b} \in U_n$, and $\mathbf{c} \in U_{n+k}$ with $\mathbf{b} \leq \mathbf{c}$. Then there is a k -tuple $\bar{c} \in M^k$ so that $\text{cov}_{n+k}(\bar{a}\bar{c}) = \mathbf{c}$.*

Proof. Arguing by induction on k , it suffices to prove this when $k = 1$. Say $\bar{a} = ([x_{i_0}], \dots, [x_{i_{n-1}}])$ and choose any $m > \max\{i_0, \dots, i_{n-1}\}$. Let $f : n \rightarrow m$ be given by $f(j) = i_j$. By the definition of cov_n , $P_{n,m}^f(\mathbf{a}_m) = \mathbf{b}$. As $\langle \mathbf{a}_k : k \in \omega \rangle$ is cofinal, there is some $\ell \geq m$ and $h : n+1 \rightarrow \ell$ such that $\text{id} \circ f = h \circ \text{id}$ and $P^h(\mathbf{a}_\ell) = \mathbf{c}$. The commutativity of the functions gives $h(j) = f(j)$ for all $j < n$, so taking $c = [x_{h(n)}]$ we have $\text{cov}_{n+1}(\bar{a}c) = \mathbf{c}$, as required. \square

Using Lemma 3.10, we now verify that the statements of the proposition. First, it is easily checked that \mathcal{F} is a sharp back-and-forth system on M . Second, taking $n = 0$ and k arbitrary, we see that $\text{cov}_k : M^k \rightarrow U_k$ is onto, so \mathbb{B} is isomorphic to the flattening of (M, \mathcal{F}) , with cov_n playing the role of π_n .

A third application of Lemma 3.10 verifies uniqueness. Suppose (M_1, \mathcal{F}_1) and (M_2, \mathcal{F}_2) , along with the maps $\{\text{cov}_n^1\}, \{\text{cov}_n^2\}$, respectively are each constructed from \mathbb{B} via different cofinal sequences. We argue that (M_1, \mathcal{F}_1) and (M_2, \mathcal{F}_2) are isomorphic as $L^\#$ -structures. To see this, consider the set of all partial bijections $\Phi : \bar{a} \rightarrow \bar{b}$, where $\bar{a} \in M_1^n$, $\bar{b} \in M_2^n$, and $\text{cov}_n^1(\bar{a}) = \text{cov}_n^2(\bar{b})$. Clearly, every such Φ preserves quantifier-free types, as, letting $\mathbf{c} = \text{cov}_n^1(\bar{a}) = \text{cov}_n^2(\bar{b})$, the L -quantifier-free types $\text{qftp}_{M_1}(\bar{a})$ and $\text{qftp}_{M_2}(\bar{b})$ can be read off from $\Delta_{\mathbf{c}}$.

But, by Lemma 3.10, this family of partial bijections Φ is a back and forth system. As both M_1 and M_2 are countable, it follows that there is an L -isomorphism $\Phi^* : M_1 \rightarrow M_2$ that satisfies $\text{cov}_n^1(\bar{a}) = \text{cov}_n^2(\Phi^*(\bar{a}))$ for every $\bar{a} \in M_1^n$. But this condition implies that Φ^* is also E_n -preserving: If $E_n(\bar{a}, \bar{a}')$ in M_1 , then by definition of E_n , we have $\text{cov}_n^1(\bar{a}) = \text{cov}_n^1(\bar{a}')$. From the sentence above, $\text{cov}_n^2(\Phi^*(\bar{a})) = \text{cov}_n^2(\Phi^*(\bar{a}'))$, so $E_n(\Phi^*(\bar{a}), \Phi^*(\bar{a}'))$ holds in M_2 as well. Thus, Φ^* is an $L^\#$ -isomorphism from (M_1, \mathcal{F}_1) onto (M_2, \mathcal{F}_2) .

3.4 Hausdorff flat structures are potential canonical Scott sentences

Fix a countable language L . The aim of this section is to distinguish a canonical class Hdf^* of Hausdorff flat structures in \mathbb{V} . We will see that every Hausdorff flat structure will be isomorphic [back-and-forth equivalent] to exactly one canonical structure, so we can

naturally identify Hdf^* with isomorphism classes [back and forth classes]. Further, we will show that Hdf^* is naturally bijective with $\text{CSS}(L)_{\text{ptl}}$.

As a preamble, we describe some expansions of L . For each $k \in \omega$, let $L(\bar{c}_k) := L \cup \{c_0, \dots, c_{k-1}\}$ (so $L_0 = L$) where $\{c_0, \dots, c_{k-1}\}$ are new constants. For any $L(\bar{c}_k)$ -structure $N = (M, \bar{a}) \in HC$, let $\text{css}_k(M, \bar{a})$ denote the canonical Scott sentence of $N = (M, \bar{a})$. For any sentence $\sigma \in (L(\bar{c}_k))_{\infty, \omega}$, let $\hat{\sigma}(\bar{x})$ be the $(L(\bar{c}_k))_{\infty, \omega}$ formula formed by replacing each c_i by x_i . That is, $\hat{\sigma}(\bar{c})$ is persistently equivalent to σ , i.e., for any forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ and any $L(\bar{c}_k)^{\mathbb{V}[G]}$ -structure $N = (M, \bar{a})$, $N \models \sigma$ if and only if $M \models \hat{\sigma}(\bar{a})$.

Definition 3.11. For any language L , call two $L_{\infty, \omega}$ -sentences σ and ψ *explicitly contradictory* if $\sigma \wedge \psi$ has no models in any forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$. An $L_{\infty, \omega}$ -sentence δ is *persistently valid* if, for every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$, every $M \in \text{Mod}_{\mathbb{V}[G]}(L)$ models δ .

As examples, note that for any $k \in \omega$, it follows from Fact 1.6 that any two canonical Scott sentences $\text{css}_k(M, \bar{a})$ and $\text{css}_k(M', \bar{a}')$ in $(L(\bar{c}_k))_{\omega_1, \omega}$ are explicitly contradictory or equal. Note that σ and ψ are explicitly contradictory $(L(\bar{c}_k))_{\infty, \omega}$ -sentences if and only if the $L_{\infty, \omega}$ -sentence $\forall \bar{x}(\hat{\sigma}(\bar{x}) \rightarrow \neg \hat{\psi}(\bar{x}))$ is persistently valid.

We begin in the countable world, i.e., in HC . Recalling Definition 3.1, let $\text{Flat}_{HC}(L) := \text{Mod}_{HC}(\text{Ax}^b(L))$, the flat L^b -structures in HC . Note that as satisfaction is absolute between transitive models of set theory, the classes $\text{Flat}_{HC}(L)$ and $\text{Mod}_{HC}(L)$ are strongly definable, i.e., are defined by HC -invariant formulas.

Definition 3.12. Let $\text{can} : \text{Mod}_{HC}(L) \rightarrow \text{Flat}_{HC}(L)$ be the following canonical map. Given $M \in \text{Mod}_{HC}(L)$, $\text{can}(M)$ is the natural flat structure isomorphic to M^b with universe

$$\{\text{css}_k(M, \bar{a}) : k \in \omega, \bar{a} \in M^k\}$$

That is, interpret U_k as $U_k(\text{can}(M)) = \{\text{css}_k(M, \bar{a}) : \bar{a} \in M^k\}$; for a quantifier-free L -formula $\alpha(x_0, \dots, x_{k-1})$, $\text{can}(M) \models \alpha^b(\text{css}_k(M, \bar{a}))$ if and only if $M \models \alpha(\bar{a})$; and for an injective $f : k \rightarrow n$, $P_{k,n}^f(\text{css}_n(M, \bar{a})) = \text{css}(M, \bar{a} \upharpoonright_f)$.

For $M \in \text{Mod}_{HC}(L)$, the structure $\text{can}(M)$ depends only on the isomorphism type of M .

Lemma 3.13. *If $M, N \in \text{Mod}_{HC}(L)$, then $M \cong N$ if and only if $\text{can}(M) = \text{can}(N)$.*

Proof. Right to left is immediate, as taking $k = 0$, $U_0(\text{can}(M)) = U_0(\text{can}(N)) = \{\text{css}(M)\}$. Thus, $M \cong N$ by the salient property of canonical Scott sentences. Now assume $h : M \rightarrow N$ is an L -isomorphism. Then, for any $k \in \omega$ and any $\bar{a} \in M^k$,

$(M, \bar{a}) \cong (N, h(\bar{a}))$ as $L(\bar{c}_k)$ -structures, hence $css_k(M, \bar{a}) = css_k(N, h(\bar{a}))$. Thus, for each k we have

$$\{css_k(M, \bar{a}) : \bar{a} \in M^k\} = \{css_k(N, \bar{b}) : \bar{b} \in N^k\}$$

so the universes of $\text{can}(M)$ and $\text{can}(N)$ are equal. It follows easily that the flat structures $\text{can}(M)$ and $\text{can}(N)$ are equal as well. \square

Using Lemma 3.13 and the uniqueness condition of Proposition 3.7, the following map is well-defined.

Definition 3.14. Let $\text{can}^b : \text{Flat}_{HC}(L) \rightarrow \text{Flat}_{HC}(L)$ be defined by $\text{can}^b(\mathbb{B}) := \text{can}(M)$, where M is any L -structure obtained from Proposition 3.7, i.e., where $(M, \mathcal{F})^b \cong \mathbb{B}$ for some \mathcal{F} .

Let Hdf_{HC}^* denote the image of can^b , which is also the image of can .

The notation is justified by the following lemma.

Lemma 3.15. 1. Every $\mathbb{B}' \in \text{Hdf}_{HC}^*$ is Hausdorff.

2. For all Hausdorff $\mathbb{B}_1, \mathbb{B}_2 \in \text{Flat}_{HC}(L)$, $\mathbb{B}_1 \cong \mathbb{B}_2$ if and only if $\text{can}^b(\mathbb{B}_1) = \text{can}^b(\mathbb{B}_2)$.
3. $\text{can}^b : \text{Flat}_{HC}(L) \rightarrow \text{Flat}_{HC}(L)$ is a projection onto Hdf_{HC}^* , i.e., $\text{can}^b(\text{can}^b(\mathbb{B})) = \text{can}^b(\mathbb{B})$ for all $\mathbb{B} \in \text{Flat}_{HC}(L)$.
4. $\text{Hdf}_{HC}^* = \{\mathbb{B} \in \text{Flat}_{HC}(L) : \text{can}^b(\mathbb{B}) = \mathbb{B}\}$.

Proof. (1) Fix distinct $\mathbf{a} \neq \mathbf{b} \in \mathbb{B}'$. For definiteness, assume $\mathbf{a}, \mathbf{b} \in U_k(\mathbb{B}')$. Choose any $M \in \text{Mod}_{HC}(L)$ with $\text{can}(M) = \mathbb{B}'$ and suppose $\mathbf{a} = css_k(M, \bar{a})$ and $\mathbf{b} = css_k(M, \bar{b})$. Let $\sigma := css_k(M, \bar{a})$ and $\psi := css_k(M, \bar{b})$. As these are distinct canonical Scott sentences, σ and ψ are explicitly contradictory. Thus, $M \models \forall \bar{x}(\hat{\sigma}(\bar{x}) \rightarrow \neg \hat{\psi}(\bar{x}))$. From this, we have $M \models \hat{\psi}(\bar{b}) \wedge \neg \hat{\psi}(\bar{a})$. As $\mathbb{B}' = M^b$, $\mathbb{B}' \models \hat{\psi}^b(\mathbf{b}) \wedge \neg \hat{\psi}^b(\mathbf{a})$. Thus, \mathbb{B}' is Hausdorff.

(2) Since for $\ell = 1, 2$, each \mathbb{B}_ℓ is Hausdorff, by Proposition 3.7 we can choose L -structures M_ℓ such that $M_\ell^b \cong \mathbb{B}_\ell$. First, assume $\text{can}^b(\mathbb{B}_1) = \text{can}^b(\mathbb{B}_2)$. Then $\text{can}(M_1) = \text{can}(M_2)$, so $M_1 \cong M_2$ by Lemma 3.13. It follows by the uniqueness of flattenings $\mathbb{B}_1 \cong \mathbb{B}_2$ as L^b -structures. Conversely, assume $\mathbb{B}_1 \cong \mathbb{B}_2$. Then, by the uniqueness clause of Proposition 3.7, $M_1 \cong M_2$, so again by Lemma 3.13, $\text{can}(M_1) = \text{can}(M_2)$. Thus, $\text{can}^b(\mathbb{B}_1) = \text{can}^b(\mathbb{B}_2)$.

(3) Given $\mathbb{B} \in \text{Flat}_{HC}(L)$, let $\mathbb{B}' = \text{can}^b(\mathbb{B})$. Note that the same model M has $M^b \cong \mathbb{B}$ and $M^b \cong \mathbb{B}'$. Thus, $\text{can}^b(\mathbb{B}) = \text{can}^b(\mathbb{B}')$ as required.

(4) Choose any $\mathbb{B} \in \text{Hdf}_{HC}^*$. That $\text{can}^b(\mathbb{B}) = \mathbb{B}$ is given by (3). Conversely, assume $\text{can}^b(\mathbb{B}) = \mathbb{B}$. Then trivially, $\mathbb{B} \in \text{Hdf}_{HC}^*$ by its definition. \square

So far, in the HC world, the surjective maps $\text{can} : \text{Mod}_{HC}(L) \rightarrow \text{Hdf}_{HC}^*$ and $\text{can}^b : \text{Flat}_{HC}(L) \rightarrow \text{Hdf}_{HC}^*$ are somewhat dual. However, this similarity will dissipate when we take potentials. This is due not only to the fact that can^b is a projection, but also because there is a preferred homomorphism $j : \mathbb{B} \rightarrow \text{can}^b(\mathbb{B})$ that we now describe.

Definition 3.16. Suppose $\mathbb{B}, \mathbb{B}' \in \text{Flat}_{HC}(L)$. A surjective homomorphism $j : \mathbb{B} \rightarrow \mathbb{B}'$ is an onto map such that, for all $\mathbf{a} \in \mathbb{B}$, $\mathbb{B} \models U_k(\mathbf{a})$ iff $\mathbb{B}' \models U_k(j(\mathbf{a}))$, for all quantifier-free L -formula $\alpha(\bar{x})$ $\mathbb{B} \models \alpha^b(\mathbf{a})$ iff $\mathbb{B}' \models \alpha_k^b(j(\mathbf{a}))$, and j commutes with each $P_{k,n}^f$.

Recall that by Definition 2.6, for any flat structure \mathbb{B} and any $\mathbf{a} \in \mathbb{B}$, $\text{b-tp}_{\mathbb{B}}(\mathbf{a}) = \{\varphi^b(z) : \mathbb{B} \models \varphi^b(\mathbf{a})\}$. The following Lemma is proved by induction on the complexity of $L_{\infty, \omega}$ -formulas.

Lemma 3.17. Let $j : \mathbb{B} \rightarrow \mathbb{B}'$ be any surjective homomorphism of flat structures. Then, for any $\mathbf{a} \in \mathbb{B}$, $\text{b-tp}_{\mathbb{B}}(\mathbf{a}) = \text{b-tp}_{\mathbb{B}'}(j(\mathbf{a}))$.

Lemma 3.18. For any $\mathbb{B}, \mathbb{B}' \in \text{Flat}_{HC}(L)$ with \mathbb{B}' Hausdorff, there is at most one surjective homomorphism $j : \mathbb{B} \rightarrow \mathbb{B}'$. In particular, for any $\mathbb{B} \in \text{Flat}_{HC}(L)$, there is a unique surjective homomorphism $j : \mathbb{B} \rightarrow \text{can}^b(\mathbb{B})$.

Proof. Choose surjective homomorphisms $j, j^* : \mathbb{B} \rightarrow \mathbb{B}'$ and choose any $\mathbf{a} \in \mathbb{B}$. Let $\mathbf{b} := j(\mathbf{a})$ and $\mathbf{b}^* := j^*(\mathbf{a})$. To show that $\mathbf{b} = \mathbf{b}^*$, by applying Lemma 3.17 twice, we have $\text{b-tp}_{\mathbb{B}'}(\mathbf{b}) = \text{b-tp}_{\mathbb{B}'}(\mathbf{b}^*)$. Thus, $\mathbf{b} = \mathbf{b}^*$ since \mathbb{B}' is Hausdorff.

For the second sentence, choose any $\mathbb{B} \in \text{Flat}_{HC}(L)$, and let (M, cov) be from Proposition 3.7 such that the corresponding $(M, \mathcal{F})^b \cong \mathbb{B}$. It is easily checked that the map $j : \mathbb{B} \rightarrow \text{can}^b(\mathbb{B})$ defined by $j(\mathbf{a}) = \text{css}_k(M, \bar{a})$ for some or any $\bar{a} \in M^k$ such that $\text{cov}_k(\bar{a}) = \mathbf{a}$ is well defined and is a surjective homomorphism. As $\text{can}^b(\mathbb{B})$ is Hausdorff from Lemma 3.15(1), the uniqueness of j follows from the first sentence. \square

Lemma 3.18 gives that the function below is well defined.

Definition 3.19. Let $\text{cmap} : \text{Flat}_{HC}(L) \rightarrow HC$ be the function described as: $\text{cmap}(\mathbb{B})$ is the unique surjective homomorphism $j : \mathbb{B} \rightarrow \text{can}^b(\mathbb{B})$.

We glean one easy consequence of this. The characterization will be useful in proving both that the class of Hausdorff structures in HC is HC -invariant, and that ‘being Hausdorff’ is absolute among forcing extensions.

Lemma 3.20. The following are equivalent for $\mathbb{B} \in \text{Flat}_{HC}(L)$.

1. \mathbb{B} is Hausdorff;
2. $\text{cmap}(\mathbb{B})$ is injective; and

3. $\text{cmap}(\mathbb{B})$ is an L^b -isomorphism.

Proof. (1) \Rightarrow (2): Choose any $\mathbb{B} \in \text{Flat}_{HC}(L)$ and denote $\text{can}^b(\mathbb{B})$ as \mathbb{B}' and $\text{cmap}(\mathbb{B})$ as $j : \mathbb{B} \rightarrow \mathbb{B}'$. First, assume \mathbb{B} is Hausdorff and choose $\mathbf{a} \neq \mathbf{b}$ from \mathbb{B} . As \mathbb{B} is Hausdorff, $b\text{-tp}_{\mathbb{B}}(\mathbf{a}) \neq b\text{-tp}_{\mathbb{B}}(\mathbf{b})$, so $b\text{-tp}_{\mathbb{B}'}(j(\mathbf{a})) \neq b\text{-tp}_{\mathbb{B}'}(j(\mathbf{b}))$ by Lemma 3.17. Thus, $j(\mathbf{a}) \neq j(\mathbf{b})$, so j is injective.

(2) \Rightarrow (3) is immediate from the definitions.

(3) \Rightarrow (1): Since \mathbb{B}' is Hausdorff by Lemma 3.15(1), (3) implies \mathbb{B} is as well. \square

Let $\text{Hdf}_{HC} := \{\mathbb{B} \in \text{Flat}_{HC}(L) : \mathbb{B} \text{ is Hausdorff}\}$. We consider forcing extensions $\mathbb{V}[G]$ of \mathbb{V} .

Lemma 3.21. 1. can , can^b , and cmap are strongly definable and are persistently functions.

2. Both Hdf_{HC}^* and Hdf_{HC} are strongly definable sets.

3. The identity map $id : \text{Hdf}_{HC}^* \rightarrow \text{Flat}_{HC}$ is a persistent, strongly definable cross section of $\text{can}^b : \text{Flat}_{HC} \rightarrow \text{Hdf}_{HC}^*$.

Proof. (1) We first show can is HC-invariant and is persistently a function. Choose any $M \in \text{Mod}_{HC}(L)$ and let $\mathbb{V}[G] \supseteq \mathbb{V}$ be any forcing extension. As css is strongly definable, for every $\bar{a} \in M^k$, we have $\text{css}_k(M, \bar{a})^{\mathbb{V}[G]} = \text{css}_k(M, \bar{a})^{\mathbb{V}}$. It follows from this that $\text{can}(M)^{\mathbb{V}[G]} = \text{can}(M)^{\mathbb{V}}$. Thus, can is strongly definable. The argument that can is well defined also holds in $\mathbb{V}[G]$, hence can is persistently a function.

The argument for can^b is similar. Given $\mathbb{B} \in \text{Flat}_{HC}(L)$, note that any $M \in \mathbb{V}$ witnessing $\text{can}^b(\mathbb{B}) = \text{can}(M)$ also witnesses this in $\mathbb{V}[G]$. Thus, $\text{can}^b(\mathbb{B})^{\mathbb{V}[G]} = \text{can}^b(\mathbb{B})^{\mathbb{V}}$. That can^b is persistently a function follows by applying Lemma 3.13 in $\mathbb{V}[G]$.

Now, for cmap , choose $\mathbb{B} \in \text{Flat}_{HC}(L)$ and let $j := \text{cmap}(\mathbb{B})$ in \mathbb{V} . Then easily, in $\mathbb{V}[G]$ we also have $j : \mathbb{B} \rightarrow \text{can}^b(\mathbb{B})$ is a surjective homomorphism. By applying Lemma 3.18 to j in $\mathbb{V}[G]$ it is the only one, hence $\text{cmap}(\mathbb{B})^{\mathbb{V}[G]} = j$ as well. Thus, cmap is strongly definable and is persistently a function.

(2) For Hdf_{HC}^* we use Lemma 3.15(4). Choose $\mathbb{B} \in HC^{\mathbb{V}}$. First, suppose $\mathbb{B} \in (\text{Hdf}_{HC}^*)^{\mathbb{V}}$ and let $\mathbb{V}[G] \supseteq \mathbb{V}$ be any forcing extension. Then $\mathbb{B} = \text{can}(\mathbb{B})$ in both \mathbb{V} and $\mathbb{V}[G]$, so $\mathbb{B} \in (\text{Hdf}_{HC}^*)^{\mathbb{V}[G]}$ by applying Lemma 3.15(4) in $\mathbb{V}[G]$. Conversely, assume $\mathbb{B} \in (\text{Hdf}_{HC}^*)^{\mathbb{V}[G]}$. As $\text{Flat}(L)$ is absolute between forcing extensions, we know $\mathbb{B} \in \text{Flat}_{HC}(L)$. By Lemma 3.15(4) in $\mathbb{V}[G]$, $\text{can}^b(\mathbb{B}) = \mathbb{B}$ in $\mathbb{V}[G]$. But, as can^b is strongly definable and $\mathbb{B} \in \mathbb{V}$, this implies $\text{can}^b(\mathbb{B}) = \mathbb{B}$ in \mathbb{V} as well. Thus, $\mathbb{B} \in (\text{Hdf}_{HC}^*)^{\mathbb{V}}$.

Showing that Hdf_{HC} is strongly definable is similar, using Lemma 3.20 in both \mathbb{V} and $\mathbb{V}[G]$. On one hand, if $\mathbb{B} \in \text{Hdf}_{HC}(L)$ in \mathbb{V} , then letting $j = \text{cmap}(\mathbb{B})$, $j : \mathbb{B} \rightarrow \text{can}^b(\mathbb{B})$ is injective, both in \mathbb{V} and (from above) in $\mathbb{V}[G]$. Thus, $\mathbb{B} \in \text{Hdf}_{HC}(L)$ in $\mathbb{V}[G]$ as well.

Conversely, if $\mathbb{B} \in HC^{\mathbb{V}}$ and $\mathbb{B} \in \text{Hdf}_{HC}(L)^{\mathbb{V}[G]}$. Then $j : \mathbb{B} \rightarrow \text{can}^b(\mathbb{B})$ is injective in both $\mathbb{V}[G]$ and \mathbb{V} , so $\mathbb{B} \in \text{Hdf}_{HC}(L)$.

(3) This is obvious by unpacking the definitions. \square

It is noteworthy that the verification of Hdf^* being strongly definable used the characterization of it in terms of the fixed points of can^b as opposed to anything about the strongly definable function can . This asymmetry will be magnified when we show that $\text{Hdf}_{\text{ptl}}^*$ is the image of $\text{can}_{\text{ptl}}^b$, but the image of can_{ptl} might be smaller.

We now leave the HC world and enter into the world of all sets by applying the ptl operator. Note that as ‘being a structure’ and ‘being a flat structure’ are absolute, $\text{Mod}(L)_{\text{ptl}} = \text{Mod}(L)$, the class of all L -structures, and similarly, $\text{Flat}(L)_{\text{ptl}}$ is the class of all flat L^b -structures, which we denote by $\text{Flat}(L)$. In light of Lemma 3.21, and Lemma 1.3 we have the following:

- $\text{can}_{\text{ptl}} : \text{Mod}(L) \rightarrow \text{Hdf}_{\text{ptl}}^*$ defined as: $\text{can}_{\text{ptl}}(M) = \text{can}(M)^{\mathbb{V}[G]}$ for some/every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ with $M \in HC^{\mathbb{V}[G]}$;
- $\text{can}_{\text{ptl}}^b : \text{Flat}(L) \rightarrow \text{Hdf}_{\text{ptl}}^*$ defined as: $\text{can}_{\text{ptl}}^b(\mathbb{B}) = \text{can}(\mathbb{B})^{\mathbb{V}[G]}$ for some/every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$; with $\mathbb{B} \in HC^{\mathbb{V}[G]}$;
- $\text{Fix}(\text{can}_{\text{ptl}}^b) = \text{Fix}(\text{can}_{\text{ptl}})$;
- $\text{cmap}_{\text{ptl}} : \text{Flat}_{\mathbb{V}}(L) \rightarrow \mathbb{V}$ defined as: $\text{cmap}_{\text{ptl}}(\mathbb{B})$ is the unique $j : \mathbb{B} \rightarrow \text{can}^b(\mathbb{B})$, as computed in some/every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ with $\mathbb{B} \in HC^{\mathbb{V}[G]}$.

We isolate the following for emphasis.

Lemma 3.22. *The class functional $\text{can}_{\text{ptl}}^b : \text{Flat}(L) \rightarrow \text{Hdf}_{\text{ptl}}^*$ is a (surjective) projection.*

Proof. This follows immediately from Lemmas 1.4 and 3.21. \square

Lemma 3.23. *For any flat structure \mathbb{B} , $\text{can}_{\text{ptl}}^b(\mathbb{B})$ is Hausdorff.*

Proof. Fix a flat structure $\mathbb{B} \in \mathbb{V}$ and choose any forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ such that $\mathbb{B} \in HC^{\mathbb{V}[G]}$. Since $\mathbb{B} \in \mathbb{V}$, $\text{can}_{\text{ptl}}^b(\mathbb{B}) \in \mathbb{V}$, as is the function $\text{cmap}_{\text{ptl}}(\mathbb{B})$. As well, since $\text{can}_{\text{ptl}}^b(\mathbb{B})$ is equal to $\text{can}^b(\mathbb{B})^{\mathbb{V}[G]}$, every element of it is equal to $\text{css}_k(M, \bar{a})$ for some $(M, \bar{a}) \in \mathbb{V}[G]$. Even though M need not be in \mathbb{V} , we do know that for every $\mathbf{a} \in \mathbb{B}$, $j(\mathbf{a})$ is a sentence of $(L(\bar{c}_k))_{\infty, \omega}$ and furthermore, for distinct \mathbf{a}, \mathbf{b} from \mathbb{B} we have that the sentences $j(\mathbf{a})$ and $j(\mathbf{b})$ are explicitly contradictory in $\mathbb{V}[G]$, and hence are explicitly contradictory in \mathbb{V} as well.

To see that \mathbb{B} is Hausdorff, we now work in \mathbb{V} . Choose distinct $\mathbf{a}, \mathbf{b} \in \mathbb{B}$. For definiteness, say $\mathbf{a}, \mathbf{b} \in U_k(\mathbb{B})$. Write $j(\mathbf{a})$ as σ and $j(\mathbf{b})$ as ψ . As these are explicitly contradictory, we have that $\forall \bar{x}(\hat{\sigma}(\bar{x}) \rightarrow \neg \hat{\psi}(\bar{x}))$ is persistently valid.

But, in $\mathbb{V}[G]$, as $(M, \bar{a}) \models \sigma$ and $(M, \bar{b}) \models \psi$, we have $M \models \hat{\sigma}(\bar{a}) \wedge \hat{\psi}(\bar{b})$. From this, it follows that in $\mathbb{V}[G]$, $\mathbb{B} \models \hat{\sigma}^b(\mathbf{a}) \wedge \hat{\psi}^b(\mathbf{b})$. As satisfaction is absolute, the same holds in \mathbb{V} . Via the implication above, $\mathbb{B} \models \neg \hat{\psi}^b(\mathbf{a})$, so $\hat{\psi}^b$ witnesses that $b\text{-tp}_{\mathbb{B}}(\mathbf{a}) \neq b\text{-tp}_{\mathbb{B}}(\mathbf{b})$. \square

Lemma 3.24. *The notion of ‘ \mathbb{B} is Hausdorff’ is absolute between forcing extensions. Thus, Hdf_{ptl} denotes the class of Hausdorff flat structures.*

Proof. Suppose $\mathbb{V}[G] \supseteq \mathbb{V}$ is any forcing extension and assume $\mathbb{B} \in \mathbb{V}$. That \mathbb{B} Hausdorff in \mathbb{V} implies \mathbb{B} Hausdorff in $\mathbb{V}[G]$ is easy, since for different elements $\mathbf{a} \neq \mathbf{b}$ of \mathbb{B} if $\mathbb{B} \models \varphi^b(\mathbf{a}) \wedge \neg \varphi^b(\mathbf{b})$ in \mathbb{V} , then, as $\varphi^b \in \mathbb{V}$, the same holds in $\mathbb{V}[G]$.

For the converse, assume $\mathbb{B} \in \mathbb{V}$ is Hausdorff in $\mathbb{V}[G]$. As $\mathbb{B} \in \mathbb{V}$, both $\text{can}_{\text{ptl}}^b(\mathbb{B})$ and $j := \text{cmap}_{\text{ptl}}(\mathbb{B})$ are in \mathbb{V} by Lemma 1.3. Choose a forcing extension $\mathbb{V}[G][H]$ in which $\mathbb{B} \in HC^{\mathbb{V}[G][H]}$. Note that by the first paragraph, \mathbb{B} is Hausdorff in $\mathbb{V}[G][H]$ as well. By definition of cmap_{ptl} , $j : \mathbb{B} \rightarrow \text{can}^b(\mathbb{B})$ is equal to $\text{cmap}(\mathbb{B})$ in $\mathbb{V}[G][H]$. Thus, working in $\mathbb{V}[G][H]$, j is an L^b -isomorphism by Lemma 3.20.

Now we work in \mathbb{V} . Since $j \in \mathbb{V}$ and ‘being an isomorphism’ is absolute, $\mathbb{B} \cong \text{can}_{\text{ptl}}^b B$. As $\text{can}_{\text{ptl}}^b(\mathbb{B})$ is Hausdorff by Lemma 3.23, so is \mathbb{B} . \square

Given this absoluteness, we can now recast our definitions as follows.

Definition 3.25. Let Hdf consist of all Hausdorff flat structures $\mathbb{B} \in \mathbb{V}$ and let Hdf^* consist of all flat structures $\mathbb{B} \in \mathbb{V}$ such that $\text{can}_{\text{ptl}}^b(\mathbb{B}) = \mathbb{B}$.

The above agrees with our previous definitions in the sense that for any $\mathbb{B} \in \text{Flat}(L)$, $\mathbb{B} \in \text{Hdf}$ (resp. $\mathbb{B} \in \text{Hdf}^*$) if and only if $(\mathbb{B} \in \text{Hdf}_{HC})^{\mathbb{V}[G]}$ (resp. $(\mathbb{B} \in \text{Hdf}_{HC}^*)^{\mathbb{V}[G]}$) for some/every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ with $\mathbb{B} \in HC^{\mathbb{V}[G]}$. That is, $\text{Hdf} = \text{Hdf}_{\text{ptl}}$ and $\text{Hdf}^* = \text{Hdf}_{\text{ptl}}^*$.

Lemma 3.26. *The following are equivalent for Hausdorff flat structures $\mathbb{B}_1, \mathbb{B}_2$ (of any size):*

1. $\mathbb{B}_1 \equiv_{\infty, \omega} \mathbb{B}_2$;
2. $\mathbb{B}_1 \cong \mathbb{B}_2$;
3. *There is a unique L^b -isomorphism $h : \mathbb{B}_1 \rightarrow \mathbb{B}_2$;*
4. $\text{can}_{\text{ptl}}^b(\mathbb{B}_1) = \text{can}_{\text{ptl}}^b(\mathbb{B}_2)$.

Proof. As (3) \Rightarrow (2) \Rightarrow (1) is obvious, it suffices to prove (1) \Rightarrow (4) and (4) \Rightarrow (3). For both parts, choose a forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ in which both $\mathbb{B}_1, \mathbb{B}_2$ are in $HC^{\mathbb{V}[G]}$.

(1) \Rightarrow (4): Assume $\mathbb{B}_1 \equiv_{\infty, \omega} \mathbb{B}_2$. Then, classically, $\mathbb{B}_1 \cong \mathbb{B}_2$ in $\mathbb{V}[G]$. Thus, by Lemma 3.15(2), $\text{can}^b(\mathbb{B}_1) = \text{can}^b(\mathbb{B}_2)$ in $\mathbb{V}[G]$. But, by definition of $\text{can}_{\text{ptl}}^b$, we have $\text{can}_{\text{ptl}}^b(\mathbb{B}_\ell) = \text{can}^b(\mathbb{B}_\ell)^{\mathbb{V}[G]}$ for $\ell = 1, 2$, hence $\text{can}_{\text{ptl}}^b(\mathbb{B}_1) = \text{can}_{\text{ptl}}^b(\mathbb{B}_2)$.

(4) \Rightarrow (3): By definition of $\text{can}_{\text{ptl}}^b$, we have $\text{can}_{\text{ptl}}^b(\mathbb{B}_\ell) = \text{can}^b(\mathbb{B}_\ell)^{\mathbb{V}[G]}$ for $\ell = 1, 2$, hence $\text{can}^b(\mathbb{B}_1) = \text{can}^b(\mathbb{B}_2)$ in $\mathbb{V}[G]$. Thus, by Lemma 3.15(2), there is some L^b -isomorphism $h : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ in $\mathbb{V}[G]$. We need to show that $h \in \mathbb{V}$, and moreover, there cannot be any other L^b -isomorphism. For both of these, note that since both \mathbb{B}_1 and \mathbb{B}_2 are Hausdorff, distinct elements have distinct b -tp's. Using h , we see that in \mathbb{V} , the sets $\{b\text{-tp}(\mathbf{a}) : \mathbf{a} \in \mathbb{B}_1\}$ and $\{b\text{-tp}(\mathbf{b}) : \mathbf{b} \in \mathbb{B}_2\}$ are equal. Let $h_0 \in \mathbb{V}$ be the unique bijection $h_0 : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ that is b -tp-preserving, i.e., $b\text{-tp}(h_0(\bar{a})) = b\text{-tp}(\bar{a})$ for all $\mathbf{a} \in \mathbb{B}_1$. As the uniqueness of h_0 is persistent in forcing extensions, it follows that $h_0 = h$. Thus, $h \in \mathbb{V}$ and is the unique isomorphism between \mathbb{B}_1 and \mathbb{B}_2 . \square

From this, we get the following useful corollary.

Corollary 3.27. *Let M be any L -structure and let M^b be the canonical flattening of M . Then for any L -structure N ,*

$$M \equiv_{\infty, \omega} N \quad \text{if and only if} \quad N^b \cong M^b$$

Proof. Say $M \equiv_{\infty, \omega} N$. Pass to any forcing extension $\mathbb{V}[G]$ in which both M and N are countable, hence $M \cong N$ in $\mathbb{V}[G]$. Thus, $M^b \cong N^b$ in $\mathbb{V}[G]$. Pulling back, we have $M^b \equiv_{\infty, \omega} N^b$ in \mathbb{V} . But, as both M^b and N^b are Hausdorff, there is a unique isomorphism between them by Lemma 3.26.

Conversely, suppose $N^b \cong M^b$, say via the (unique) map h . Let $\mathcal{F} = \{(\bar{a}, \bar{b}) \in N^{<\omega} \times M^{<\omega} : h(\pi(\bar{a})) = \pi(\bar{b})\}$, and it suffices to show that \mathcal{F} is a back-and-forth system between N and M . But this follows easily from Lemma 3.10. \square

The following summarizes the key relationships.

Conclusion 3.28. *There are natural bijections between three classes:*

1. (Hdf, \cong), the class of isomorphism types of Hausdorff flat structures;
2. The class Hdf* ; and
3. The class $CSS(L)_{\text{ptl}}$ discussed in [16].

By contrast, the subclass of Hdf* consisting of all $\text{can}_{\text{ptl}}(M)$ for $M \in \text{Mod}(L)$ is less interesting. In the notation of [16], it is bijective with $CSS(L)_{\text{sat}}$.

3.5 Relativizing to $\text{Mod}(\Phi)$

In the whole of the previous subsection, we were working with the class of all L -structures and all flat L^b -structures. However, for applications we want to restrict to e.g., models of a particular sentence of $L_{\omega_1, \omega}$. For a fixed $\Phi \in L_{\omega_1, \omega}$, let $\text{Mod}(\Phi)$ and $\text{Flat}(\Phi)$ denote the class of models of Φ and flat models of Φ^b , respectively. Similarly, $\text{Hdf}(\Phi)$ denotes the class of Hausdorff flat structures that model Φ^b and $\text{Hdf}^*(\Phi)$ is the image of $\text{Flat}(\Phi)$ under $\text{can}_{\text{ptl}}^b$. Owing to the fact that satisfaction is absolute, all of the results in the previous subsection go through, which leads to the following conclusion.

Theorem 3.29. *Let $\Phi \in L_{\omega_1, \omega}$. There are natural bijections between three classes:*

1. $(\text{Hdf}(\Phi), \cong)$, the class of isomorphism types of Hausdorff flat models of Φ ;
2. The class $\text{Hdf}^*(\Phi)$; and
3. The class $\text{CSS}(\Phi)_{\text{ptl}}$ discussed in [16].

In [16], we defined a sentence $\Phi \in L_{\omega_1, \omega}$ to be *grounded* if $\text{CSS}(\Phi)_{\text{sat}} = \text{CSS}(\Phi)_{\text{ptl}}$. Here, we obtain a more malleable equivalence. Recall that by Proposition 3.7, every *countable* Hausdorff flat structure is isomorphic to M^b for some $M \in \text{Mod}(\Phi)$.

Corollary 3.30. *A sentence $\Phi \in L_{\omega_1, \omega}$ is grounded if and only if every Hausdorff flat structure (of any size) is isomorphic to M^b for some $M \in \text{Mod}(\Phi)$.*

4 Flat structures, sharp expansions of a countable M , and $\text{Aut}(M)$

Let M be a countable L -structure with universe ω and let $\varphi_M \in L_{\omega_1, \omega}$ be any Scott sentence for M . Among all flattenings $(M, \mathcal{F})^b$, we know there is a unique (up to unique isomorphism) countable, Hausdorff flat structure \mathbb{B} , namely the canonical flattening M^b of M . As $M \models \varphi_M$, it follows that $\mathbb{B} \models \varphi_M^b \wedge \text{Ax}^b(L)$. We will see that the other countable models of $\varphi_M^b \wedge \text{Ax}^b(L)$ correspond to closed subgroups of $\text{Aut}(M)$ and we analyze the complexity of this correspondence.

In what follows, for $\Phi \in L_{\omega_1, \omega}$, $\text{Mod}_\omega(\Phi)$ denotes the Polish space of models of Φ with universe ω . Note that $(\text{Mod}_\omega(\varphi_M \wedge \text{Ax}^\sharp(L)), \cong_\sharp)$ is HC-bireducible to $\{\text{sharp expansions } (M, \mathcal{F})\} / \cong_{L^\sharp}$.

From Proposition 3.7 there is an HC-bireduction between $\text{Mod}_\omega(\varphi_M^b \wedge \text{Ax}^b(L)) / \cong_{L^b}$ and $\{\text{sharp expansions } (M, \mathcal{F})\} / \cong_{L^\sharp}$, so to obtain a correspondence between $\text{Mod}_\omega(\varphi_M^b \wedge \text{Ax}^b(L)) / \cong_{L^b}$ and closed subgroups of $\text{Aut}(M)$, it would suffice to find an HC-invariant

bijection between sharp expansions (M, \mathcal{F}) of M and closed subgroups of $\text{Aut}(M)$. However, such a mapping cannot exist, for the simple reason that closed subgroups of $\text{Aut}(M)$ are rarely elements of HC . Thus, we instead derive a correspondence between sharp expansions and *codes* of closed subgroups of $\text{Aut}(M)$.

4.1 Codes of closed subgroups

We begin by defining the codes. It is natural to work with the topological group $S_\infty = \text{Sym}(\omega)$, which we can view as $\text{Aut}(\mathcal{E})$, where $\mathcal{E} := (\omega, =)$.

Definition 4.1. For each $n \in \omega$, let $I_n = \omega^n \times \omega^n$ and let $I = \bigcup \{I_n : n \in \omega\}$. For each $(\bar{a}, \bar{b}) \in I$, let $U_{\bar{a}, \bar{b}} = \{\sigma \in S_\infty : \sigma(\bar{a}) = \bar{b}\}$.

Then $\{U_{\bar{a}, \bar{b}} : (\bar{a}, \bar{b}) \in I\}$ is the standard basis for the topology of uniform convergence on S_∞ . With respect to this topology, for any closed $C \subseteq S_\infty$, let

$$\mathcal{F}(C) = \{(\bar{a}, \bar{b}) \in I : C \cap U_{\bar{a}, \bar{b}} \neq \emptyset\}$$

denote the *code* of C . Note that the mapping $C \mapsto \mathcal{F}(C)$ is injective on closed sets and $\mathcal{F}(C) \in HC$ for every $C \subseteq S_\infty$. Moreover, recalling Definitions 2.1 and 2.2, $\mathcal{F}(C)$ is a downward closed, back and forth system on \mathcal{E} for every closed $C \subseteq S_\infty$.

We endow $\mathcal{P}(I)$ with the usual Cantor topology, i.e., the sets $\{V_{\bar{a}, \bar{b}} : (\bar{a}, \bar{b}) \in I\}$ (where $V_{\bar{a}, \bar{b}} := \{A \subseteq I : (\bar{a}, \bar{b}) \in A\}$) and their complements form a sub-basis for the topology. As I is countable, this topology is Polish.

Now suppose $\mathcal{F} \subseteq I$ is a downward closed, back and forth system. The inverse operation is given by

$$C_{\mathcal{F}} = \{\sigma \in S_\infty : \text{graph}(\sigma) \subseteq \mathcal{F}\}$$

The proof of the following lemma is routine, once one notes that for \mathcal{F} any downward closed, back and forth system, if $(\bar{a}, \bar{b}) \in \mathcal{F}$ then there is some $\sigma \in S_\infty$ with $\text{graph}(\sigma) \subseteq \mathcal{F}$.

Lemma 4.2. *The set $\{\mathcal{F}(C) : C \subseteq S_\infty \text{ closed}\}$ is equal to the set of downward closed, back and forth systems on \mathcal{E} . Specifically, $C_{\mathcal{F}(C)} = C$ for every closed $C \subseteq S_\infty$, and $\mathcal{F}(C_{\mathcal{F}}) = \mathcal{F}$ for every downward closed, back and forth system \mathcal{F} on \mathcal{E} .*

It is easily seen from Lemma 4.2 that the subspace $\{\mathcal{F}(C) : C \text{ closed in } S_\infty\} \subseteq \mathcal{P}(I)$ is closed, hence is also a Polish space. The correspondence given in Lemma 4.2 explains our interest in Definitions 2.1 and 2.2. Continuing, we are not so much interested in arbitrary closed sets C , but rather closed subgroups $H \leq \text{Aut}(\mathcal{E})$. In the proof of Lemma 4.3, the three clauses of being a subgroup exactly align with the reflexivity, symmetry, and transitivity clauses of being an equivalence relation.

Lemma 4.3. *A closed subset $H \subseteq \text{Aut}(\mathcal{E})$ is a closed subgroup if and only if $\mathcal{F}(H)$ is a sharp back and forth system on $\text{Aut}(\mathcal{E})$.*

Whittling down to closed subgroups of a given countable M , we have the following fundamental result.

Proposition 4.4. *Let M be any countable structure. The map $(M, \mathcal{F}) \mapsto \mathcal{F}$ is a persistent, HC-invariant bijection between $\{\text{sharp expansions of } M\}$ and $\{\text{codes of closed subgroups of } \text{Aut}(M)\}$.*

4.2 Conjugacy and sharp expansions

Fix an L -structure M with universe ω . Classically, any expansion M^* of M can be ‘seen’ by looking at $\text{Aut}(M^*)$, which is a closed subgroup of $\text{Aut}(M)$. Conversely, any closed subgroup $H \leq \text{Aut}(M)$ gives rise to an expansion M_H by letting $L^* = L \cup \{P^{\bar{a}}(\bar{x}) : \bar{a} \in M^{<\omega}\}$ and letting M^* be the expansion of M formed by interpreting each $P^{\bar{a}}$ as the H -orbit of \bar{a} in $M^{\text{lg}(\bar{a})}$. In such an expansion we have $\text{Aut}(M_H) = H$.

Proposition 4.4 gives a different correspondence between sharp expansions and closed subgroups. Given a sharp expansion (M, \mathcal{F}) of M , the relevant closed subgroup is

$$\text{Fix}(M, \mathcal{F}) = \{\sigma \in \text{Aut}(M) : (M, \mathcal{F}) \models E_k(\bar{a}, \sigma(\bar{a})) \text{ for all } k \text{ and all } \bar{a} \in M^k\}$$

which is a closed subgroup of $\text{Aut}(M, \mathcal{F})$, but is typically not equal to it. The relevance of $\text{Fix}(M, \mathcal{F})$ is largely explained by the following easy lemmas.

Lemma 4.5. *Let M be any countable L -structure and let \mathcal{F} be any sharp back-and-forth system on M . Then, for any $(\bar{a}, \bar{b}) \in \mathcal{F}$ (i.e., $(M, \mathcal{F}) \models E_k(\bar{a}, \bar{b})$) there is some $\sigma \in \text{Fix}(M, \mathcal{F})$ such that $\sigma(\bar{a}) = \bar{b}$.*

Proof. An appropriate σ can be constructed by finite approximations using the back-and-forth property of \mathcal{F} . □

Lemma 4.6. *For any sharp back and forth system \mathcal{F} on M , $\text{Fix}(M, \mathcal{F}) = C_{\mathcal{F}}$ from Lemma 4.2.*

Proof. Choose $\sigma \in \text{Fix}(M, \mathcal{F})$. To see $\text{graph}(\sigma) \subseteq \mathcal{F}$, choose any $\bar{a} \in M^k$. Since $(M, \mathcal{F}) \models E_k(\bar{a}, \sigma(\bar{a}))$, we have $(\bar{a}, \sigma(\bar{a})) \in \mathcal{F}$. Conversely, choose $\sigma \in C_{\mathcal{F}}$. To see that $\sigma \in \text{Aut}(M)$, choose any $(\bar{a}, \bar{b}) \in \text{graph}(\sigma)$. Since $(\bar{a}, \bar{b}) \in \mathcal{F}$, it follows from Lemma 4.5 that there is an automorphism $\psi \in \text{Aut}(M)$ with $(\bar{a}, \bar{b}) \in \text{graph}(\psi)$. As this holds for all (\bar{a}, \bar{b}) , we conclude $\sigma \in \text{Aut}(M)$. To see $\sigma \in \text{Fix}(M, \mathcal{F})$, choose any $\bar{a} \in M^k$. Since $(\bar{a}, \sigma(\bar{a})) \in \mathcal{F}$, we have $(M, \mathcal{F}) \models E_k(\bar{a}, \sigma(\bar{a}))$, as required. □

Definition 4.7. Given an L -structure M with universe ω , let

$$\mathcal{C}_M = \{\mathcal{F}(H) \in \mathcal{P}(I) : H \text{ a closed subgroup of } \text{Aut}(M)\}.$$

Conjugacy induces an equivalence relation \sim_{conj} on \mathcal{C}_M as: $\mathcal{F}(H_2) \sim_{\text{conj}} \mathcal{F}(H_1)$ if and only if H_2 and H_1 are $\text{Aut}(M)$ -conjugate, i.e., $H_2 = \delta H_1 \delta^{-1}$ for some $\delta \in \text{Aut}(M)$.

In terms of descriptive set theory, being a closed subspace of $\mathcal{P}(I)$, \mathcal{C}_M is a Polish space, and \sim_{conj} is an analytic equivalence relation on \mathcal{C}_M . We summarize our correspondence with the following. Recall the definitions of $\text{Ax}^\sharp(L)$ and $\text{Ax}^\flat(L)$ from Definitions 2.3 and 3.1, respectively.

Proposition 4.8. *Let M be any L -structure with universe ω and let $\varphi_M \in L_{\omega_1, \omega}$ be any Scott sentence for M . The following three strongly definable quotients are HC-bireducible.*

1. $(\text{Mod}_\omega(\varphi_M^\flat \wedge \text{Ax}^\flat(L)), \cong_{L^\flat})$;
2. $(\text{Mod}_\omega(\varphi_M \wedge \text{Ax}^\sharp(L)), \cong_{L^\sharp})$; and
3. $(\mathcal{C}_M, \sim_{\text{conj}})$.

Proof. The correspondence between (1) and (2) is given by the flattening operation and Proposition 3.7. To get the HC-bireduction between (2) and (3), we use the map $(M, \mathcal{F}) \mapsto \mathcal{F}$ from Proposition 4.4. We must verify that this map preserves quotients in both directions, i.e.

$$(M, \mathcal{F}_1) \cong_{L^\sharp} (M, \mathcal{F}_2) \quad \text{if and only if} \quad \mathcal{F}_1 \sim_{\text{conj}} \mathcal{F}_2$$

To establish both directions, choose sharp back and forth systems $\mathcal{F}_1, \mathcal{F}_2$ on M . By Lemma 4.6, $H_1 := \text{Fix}(M, \mathcal{F}_1)$ and $H_2 := \text{Fix}(M, \mathcal{F}_2)$ are the closed subgroups of $\text{Aut}(M)$ coded by \mathcal{F}_1 and \mathcal{F}_2 , respectively.

First, assume $\delta : (M, \mathcal{F}_1) \rightarrow (M, \mathcal{F}_2)$ is an L^\sharp -isomorphism. We claim that H_1 and H_2 are conjugate via δ , i.e., $H_2 = \delta H_1 \delta^{-1}$. To see this, choose any $\sigma \in H_1$, and we must show $\delta \sigma \delta^{-1} \in H_2$. Clearly, $\sigma \in \text{Aut}(M)$, so to see $\delta \sigma \delta^{-1} \in \text{Fix}(M, \mathcal{F}_2)$, choose any k and $\bar{c} \in M^k$. We must show $(M, \mathcal{F}_2) \models E_k(\bar{c}, \delta \sigma \delta^{-1}(\bar{c}))$. Let $\bar{a} = \delta^{-1}(\bar{c})$. As $\sigma \in H_1$, $(M, \mathcal{F}_1) \models E_k(\bar{a}, \sigma(\bar{a}))$. As δ is an L^\sharp -isomorphism, $(M, \mathcal{F}_2) \models E_k(\delta(\bar{a}), \delta \sigma(\bar{a}))$, hence $(M, \mathcal{F}_2) \models E_k(\bar{c}, \delta \sigma \delta^{-1}(\bar{c}))$.

Conversely, assume $\mathcal{F}_1 \sim_{\text{conj}} \mathcal{F}_2$ as witnessed by $\delta \in \text{Aut}(M)$, i.e., $H_2 = \delta H_1 \delta^{-1}$. We argue that $\delta : (M, \mathcal{F}_1) \rightarrow (M, \mathcal{F}_2)$ is an L^\sharp -isomorphism. As $\delta \in \text{Aut}(M)$, choose any k and suppose $(M, \mathcal{F}_1) \models E_k(\bar{a}, \bar{b})$. It suffices to show $(M, \mathcal{F}(H_2)) \models E_k(\delta(\bar{a}), \delta(\bar{b}))$. Let $\bar{c} = \delta(\bar{a})$. By Lemma 4.5, choose $\sigma \in \text{Fix}(M, \mathcal{F}_1)$ such that $\sigma(\bar{a}) = \bar{b}$. As $\delta \sigma \delta^{-1} \in H_2 = \text{Fix}(M, \mathcal{F}_2)$, we have $(M, \mathcal{F}_2) \models E_k(\bar{c}, \delta \sigma \delta^{-1}(\bar{c}))$, hence $(M, \mathcal{F}_2) \models E_k(\delta(\bar{a}), \delta(\bar{b}))$. \square

4.3 Comparing families \mathcal{C}_M and the completeness of $\mathcal{C}_{(\omega,=)}$

Here we compare the quotients \mathcal{C}_M and \mathcal{C}_N , where M and N are each structures with universe ω , although possibly in different languages. The following Lemma must be very standard.

Lemma 4.9. *Suppose M and N are structures (possibly in different countable languages), each with universe ω . Suppose there is a continuous, surjective homomorphism $f : \text{Aut}(N) \rightarrow \text{Aut}(M)$. Then the map $H \mapsto f^{-1}(H)$ is an injection of closed subgroups that respects conjugacy. Thus, the induced mapping $h : (\mathcal{C}_M, \sim_{\text{conj}}) \rightarrow (\mathcal{C}_N, \sim_{\text{conj}})$ of the codes given by $h(\mathcal{F}(H)) = \mathcal{F}(f^{-1}(H))$ is an HC-reduction.*

Proof. We first show that the map $H \mapsto f^{-1}(H)$ factors through conjugacy. That is, suppose $H_1, H_2 \in \mathcal{C}_M$ satisfy $H_2 = \delta H_1 \delta^{-1}$ for some $\delta \in \text{Aut}(M)$. As f is onto, choose $\gamma \in \text{Aut}(N)$ such that $f(\gamma) = \delta$. We argue that $f^{-1}(H_2) = \gamma f^{-1}(H_1) \gamma^{-1}$. To see this, choose $\sigma \in f^{-1}(H_1)$. It suffices to show $\gamma \sigma \gamma^{-1} \in f^{-1}(H_2)$, i.e., that $f(\gamma \sigma \gamma^{-1}) \in H_2$. But this is immediate, as f is a homomorphism and $\delta f(\sigma) \delta^{-1} \in H_2$. Thus, h is well-defined.

To see that h is injective, suppose $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$ in \mathcal{C}_N are conjugate, i.e., $G_2 = \gamma G_1 \gamma^{-1}$ for some $\gamma \in \text{Aut}(N)$. As f is a homomorphism, it follows that $H_2 = f(\gamma) H_1 f(\gamma)^{-1}$, so H_1 and H_2 are conjugate in \mathcal{C}_M .

To show HC-invariance, choose a closed $H \leq \text{Aut}(M)$ and a forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$. Let H' be the closed subgroup of $\text{Aut}(M)^{\mathbb{V}[G]}$ with the same code as H . Since they have the same codes, $H \subseteq H'$ is dense. Because analytic statements are absolute, we have $f^{-1}(H)^{\mathbb{V}}$ is dense in $f^{-1}(H')^{\mathbb{V}[G]}$. \square

For this to be interesting, we need to know that (S_∞, conj) is sufficiently complicated. Recall that we can identify S_∞ with $\text{Aut}(\omega, =)$.

Lemma 4.10. *$(\text{Mod}_\omega(\text{Graphs}), \cong)$ is HC-reducible to $(\mathcal{C}_{(\omega,=)}, \sim_{\text{conj}})$.*

Proof. Call an ω -tree (T, \leq) *padded* if, for all $a \in T$ and $b \in \text{Succ}(a)$, there are infinitely many $\{b_i : i \in \omega\} \subseteq \text{Succ}(a)$ such that $T_{\geq b_i} \cong T_{\geq b}$ for every i . An easy alteration of the original Friedman-Stanley proof that $\{\text{Subtrees of } \omega^{<\omega}\}$ are Borel complete shows that there is a Borel reduction

$$f : \text{Graphs} \rightarrow \text{Padded subtrees of } (\omega \times \omega)^{<\omega}$$

Recall $\mathcal{E} = (\omega, =)$, so $\text{Aut}(\mathcal{E}) = S_\infty$ and $\mathcal{C}_\mathcal{E}$ denotes the codes of closed subgroups of S_∞ . Now let $g : \{\text{Graphs}\} \rightarrow \mathcal{C}_\mathcal{E}$ be the map $g(G) := \mathcal{F}(\text{Aut}(f(G)))$.

We must show that if G, H are graphs with universe ω and $\text{Aut}(f(G)), \text{Aut}(f(H))$ are conjugate by some $\delta \in S_\infty$, then G and H are isomorphic. [The converse is easy.] We verify this by way of the following Claim.

Claim. If (T, \trianglelefteq) is any padded ω -tree, then for any $a, b \in T$, $a \trianglelefteq b$ if and only if every $\sigma \in \text{Aut}(T, \trianglelefteq)$ that fixes b also fixes a .

Proof. Left to right is obvious, as for any ω -tree (T, \trianglelefteq) , if $a \trianglelefteq b$, then $a \in \text{dcl}(b)$. For the converse, note that for any $c \in T$ and any $d, d' \in \text{Succ}(c)$ with $T_{\geq d} \cong T_{\geq d'}$, there is $\sigma \in \text{Aut}(T, \trianglelefteq)$ fixing $T \setminus (T_{\geq d} \cup T_{\geq d'})$ with $\sigma(d) = d'$.

Now assume that $a \not\trianglelefteq b$ and let $c = a \wedge b$. There are two cases. First, if $c = b$ (i.e., if $b \trianglelefteq a$) then let $d \in \text{Succ}(b)$ with $d \trianglelefteq a$. As (T, \trianglelefteq) is padded, choose $d' \in \text{Succ}(b)$ with $d' \neq d$ but $T_{\geq d'} \cong T_{\geq d}$. Then any $\sigma \in \text{Aut}(T, \trianglelefteq)$ fixing b with $\sigma(d) = d'$ will move a as well. Second, assume $c \neq b$, hence $c \triangleleft b$. Let $d \in \text{Succ}(c)$ with $d \trianglelefteq a$. Choose $d' \in \text{Succ}(c)$ such that $d' \neq d$ and $d' \not\trianglelefteq b$, but with $T_{\geq d'} \cong T_{\geq d}$. Any automorphism $\sigma \in \text{Aut}(T, \trianglelefteq)$ with $\sigma(d) = d'$ that fixes $T \setminus (T_{\geq d} \cup T_{\geq d'})$ will fix b , but move a . \square

To see that this Claim suffices, suppose G, H are graphs with universe ω and $\text{Aut}(f(H)) = \delta \text{Aut}(f(G)) \delta^{-1}$. Because of the Borel reduction, it suffices to show that $\delta : (f(G), \trianglelefteq) \cong (f(H), \trianglelefteq)$ is a tree isomorphism. As $\delta \in S_\infty$ and by symmetry, it suffices to show that if $a, b \in f(G)$ and $(f(G), \trianglelefteq) \models a \trianglelefteq b$, then $(f(H), \trianglelefteq) \models \delta(a) \trianglelefteq \delta(b)$. But this is clear: Let $c = \delta(a)$, $d = \delta(b)$, and choose any $\tau \in \text{Aut}(f(H), \trianglelefteq)$ with $\tau(d) = d$. Put $\sigma := \delta^{-1} \tau \delta \in \text{Aut}(f(G), \trianglelefteq)$. Since $\tau(d) = d$, $\delta(\sigma(b)) = \delta(b)$, hence $\sigma(b) = b$. By the Claim, $\sigma(a) = a$ as well, hence $\tau(c) = c$. Thus, $(f(H), \trianglelefteq) \models c \trianglelefteq d$ by the Claim. \square

5 Characterizing Borel complete expansions

With the extensive preliminaries out of the way, we are now able to obtain our characterization of sentences of $L_{\omega_1, \omega}$ that have Borel complete expansions. We begin with a classical result of Morley [14] that characterizes which sentences of $L_{\omega_1, \omega}$ have Ehrenfeucht-Mostowski models. The novelty will be that we apply this result to a class of flat structures.

Fact 5.1. *The following are equivalent for $\psi \in L_{\omega_1, \omega}$.*

1. *For all $\alpha < \omega_1$, ψ has a model of size at least \beth_α ;*
2. *ψ has arbitrarily large models;*
3. *There is a countable $L^* \supseteq L$, a countable, Skolemized fragment \mathcal{F}^* containing ψ , and a countable, L^* -structure $N^* \models \psi$ that is the Skolem hull of a sequence $\langle a_q : q \in \mathbb{Q} \rangle$ of \mathcal{F}^* -order indiscernibles; and*
4. *For all linear orderings (J, \leq) , there is a model $N_J \models \psi$ containing a sequence $\langle a_j : j \in J \rangle$ of order indiscernibles.*

For brevity, we say that ψ admits *Ehrenfeucht-Mostowski models* if any of the four conditions hold. Next, we have a group theoretic notion.

Definition 5.2. We say that the topological group H divides the topological group G if there is a closed subgroup $G' \leq G$ and a continuous surjection $h : G' \rightarrow H$.

Obviously, if $H \leq G$ is closed, then H divides G . Here are two useful examples of dividing among automorphism groups.

Lemma 5.3. 1. For N^* chosen as in Fact 5.1(3), $Aut(\mathbb{Q}, \leq)$ divides $Aut(N^*)$.

2. Let (M, \mathcal{F}) be any countable sharp structure and let $\mathbb{B} = (M, \mathcal{F})^b$ be its flattening. Then $Aut(\mathbb{B})$ divides $Aut(M)$.

Proof. (1) Given N^* as in Fact 5.1(3), let $G' \leq Aut(N^*)$ denote the setwise stabilizer of $\{a_q : q \in \mathbb{Q}\}$. Then, as $\langle a_q : q \in \mathbb{Q} \rangle$ are order indiscernible and $N^* = \text{dcl}(\{a_q : q \in \mathbb{Q}\})$, G' is homeomorphic to $Aut(\mathbb{Q}, \leq)$.

(2) As $Aut(M, \mathcal{F})$ is a closed subgroup of $Aut(M)$, it suffices to verify that the map $h : Aut(M, \mathcal{F}) \rightarrow Aut(\mathbb{B})$ is a continuous surjection, where h is defined by $h(\sigma)$ is the automorphism of \mathbb{B} satisfying $h(\sigma)(\mathbf{a}) = \pi_k(\sigma(\bar{a}))$ for every $\mathbf{a} \in U_k(\mathbb{B})$ and some $\bar{a} \in \pi_k^{-1}(\mathbf{a})$. Showing that h is well defined amounts to verifying that $\pi_k(\sigma(\bar{a})) = \pi_k(\sigma(\bar{a}'))$ whenever $(\bar{a}, \bar{a}') \in \mathcal{F}$, which is immediate since $\sigma \in Aut(M, \mathcal{F})$. That h is continuous follows from the fact that for any $\mathbf{a} \in \mathbb{B}$, $h^{-1}(Fix(\mathbf{a}))$ is equal to the set of $\sigma \in Aut(M, \mathcal{F})$ fixing the E_k -class $[\bar{a}]_k$. To see that h is onto, choose any $\psi \in Aut(\mathbb{B})$ and we construct a $\sigma \in Aut(M, \mathcal{F})$ such that $h(\sigma) = \psi$ by back-and-forth. Given ψ , let $\mathcal{G} = \{(\bar{a}, \bar{b}) \in M^{2k} : \psi(\pi_k(\bar{a})) = \pi_k(\bar{b})\}$. The verification that \mathcal{G} is a back-and-forth system follows easily by Lemma 3.10. \square

It is easily verified that the relation of dividing is transitive. What is more surprising is the following, which largely appears as Lemma 2.5 of [5].

Lemma 5.4. S_∞ divides $Aut(\mathbb{Q}, \leq)$. For any topological group G , S_∞ divides G if and only if $Aut(\mathbb{Q}, \leq)$ divides G .

Proof. To see that S_∞ divides $Aut(\mathbb{Q}, \leq)$, choose a partition $\mathbb{Q} = \sqcup\{D_n : n \in \omega\}$ into dense sets and let $E \subseteq \mathbb{Q}^2$ be the equivalence relation given by $E(a, b)$ iff $\bigwedge_{n \in \omega} D_n(x) \leftrightarrow D_n(y)$. Let $G' = Aut(\mathbb{Q}, \leq, D_n, E)$, which is a closed subgroup of $Aut(\mathbb{Q}, \leq)$. But it is easily checked that the map $h : G' \rightarrow S_\infty$, where $h(\sigma)$ is the element of S_∞ given by $h(\sigma)(n) =$ the (unique) m such that $\sigma(a) \in D_m$ for any/all $a \in D_n$ is a topological group homeomorphism. The fact that h is surjective is by a standard back-and-forth argument.

As $Aut(\mathbb{Q}, \leq)$ is a closed subgroup of S_∞ , it trivially divides it, so the second sentence follows by the transitivity of dividing. \square

We can now state and prove our main theorem. Recall that by Definition 3.1 flat L^b -structures are axiomatized by $\text{Ax}^b(L)$, hence $\text{Mod}(\varphi^b \wedge \text{Ax}^b(L)) = \text{Flat}(\varphi)$ for any infinitary L -sentence φ .

Theorem 5.5. *The following are equivalent for a sentence $\varphi \in L_{\omega_1, \omega}$.*

1. φ has a Borel complete expansion (i.e., there is some countable $L' \supseteq L$ and $\psi \in (L')_{\omega_1, \omega}$ such that $\psi \vdash \varphi$ and ψ is Borel complete);
2. $\varphi^b \wedge \text{Ax}^b(L)$ has arbitrarily large models;
3. $\varphi^b \wedge \text{Ax}^b(L)$ admits Ehrenfeucht-Mostowski models;
4. S_∞ divides $\text{Aut}(M)$ for some countable $M \models \varphi$.

Proof. The equivalence (2) \Leftrightarrow (3) is by Fact 5.1 applied to the $L_{\omega_1, \omega}$ -sentence $\varphi^b \wedge \text{Ax}^b(L)$. We prove the equivalence of (1) and (4) by independently showing that each is equivalent to (2).

(1) \Rightarrow (2): Suppose $\psi \in (L')_{\omega_1, \omega}$ is Borel complete with $\psi \vdash \varphi$. As ψ is Borel complete, $|\psi| = |\psi_{\text{ptl}}| = \infty$, i.e., there are a proper class of back-and-forth inequivalent models of ψ . Thus, $\psi^b \wedge \text{Ax}^b(L')$ has a proper class of non-isomorphic models by Corollary 3.27. This implies $\psi^b \wedge \text{Ax}^b(L')$ has arbitrarily large models. But $\psi \vdash \varphi$ implies $\psi^b \wedge \text{Ax}^b(L') \vdash \varphi^b \wedge \text{Ax}^b(L)$, so $\varphi^b \wedge \text{Ax}^b(L)$ also has arbitrarily large models.

(2) \Rightarrow (1): Suppose $\varphi^b \wedge \text{Ax}^b(L)$ has arbitrarily large models, hence admits EM models. Applying Fact 5.1(3) to $\varphi^b \wedge \text{Ax}^b(L)$, choose a countable model $\mathbb{B}^* \models \varphi^b \wedge \text{Ax}^b(L)$ in an expanded language $L^* \supseteq L^b$, the Skolem hull of a sequence $\langle \mathbf{a}_q : q \in \mathbb{Q} \rangle$ of order indiscernibles. Let \mathbb{B} be the L^b -reduct of \mathbb{B}^* , which is a countable, flat structure in which $\langle \mathbf{a}_q : q \in \mathbb{Q} \rangle$ remain order indiscernible. Using Proposition 3.7, let (M, \mathcal{F}) be a sharp structure whose flattening is \mathbb{B} . So M is a countable L -structure and $M \models \varphi$ since $\mathbb{B} \models \varphi^b$. Without loss, assume M has universe ω and choose a Scott sentence $\Phi_M \in L_{\omega_1, \omega}$ for M .

For definiteness, suppose $\langle \mathbf{a}_q : q \in \mathbb{Q} \rangle$ are from $U_k(\mathbb{B})$. As in the proof of Lemma 5.4, partition $\mathbb{Q} = \sqcup \{D_n : n \in \omega\}$ into dense pieces and, for each n , let $X_n := \bigcup \{\text{cov}^{-1}(\mathbf{a}_q) : q \in D_n\}$. Let $X^* := \bigcup \{X_n : n \in \omega\} \subseteq M^k$ and let E^* be the equivalence relation on X whose classes are X_n . Let $L' = L \cup \{X^*, E^*, R\}$ and let ψ be the same sentence φ , but viewed as an $(L')_{\omega_1, \omega}$ -sentence. We can see that ψ is Borel complete directly: Given any directed graph $G = (\omega, r)$, let M_G be the expansion (M, X^*, E^*, R) of M with $R(\bar{a}, \bar{b})$ holding if and only if for some $n, m \in \omega$, $\bar{a} \in X_n$, $\bar{b} \in X_m$, and $G \models r(n, m)$.

(4) \Rightarrow (2): Suppose $M \models \varphi$ is countable and S_∞ divides $\text{Aut}(M)$. Choose an expansion M^* of M for which there is a continuous surjection $f : \text{Aut}(M^*) \rightarrow \text{Aut}(\mathcal{E})$

(recall that $\mathcal{E} = (\omega, =)$ and $S_\infty = \text{Aut}(\omega, =)$). By Lemma 4.9, there is an HC-reduction $(\mathcal{C}_{\mathcal{E}}, \sim_{\text{conj}}) \leq_{\text{HC}} (\mathcal{C}_{M^*}, \sim_{\text{conj}})$. Let φ_{M^*} be a Scott sentence for M^* . Composing the above with the HC-reduction from Lemma 4.10 and applying Lemma 4.8 to the right hand side yields an HC-reduction

$$(\text{Mod}_\omega(\text{Graphs}), \cong) \leq_{\text{HC}} (\text{Mod}_\omega(\varphi_{M^*}^b \wedge \text{Ax}^b(L)), \cong_{L^b})$$

As $\|(\text{Graphs}, \cong)\| = \infty$, it follows from Fact 1.5 that there is a proper class of non- L^b -isomorphic models of $\varphi_{M^*}^b \wedge \text{Ax}^b(L)$. Since $\varphi_{M^*}^b \vdash \varphi_M^b$, $\varphi_M^b \wedge \text{Ax}^b(L)$ has arbitrarily large models.

(2) \Rightarrow (4): $\varphi^b \wedge \text{Ax}^b(L)$ has arbitrarily large models. By Fact 5.1(3), choose $L^* \supseteq L^b$ and a countable L^* -structure $\mathbb{B}^* \models \varphi^b \wedge \text{Ax}^b(L)$ that is the Skolem hull of a sequence $\langle \mathbf{a}_q : q \in \mathbb{Q} \rangle$ of order indiscernibles and let \mathbb{B} be the L^b -reduct of \mathbb{B}^* . By Lemma 5.3(1), $\text{Aut}(\mathbb{Q}, \leq)$ divides $\text{Aut}(\mathbb{B}^*)$, hence also divides $\text{Aut}(\mathbb{B})$ since $\text{Aut}(\mathbb{B}^*) \leq \text{Aut}(\mathbb{B})$ is closed. As $\mathbb{B} \models \varphi^b$ is countable, choose (M, \mathcal{F}) as in Proposition 3.7. Then $M \models \varphi$ and is countable. By Lemma 5.3(2) $\text{Aut}(\mathbb{B})$ divides $\text{Aut}(M)$, hence $\text{Aut}(\mathbb{Q}, \leq)$ divides $\text{Aut}(M)$ by transitivity. Thus S_∞ divides $\text{Aut}(M)$ by Lemma 5.4. \square

Corollary 5.6. *Suppose $\varphi \in L_{\omega_1, \omega}$ admits Ehrenfeucht-Mostowski models. Then φ has a Borel complete expansion.*

Proof. Apply Fact 5.1 directly to φ and get $L^* \supseteq L$ countable and a countable L^* -structure $M^* \models \varphi$ that is the Skolem hull of an order-indiscernible sequence $\langle a_q : q \in \mathbb{Q} \rangle$. By Lemma 5.3(1), $\text{Aut}(\mathbb{Q}, \leq)$ divides $\text{Aut}(M^*)$, hence also divides $\text{Aut}(M)$, where M is the L -reduct of M^* . Thus, by Lemma 5.4 S_∞ divides $\text{Aut}(M)$, so Theorem 5.5 applies. \square

In light of Corollary 5.6 we are interested in $\varphi \in L_{\omega_1, \omega}$ that do not admit Ehrenfeucht-Mostowski models, yet have Borel complete expansions. One example is given in Theorem 6.2 of this paper, where the Scott sentence φ_h of the unique atomic model of T_h has these properties. More generally, if φ is *receptive* (see Definition 1.11 of [1]) then S_∞ divides the automorphism group of some expansion M^* of a countable $M \models \varphi$, hence φ has a Borel complete expansion by Theorem 5.5. By results in Section 1 of [1], receptive sentences include the Scott sentences of Fraïssé limits of families \mathbf{K} of finite structures satisfying disjoint amalgamation. Thus, any of the examples of Scott sentences φ characterizing \aleph_α in either [2] or [6] do not admit Ehrenfeucht-Mostowski models, but have Borel complete expansions.

Even though Knight's example [9] of a complete sentence φ_{K_n} characterizing \aleph_1 is not obtained as a Fraïssé limit, Hjorth proves that S_∞ divides the automorphism group of the countable model of φ_{K_n} by a different argument [6]. Thus, φ_{K_n} also has a Borel complete expansion by Theorem 5.5.

6 Applications

We record two instances where the conditions of Theorem 5.5 hold.

Corollary 6.1. *1. If T is a first-order theory with an infinite model, then T has a Borel complete expansion.*

2. If $\|\varphi_{\text{ptl}}\| \geq \beth_{\omega_1}$, then φ has a Borel complete expansion.

Proof. (1) By Ramsey and compactness, T admits Ehrenfeucht-Mostowski models, so apply Corollary 5.6 to $\varphi := \bigwedge T$.

For (2), it follows from $\|\varphi_{\text{ptl}}\| \geq \beth_{\omega_1}$ that $\varphi^b \wedge \text{Ax}^b(L)$ has at least \beth_{ω_1} Hausdorff models. Thus, by applying Fact 5.1 to $\varphi^b \wedge \text{Ax}^b(L)$, there are arbitrarily large models of $\varphi^b \wedge \text{Ax}^b(L)$, so Theorem 5.5 applies. \square

More interesting are cases where the conditions of Theorem 5.5 do not hold. In [12], the authors considered families of cross-cutting equivalence relations, each with finitely many classes. More formally, fix the language $L = \{E_n : n \in \omega\}$. For any function $h : \omega \rightarrow (\omega - \{0, 1\})$, let T_h^\forall be the universal L -theory asserting that each E_n is an equivalence relation with at most $h(n)$ classes. It is easily checked that T_h^\forall has a model completion T_h , which asserts that each E_n has exactly $h(n)$ classes, and the relations cross-cut, i.e., for any finite $F \subseteq \omega$, if we let $E_F(x, y) := \bigwedge_{n \in F} E_n(x, y)$, then E_F has exactly $\prod_{n \in F} h(n)$ classes.

In [12], the authors proved that if the set of integers $\{h(n) : n \in \omega\}$ was unbounded, then T_h is Borel complete. Here, we use Theorem 5.5 to obtain the converse.

Theorem 6.2. *If the set of integers $\{h(n) : n \in \omega\}$ is uniformly bounded, then T_h is not Borel complete.*

Proof. As T_h is first-order, Corollary 6.1 shows we cannot apply Theorem 5.5 to T_h directly. Rather, consider an arbitrary countable model $M \models T_h$. Consider the infinitary formula $E_\infty(x, y) := \bigwedge_{n \in \omega} E_n(x, y)$. Then E_∞ is an equivalence relation. For each $a \in M$, $[a]_\infty := \{b \in M : E_\infty(a, b)\}$, which ostensibly can have any cardinality m , $1 \leq m \leq \omega$. It is easily checked by the quantifier elimination that a model $M \models T_h$ is atomic if and only if $|[a]_\infty| = 1$ for every $a \in M$. Thus, the class \mathbf{At}_h of atomic models of T_h is axiomatized by the $L_{\omega_1, \omega}$ -sentence φ_h asserting that T_h holds, and every $[a]_\infty$ -class is a singleton. The automorphism groups $\text{Aut}(M)$ of atomic models are easy to analyze.

Claim. For $M \in \mathbf{At}_h$, $\text{Aut}(M)$ has bounded exponent.

Proof. Choose an integer K so that $h(n) \leq K$ for all $n \in \omega$. As $M \models T_h$ is atomic, $\text{Aut}(M)$ is isomorphic to a subgroup of $\prod_{n \in \omega} \mathbb{Z}/h(n)\mathbb{Z}$. As each of the finite groups $\mathbb{Z}/h(n)\mathbb{Z}$ has exponent dividing $K!$, so does $\text{Aut}(M)$. \square

It follows from the Claim that S_∞ cannot divide any countable model $M \in \mathbf{At}_h$. As $\text{Mod}(\mathbf{At}_h)$ is expressible by a sentence φ_h of $L_{\omega_1, \omega}$, we conclude from Theorem 5.5 that no expansion of φ_h can be Borel complete.

We define an expansion ψ_h of φ_h , formed by adding unary predicates to encode $|[a]_\infty|$ for elements a in an arbitrary countable model of T_h . In more detail, let $L^* = L \cup \{U_m : 1 \leq m \leq \omega\}$, where each U_m is a unary predicate, and let ψ_h be the $(L^*)_{\omega_1, \omega}$ -sentence asserting φ_h and that the predicates $\{U_m : 1 \leq m \leq \omega\}$ partition the universe. Thus, ψ_h is not Borel complete.

But easily, T_h and ψ_h are Borel equivalent, since given any countable $M \models T_h$, let $f(M)$ be the L^* -structure with universe M/E_∞ , where the equivalence relations E_n interpreted naturally, and where $U_m([a]_{E_\infty})$ holds if and only if $|[a]_{E_\infty}| = m$ (i.e., we ‘color the branches by the number of realizations of the branch’). Since ψ_h is not Borel complete, neither is T_h . \square

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