Mutually algebraic structures and expansions
by predicates

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Abstract

We introduce the notions of a mutually algebraic structures and
theories and prove many equivalents. A theory $T$ is mutually algebraic
if and only if it is weakly minimal and trivial if and only if no model
$M$ of $T$ has an expansion $(M, A)$ by a unary predicate with the finite
cover property. We show that every structure has a maximal mutually
algebraic reduct, and give a strong structure theorem for the class of
elementary extensions of a fixed mutually algebraic structure.

1 Introduction

This paper is written with two objectives in mind. On one hand, it is a
continuation of [5], where a strong quantifier elimination theorem was proved
for elementary diagrams of models of a weakly minimal, trivial theory. Here,
we show that the crucial notion of mutual algebraicity of a formula (see
Definition 2.2) has meaning in arbitrary structures, and in fact describes a
specific reduct of any structure. As well, Theorem 3.3 reverses the argument
in [5]. The quantifier elimination result described there can only occur as the
elementary diagram of a weakly minimal, trivial theory.

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On the other hand, there has been a large body of research about whether an expansion \((M, A)\) of a given stable structure \(M\) by a unary predicate \(A\) remains stable. Sufficient conditions abound, but the general question remains open. Here, also with Theorem 3.3, we characterize those structures \(M\) with the property that every unary expansion \((M, A)\) satisfies the non-finite cover property (nfcp), which is a strengthening of stability.

The motivation for this came from the author’s reading [1], where Baldwin and Baizhanov showed that a non-trivial, strongly minimal structure \(M\) has an unstable expansion \((M, A)\). Thanks are due to John Baldwin for a careful reading of this paper, and for pointing out that an alternate treatment of a portion of Section 4 appears in Section 6 of [2].

2 The mutually algebraic reduct of a structure

We begin by recalling the definition of a mutually algebraic formula. This notion was introduced by Dolich, Raichev, and the author in [4] and further developed in [5]. However, in both of those papers, the ambient theory was assumed to have the non-finite cover property (nfcp). Here, we define the notions without any ambient assumptions. We begin by formulating the notion of a mutually algebraic set.

**Definition 2.1** Given an arbitrary set \(A\) and an integer \(n \geq 1\), a proper partition of \(n\) is a partition \(X \sqcup Y = \{1, \ldots, n\}\) where \(X, Y\) are disjoint and each is non-empty. Given such a partition, \(\pi_Y\) denotes the projection of \(A^n\) onto the coordinates in \(Y\).

A subset \(B \subseteq A^n\) is mutually algebraic if there is a number \(K\) so that for any proper partition of the coordinates \(X \sqcup Y = \{1, \ldots, n\}\), the projection \(\pi_Y\) restricted to \(B\) is at most \(K\)-to-1. That is, \(|\pi_Y^{-1}(b_Y) \cap B| \leq K\) for any \(b_Y \in \pi_Y(B)\).

As special cases, note that if either \(A\) is finite or \(B\) is empty, then \(B\) is mutually algebraic. Furthermore, for any set \(A\), every subset \(B \subseteq A^1\) is mutually algebraic as there are no proper partitions of a one element set.

**Definition 2.2** Let \(M\) denote any \(L\)-structure. An \(L(M)\)-formula \(\varphi(\bar{z})\) is mutually algebraic if \(\varphi(M) := \{\bar{a} \in M^{\bar{z}(\bar{a})} : M \models \varphi(\bar{a})\}\) is a mutually
algebraic subset of $M^{|\bar{z}|}$. We let $\mathcal{MA}(M)$ denote the set of all mutually algebraic $L(M)$-formulas. When $M$ is understood, we simply write $\mathcal{MA}$.

To clarify this concept and to set notation, given a formula $\varphi(\bar{z})$, a proper partition of $\bar{z}$ has the form $\bar{z} = \bar{x} \hat{\bar{y}}$, where $\bar{x}, \bar{y}$ are disjoint and $\lg(\bar{x}), \lg(\bar{y}) \geq 1$. We do not require $\bar{x}$ be an initial segment of $\bar{z}$ but to simplify notation, we write it as if it were. Then, for any $L$-structure $M$, an $L(M)$-formula $\varphi(\bar{z})$ is mutually algebraic if and only if there is an integer $K$ so that $M \models \forall \bar{y} \exists^{\leq K} \bar{x} \varphi(\bar{x}, \bar{y})$ for every proper partition $\bar{x} \hat{\bar{y}}$ of $\bar{z}$.

The reader is cautioned that whether a formula $\varphi(\bar{z})$ is mutually algebraic or not depends on the choice of free variables. In particular, mutual algebraicity is not preserved under adjunction of dummy variables. The special cases mentioned above imply that if $M$ is finite, then every $L(M)$-formula is in $\mathcal{MA}(M)$, and for an arbitrary $M$, every inconsistent formula and every $L(M)$-formula $\varphi(z)$ with exactly one free variable symbol is mutually algebraic. Our first easy Lemma gives a semantic interpretation to this notion when $\lg(\bar{z}) \geq 2$:

**Lemma 2.3** Let $M$ be any $L$-structure. The following are equivalent for any $L(M)$-formula $\varphi(\bar{z})$ with $\lg(\bar{z}) \geq 2$:

1. $\varphi(\bar{z}) \in \mathcal{MA}(M)$;

2. There is an integer $K$ so that $M \models \forall x \exists^{\leq K} \bar{y} \varphi(x, \bar{y})$ for all partitions $\bar{z} = x \hat{\bar{y}}$ with $\lg(x) = 1$;

3. For all $N \succeq M$, for all $\bar{e} \in N^{|\bar{z}|}$ realizing $\varphi$, and for all $e \in \bar{e}$, $\bar{e} \subseteq \acl(M \cup \{e\})$ (i.e, every $e' \in \bar{e}$ is in $\acl(M \cup \{e\})$).

**Proof.** (1) $\Rightarrow$ (2) is immediate.

(2) $\Rightarrow$ (3) Fix any $N \succeq M$ and assume $N \models \varphi(\bar{e})$. Fix any variable symbol $x \in \bar{z}$ and let $e$ be the corresponding element of $\bar{e}$. By elementarity, $N \models \exists^{\leq K} \bar{y} \varphi(e, \bar{y})$, so $\bar{e} \subseteq \acl(M \cup \{e\})$.

(3) $\Rightarrow$ (1) If (1) fails, then for some proper partition $\bar{z} = \bar{x} \hat{\bar{y}}$ we have $M \models \exists \bar{y} \exists^{\geq r} \bar{x} \varphi(\bar{x}, \bar{y})$. Thus, by compactness, there is $N \succeq M$ and $\bar{b}$ from $N$ such that $N \models \exists^{\geq r} \bar{x} \varphi(\bar{x}, \bar{b})$ for each $r \in \omega$. By compactness again, there is $N^* \succeq N$ and $\bar{a} \in (N^*)^{|\bar{z}|}$ such that $\bar{a} \not\subseteq \acl(M \cup \bar{b})$, contradicting (3).
The following Lemma indicates some of the closure properties of the set $\mathcal{M}A$. In what follows, when we write $\varphi(\bar{x}, \bar{y}) \in \mathcal{M}A$, we mean that $\bar{x}$ and $\bar{y}$ are disjoint sets of variable symbols and $\varphi(\bar{z}) \in \mathcal{M}A$ where $\bar{z} = \bar{\hat{z}} \bar{y}$, but that we are concentrating on a specific proper partition of $\varphi(\bar{z})$.

**Lemma 2.4** Let $M$ be any structure in any language $L$.

1. If $\varphi(\bar{z}) \in \mathcal{M}A$, then $\varphi(\sigma(\bar{z})) \in \mathcal{M}A$ for any permutation $\sigma$ of the variable symbols;

2. If $\varphi(\bar{x}, \bar{y}) \in \mathcal{M}A$ and $\bar{a} \in M^{lg(\bar{y})}$, then both $\exists \bar{y} \varphi(\bar{x}, \bar{y})$ and $\varphi(\bar{x}, \bar{a}) \in \mathcal{M}A$;

3. If $\varphi(\bar{z}) \vdash \psi(\bar{z})$ and $\psi(\bar{z}) \in \mathcal{M}A$, then $\varphi(\bar{z}) \in \mathcal{M}A$;

4. If $\{\varphi_i(\bar{z}_i) : i < k\} \subseteq \mathcal{M}A$, and there is some variable $x$ common to every $\bar{z}_i$, then $\psi(\bar{w}) := \bigwedge_{i<k} \varphi_i(\bar{z}_i) \in \mathcal{M}A$, where $\bar{w} = \bigcup_{i<k} \bar{z}_i$;

5. If $\varphi(\bar{x}, \bar{y}) \in \mathcal{M}A$ and $r \in \omega$, then $\theta_r(\bar{y}) := \exists^{\geq r} \bar{x} \varphi(\bar{x}, \bar{y}) \in \mathcal{M}A$.

**Proof.** The verification of (1), (2), and (3) are immediate. Concerning (4), we apply Lemma 2.3. Fix $N \succeq M$ and $\bar{e}$ such that $N \models \psi(\bar{e})$. Let $x$ denote a variable symbol that appears in every $\bar{z}_i$ and let $e_x$ denote the element of $\bar{e}$ corresponding to $x$. Similarly, for each $i < k$ let $\bar{e}_i$ be the subsequence corresponding to $\bar{z}_i$. As each $\varphi_i(\bar{z}_i) \in \mathcal{M}A$, $e_x \in acl(M \cup \{e\})$ for every $e \in \bar{e}_i$, so $e_x \in acl(M \cup \{e\})$ for every $e \in \bar{e}$. But also, $e \in acl(M \cup \{x\})$ for every $e \in \bar{e}$. Thus, by the transitivity of algebraic closure, $e \in acl(M \cup \{e\})$ for all pairs $e, e' \in \bar{e}$. So $\psi(\bar{w}) \in \mathcal{M}A$ by Lemma 2.3.

To establish (5), let $\{\bar{x}_i : i < r\}$ be disjoint sequences of variable symbols, each disjoint from $\bar{y}$. Then $\theta_r(\bar{y})$ is equivalent to

$$\exists \bar{x}_0 \exists \bar{x}_1 \ldots \exists \bar{x}_{r-1} \left( \bigwedge_{i<i} \varphi(\bar{x}_i, \bar{y}) \wedge \bigwedge_{i<j<i} \bar{x}_i \neq \bar{x}_j \right)$$

That this formula is in $\mathcal{M}A$ follows by successively applying Clauses (4), (3), and (2).
Definition 2.5 For any $L$-structure $M$, let $M_M$ denote the canonical expansion of $M$ to an $L(M)$-structure formed by adding a constant symbol $c_a$ for each $a \in M$. We let $\mathcal{MA}^*(M)$ denote the set of all $L(M)$-formulas that are $\text{Th}(M_M)$-equivalent to a boolean combination of formulas from $\mathcal{MA}(M)$. When $M$ is understood, we simply write $\mathcal{MA}^*$.

Whereas the definition of $\mathcal{MA}$ was rather fussy, membership in $\mathcal{MA}^*$ is more relaxed, mostly owing to the fact that $\mathcal{MA}^*$ is closed under adjunction of dummy variables. Indeed, we will see with Proposition 2.7 below, for any structure $M$, $\mathcal{MA}^*(M)$ specifies a reduct of the canonical expansion $M_M$.

Lemma 2.6 Let $M$ denote any $L$-structure.

1. $\mathcal{MA}^*$ is closed under boolean combinations;
2. $\mathcal{MA}^*$ is closed under adjunction of dummy variables, i.e., if $\varphi(z) \in \mathcal{MA}^*$ then $\varphi(x, z) \in \mathcal{MA}^*$;
3. For each $k \geq 1$, if $\{\varphi_i(x, \bar{y}_i) : i < k\} \subseteq \mathcal{MA}$ and $r \in \omega$, then each of $\exists^=r x \bigvee_{i<k} \varphi_i(x, \bar{y}_i)$, $\exists^{\leq r} x \bigvee_{i<k} \varphi_i(x, \bar{y}_i)$, and $\exists^{\geq r} x \bigvee_{i<k} \varphi_i(x, \bar{y}_i)$ are in $\mathcal{MA}^*$.

Proof. The proof of (1) is immediate. For (2), note that $\psi(x) := \bar{x} = x$ is in $\mathcal{MA}$, hence in $\mathcal{MA}^*$, but $\varphi(x, z)$ is equivalent to $\varphi(z) \land \psi(x)$. The verification of (3) is more substantial. We argue by induction on $k$ that for every $r \in \omega$, $\exists^{=r} x \bigvee_{i<k} \varphi_i(x, \bar{y}_i) \in \mathcal{MA}^*$ for every $k$-element subset $\{\varphi_i(x, \bar{y}_i) : i < k\}$ from $\mathcal{MA}$. This suffices, as $\mathcal{MA}^*$ is closed under boolean combinations and the trivial facts that $\exists^{\leq r} x \theta$ is equivalent to $\bigvee_{s \leq r} \exists^{=s} x \theta$ and $\exists^{\geq r} x \theta$ is equivalent to $\neg \exists^{\leq r-1} x \theta$.

To handle the case when $k = 1$, fix any $\varphi(x, \bar{y}) \in \mathcal{MA}$ and any $r \in \omega$. By Lemma 2.4(5), both $\exists^{=r} x \varphi(x, \bar{y}) \in \mathcal{MA}$ and $\exists^{\geq r} x \varphi(x, \bar{y}) \in \mathcal{MA}$ and $\exists^{\leq r} x \varphi(x, \bar{y})$ is a boolean combination of these.

Next, inductively assume that for every $r \in \omega$, $\exists^{=r} x \bigvee_{i<k} \varphi_i(x, \bar{y}_i) \in \mathcal{MA}^*$ for every $k$-element subset $\{\varphi_i(x, \bar{y}_i) : i < k\}$ from $\mathcal{MA}$. Choose any $(k+1)$-element subset $\{\varphi_i(x, \bar{y}_i) : i \leq k\}$ from $\mathcal{MA}$ and choose any $r \in \omega$. As notation, let $\psi(x, \bar{w}) := \bigvee_{i \leq k} \varphi_i(x, \bar{y}_i)$. By the inclusion/exclusion
principle of integers, the formula \( \exists^r x \bigvee_{i \leq k} \varphi(x, \bar{y}_i) \), which is equivalent to 
\( \exists^r x (\psi(x, \bar{w}) \lor \varphi_k(x, \bar{y}_k)) \), is equivalent to 
\[
\bigvee_{a,b \leq r \atop a+b-c=r} \left( \exists^a x \psi(x, \bar{w}) \land \exists^b x \varphi_k(x, \bar{y}_k) \land \exists^c x [\psi(x, \bar{w}) \land \varphi_k(x, \bar{y}_k)] \right)
\]

By the inductive hypothesis \( \exists^a x \psi(x, \bar{w}) \in \mathcal{MA}^* \) and \( \exists^b x \varphi_k(x, \bar{y}_k) \in \mathcal{MA}^* \) by the case \( k = 1 \). Also, note that \( \psi(x, \bar{w}) \land \varphi_k(x, \bar{y}_k) \) is equivalent to 
\( \bigvee_{i<k} \delta_i(x, \bar{y}_i, \bar{y}_k) \), where each \( \delta_i(x, \bar{y}_i, \bar{y}_k) := \varphi_i(x, \bar{y}_i) \land \varphi_k(x, \bar{y}_k) \) is in \( \mathcal{MA} \) by Lemma 2.4(4). Thus, by applying the inductive hypothesis to this \( k \)-element subset from \( \mathcal{MA} \), we conclude that 
\( \exists^c x (\psi(x, \bar{w}) \land \varphi_k(x, \bar{y}_k)) \in \mathcal{MA}^* \), completing the proof.

**Proposition 2.7** For any structure \( M \), the set \( \mathcal{MA}^*(M) \) is closed under existential quantification. Thus, the structure with universe \( M \), together with the definable sets \( \mathcal{MA}^*(M) \), is a reduct of the canonical expansion \( M_M \).

**Proof.** The second sentence follows from the first, since \( \mathcal{MA}^* \) is a set of \( L(M) \)-formulas closed under boolean combinations. To establish the first sentence, there are two cases. First, if the structure \( M \) is finite, then every \( L(M) \)-formula \( \varphi(\bar{z}) \in \mathcal{MA} \), so \( \mathcal{MA}^* \) is precisely the elementary diagram of \( M \) and there is nothing to prove. So assume that \( M \) is infinite.

Choose \( \varphi(x, \bar{y}) \in \mathcal{MA}^* \) and we argue that \( \exists x \varphi(x, \bar{y}) \) is equivalent to a formula in \( \mathcal{MA}^* \). By writing \( \varphi \) in Disjunctive Normal Form and noting that disjunction commutes with existential quantification, we may assume that \( \varphi(x, \bar{y}) \) has the form 
\[
\bigwedge_{i<k} \beta_i(x, \bar{y}_i) \land \bigwedge_{j<m} \neg \gamma_j(x, \bar{y}_j)
\]
where each \( \beta_i \) and \( \gamma_j \) are in \( \mathcal{MA} \) and the variable \( x \) occurs in each of these subformulas. By Lemma 2.4(4), if \( k \geq 1 \), then \( \bigwedge_{i<k} \beta_i(x, \bar{y}_i) \in \mathcal{MA} \), so we may assume there is at most one \( \beta \). If there is no \( \beta \), then since the model \( M \) is infinite, then for any choice of \( \bar{y} \), \( \exists x \varphi(x, \bar{y}) \) always holds. Thus, we assume that there is exactly one \( \beta \), i.e., that \( \varphi(x, \bar{y}) \) has the form \( \beta(x, \bar{y}^*) \land \bigwedge_{j<m} \neg \gamma_j(x, \bar{y}_j) \), where \( \bar{y}^* \) and each \( \bar{y}_j \) are subsequences of \( \bar{y} \), and both \( \beta \) and each \( \gamma_j \) are from \( \mathcal{MA} \).
We first consider the case where \( \bar{y}^* \) is empty. In this case, we may additionally assume that no \( \bar{y}_j \) is empty, since we could replace \( \beta(x) \) by \( \beta(x) \land \neg \gamma_j(x) \). Thus, for any choice of \( \bar{y} \), the solution set of \( \bigwedge_{j<m} \neg \gamma_j(x, \bar{y}_j) \) is a cofinite subset of \( M \). We have two subcases: On one hand, if \( \beta(x) \) were algebraic, then every solution to \( \beta \) lies in \( M \), hence \( \varphi(x, \bar{y}) \) would be equivalent to \( \bigvee_{m \in \beta(M)} \varphi(m, \bar{y}) \), which would be in \( \mathcal{MA}^* \) by Lemma 2.4(2). On the other hand, if \( \beta(x) \) were non-algebraic, then \( \beta(x) \) would have infinitely many solutions in \( M \), so \( \varphi(x, \bar{y}) \) would have a solution in \( M \) for any choice of \( \bar{y} \). Thus, \( \exists x \varphi(x, \bar{y}) \) would always hold.

Finally, assume that \( \bar{y}^* \neq \emptyset \). By the definition of mutual algebraicity, there is an integer \( K \) so that \( M \models \forall \bar{y}^* \exists < K x \beta(x, \bar{y}^*) \). For each \( j < m \), let \( \theta_j(x, \bar{y}^*, \bar{y}_j) := \beta(x, \bar{y}^*) \land \gamma_j(x, \bar{y}_j) \). By Lemma 2.4(4), each \( \theta_j(x, \bar{y}^*, \bar{y}_j) \in \mathcal{MA} \). Thus, the formula \( \exists x \varphi(x, \bar{y}) \) is equivalent to

\[
\bigvee_{r \leq K} \left( \exists^{= r} x \beta(x, \bar{y}^*) \land \exists^{< r} x \bigvee_{j<m} \theta_j(x, \bar{y}^*, \bar{y}_j) \right)
\]

which is in \( \mathcal{MA}^* \) by Lemma 2.6.

The previous Proposition inspires the following two definitions:

**Definition 2.8** A structure \( M \) is **mutually algebraic** if every \( L(M) \)-formula is in \( \mathcal{MA}^*(M) \).

**Definition 2.9** Let \( M \) be any structure. The **mutually algebraic reduct of \( M \)** is the structure with the same universe as \( M \), and whose definable sets are precisely \( \mathcal{MA}^*(M) \).

Proposition 2.7 immediately implies that the mutually algebraic reduct of a structure \( M \) is a mutually algebraic structure.

**Lemma 2.10** Mutual algebraicity of structures is preserved under elementary equivalence.

**Proof.** Suppose that \( M \) is a mutually algebraic structure and that \( N \) is elementarily equivalent to \( M \). It suffices to show that \( \varphi(\bar{x}, \bar{y}) \in \mathcal{MA}^*(N) \) for any \( L \)-formula \( \varphi(\bar{x}, \bar{y}) \) (with \( \bar{x} \) and \( \bar{y} \) disjoint and there are no hidden
parameters) and any \( \bar{h} \in N^{\lg(\theta)} \). Given this data, let \( \bar{z} = \bar{x} \ast \bar{y} \) and consider the \( L \)-formula \( \varphi(\bar{z}) \). As \( M \) is mutually algebraic, \( \varphi(\bar{z}) \in \mathcal{MA}^\ast(M) \), so there are (finitely many) \( L \)-formulas \( \delta_i(\bar{z}, \bar{w}_i) \) and \( \bar{c}_i \) from \( M \) so that (1) \( \varphi(\bar{z}) \) is \( Th(M_M) \)-equivalent to a boolean combination \( \theta(\bar{z}, \bar{c}_i^\ast) \) of the \( \delta_i(\bar{z}, \bar{c}_i) \) (\( \bar{c}_i^\ast \) denotes the concatenation of the \( \bar{c}_i \)'s); and (2) There is a number \( K \) so that each of the formulas \( \delta_i(\bar{z}, \bar{c}_i) \) satisfy \( M \models \forall \bar{y} \exists^\leq K \bar{x}' \delta_i(\bar{x}', \bar{y}', \bar{c}_i) \) for every proper partition \( \bar{z} = \bar{x}' \ast \bar{y}' \). Thus, by quantifying out the \( \bar{c}_i^\ast \), there is an \( L \)-sentence \( \sigma \) asserting that

\[
\exists \bar{w}^\ast \left( \forall \bar{z} [\varphi(\bar{z}) \leftrightarrow \theta(\bar{z}, \bar{w}^\ast)] \land \text{‘each } \delta_i(\bar{z}, \bar{w}_i) \text{ is } K\text{-mutually algebraic’} \right)
\]

As \( M \models \sigma \), so does \( N \). Choose \( \bar{c}^\ast \) from \( N \) so that \( \varphi(\bar{z}) \) is \( Th(N_N) \)-equivalent to \( \theta(\bar{z}, \bar{c}^\ast) \) and \( \theta(\bar{z}, \bar{c}^\ast) \) is equivalent to a boolean combination of \( \delta_i(\bar{z}, \bar{c}_i) \in \mathcal{MA}(N) \), where each \( \bar{c}_i \) is the corresponding subsequence of \( \bar{c}^\ast \). Finally, rewrite \( \bar{z} \) as \( \langle \bar{x}, \bar{y} \rangle \) and substitute \( \bar{h} \) for \( \bar{y} \). By Lemma 2.4(2), each of the formulas \( \delta_i(\bar{x}, \bar{h}, \bar{c}_i) \in \mathcal{MA}(N) \) and \( \varphi(\bar{x}, \bar{h}) \) is \( Th(N_N) \)-equivalent to the boolean combination \( \theta(\bar{x}, \bar{h}, \bar{c}^\ast) \). Thus, \( \varphi(\bar{x}, \bar{h}) \in \mathcal{MA}^\ast(N) \), as required.

The following Lemma is folklore, but a proof is included for the convenience of the reader. Recall that a partitioned formula \( \varphi(\bar{x}, \bar{y}) \) does not have the finite cover property (i.e., has nfcp) with respect to a theory \( T \) if there is a number \( k \) so that for all sets \( \{ \bar{c}_i : i \in I \} \), the type \( \Gamma := \{ \varphi(\bar{x}, \bar{c}_i) : i \in I \} \) is consistent with \( T \) whenever every \( k \)-element subset of \( \Gamma \) is consistent with \( T \).

**Lemma 2.11** Let \( M \) be any structure, and let \( \varphi(\bar{x}, \bar{y}) \) be any partitioned \( L(M) \)-formula. If, for some integer \( K \), either \( M \models \forall \bar{y} \exists^\leq K \bar{x} \varphi(\bar{x}, \bar{y}) \), or \( M \models \forall \bar{y} \exists^\leq K \bar{x} \neg \varphi(\bar{x}, \bar{y}) \), then \( \varphi(\bar{x}, \bar{y}) \) does not have the finite cover property with respect to \( Th(M_M) \).

**Proof.** If \( M \) is finite, then every partitioned formula \( \varphi(\bar{x}, \bar{y}) \) has nfcp for trivial reasons, so assume that \( M \) is infinite. First, assume that \( M \models \forall \bar{y} \exists^\leq K \bar{x} \varphi(\bar{x}, \bar{y}) \). Choose tuples \( \{ \bar{c}_i : i \in I \} \) from some elementary extension of \( M \) and assume that the type \( \Gamma := \{ \varphi(\bar{x}, \bar{c}_i) : i \in I \} \) is inconsistent. It suffices to find a subtype of at most \( K \) elements that is inconsistent as well. Choose a maximal sequence \( \langle i_j : j \leq n \rangle \) from \( I \) such that \( i_0 \in I \) is arbitrary
and for each $1 \leq m \leq n$, 
\[
\models \exists \bar{x} \left( \bigwedge_{j<m} \varphi(\bar{x}, \bar{c}_{i_j}) \land \neg \varphi(\bar{x}, \bar{c}_{i_m}) \right)
\]
By our hypotheses on $\varphi(\bar{x}, \bar{c}_{i_0}), n \leq K$. But now, if $\bigwedge_{j\leq n} \varphi(\bar{x}, \bar{c}_{i_j})$ were consistent but $\Gamma$ were not, we would contradict the maximality of the sequence.

In the other case, as $M$ is infinite, every partial type of the form $\{\varphi(\bar{x}, \bar{c}_i) : i \in I\}$ is consistent, so the nfcp of $\varphi(\bar{x}, \bar{y})$ is vacuously true.

**Proposition 2.12** For any structure $M$, the theory of the mutually algebraic reduct of $M$ has nfcp.

**Proof.** By the equivalence of (1) and $\forall m(2)_m$ in Theorem II 4.4 of [6] (whose proof does not use stability) it suffices to show that no partitioned formula of the form $\varphi(x, \bar{y}) \in MA^*$ with $\text{lg}(x) = 1$ has the finite cover property.

Consider any formula $\theta(x, \bar{y})$ of the form
\[
\bigwedge_{i<k} \beta_i(\bar{z}_i) \land \bigwedge_{j<m} \neg \gamma_j(\bar{z}_j)
\]
with each $\beta_i$ and $\gamma_j$ from $MA$. First, if the variable $x$ occurs in any $\beta_i$, then it follows that there is a number $K$ so that $M \models \forall \bar{y} \exists x^<K x \theta(x, \bar{y})$. Second, if $x$ does not occur in any $\beta_i$, then there is a number $K$ so that there is a number $K$ so that $M \models \forall \bar{y} \exists x^<K x \neg \theta(x, \bar{y})$. But, any formula $\varphi(x, \bar{y}) \in MA^*$ is a finite disjunction of formulas $\theta(x, \bar{y})$ described above. It follows that for some $K$, either $M \models \forall \bar{y} \exists x^<K x \varphi(x, \bar{y})$ or there is a number $K$ so that $M \models \forall \bar{y} \exists x^<K x \varphi(x, \bar{y})$ holds. Thus, $\varphi(x, \bar{y})$ has the nfcp by Lemma 2.11.

## 3 Characterizing theories of mutually algebraic structures

We begin with two definitions indicating that the forking behavior of 1-types (types with a single free variable) is particularly simple.
Definition 3.1 A complete, stable theory with an infinite model is weakly minimal if every forking extension of a 1-type is algebraic (equivalently if $R^\infty(x = x) = 1$) and is trivial if there do not exist a set $D$ and three elements $\{a, b, c\}$ that are dependent, but pairwise independent over $D$. A type $p \in S(D)$ is trivial if there do not exist a set $\{a, b, c\}$ of realizations of $p$ that are dependent, but pairwise independent over $D$.

It is well known that a weakly minimal theory is trivial if and only if every minimal type is trivial. The following Lemma generalizes the analogous result for non-trivial, strongly minimal theories that was proved by Baldwin and Baizhanov in [1].

Lemma 3.2 If $T$ is weakly minimal and non-trivial, then there is a model $M$ of $T$ and a subset $A \subseteq M$ such that $(M, A)$ is unstable.

Proof. Among all minimal types $p \in S(D)$ and formulas $\varphi(z, xy)$ over $D$ that contain a dependent, but pairwise independent triple $\{a, b, c\}$ of realizations of $p$, with the dependency witnessed by the algebraic formula $\varphi(z, ab) \in \text{tp}(c/Dab)$, choose one with the multiplicity of $\varphi(z, ab)$ as small as possible. It follows from this multiplicity condition that $\text{acl}(D \cup \{a\}) \cup \text{acl}(D \cup \{b\})$ does not contain any realizations of $\varphi(z, ab)$.

Fix $p \in S(D)$ and $\varphi(z, xy)$ as above, and let $M$ be a sufficiently saturated model containing $D$. To ease notation, we may assume $D = \emptyset$. Let $\{(a_i, b_i) : i \in \omega\}$ be a Morley sequence in $p^{(2)}$. That is, $\{a_i : i \in \omega\} \cup \{b_j : j \in \omega\}$ is an independent set of realizations of $p$. For each pair $(i, j) \in \omega^2$, choose $c_{i,j} \in p(M)$ realizing $\varphi(z, a_ib_j)$. Let $A = \{c_{i,j} : i \leq j < \omega\}$. We argue that the $L_P$-formula $\Phi(x, y) := \exists z (P(z) \land \varphi(z, xy))$ has the order property in $(M, A)$.

To see this, it is clear that the element $c_{i,j}$ witnesses $\Phi(a_i, b_j)$ whenever $i \leq j$. On the other hand, suppose some $c_{k,\ell}$ witnessed $\varphi(x, a_i, b_j)$. We argue that we must have $k = i$ and $\ell = j$: If neither equality held, then we would have $c_{k,\ell}$ forking with both sets $\{a_i, b_j\}$ and $\{a_k, b_\ell\}$. This is impossible, as the doubletons are independent from each other and the type $p$ is minimal, hence regular, hence of weight one. Similarly, suppose that $k = i$ but $\ell \neq j$. Then, working over $a_i$, $c_{i,\ell}$ is not algebraic over $a_i$, so $\text{tp}(c_{i,\ell}/a_i)$ is parallel to $p$, hence is also regular, so of weight one. But, working over $a_i$, $c_{i,\ell}$ forks with each of $b_j$ and $b_\ell$, which are independent over $a_i$. The case where $j = \ell$ is symmetric, completing the proof.
In what follows, a *mutually algebraic expansion* of a structure \( M \) is an expansion formed by adding arbitrarily many new relation symbols \( R_i \), whose interpretation is a mutually algebraic subset of \( M^{\text{arity}(R_i)} \) (see Definition 2.1).

In the Theorem that follows, we do not require that the theory \( T \) be complete.

**Theorem 3.3** The following are equivalent for any theory \( T \):

1. Every model of \( T \) is a mutually algebraic structure;

2. Every mutually algebraic expansion of every model of \( T \) is a mutually algebraic structure;

3. \( \text{Th}((M, A)) \) has the nfcp for every \( M \models T \) and every expansion \((M, A)\) by a unary predicate;

4. Every complete extension of \( T \) having an infinite model is weakly minimal and trivial.

**Proof.** (1) \( \Rightarrow \) (2) Fix \( M \models T \) and let \( \overline{M} = (M, R_i)_{i \in I} \) be any expansion of \( M \), where each \( R_i \) is a \( k(i) \)-ary relation symbol whose interpretation in \( \overline{M} \) is a mutually algebraic subset \( B_i \subseteq M^{k(i)} \). By definition, the \( M \)-definable subsets are the smallest class of subsets of \( M^\ell \) for various \( \ell \) that contain every \( M \)-definable set and every \( B_i \) and are closed under boolean combinations and projections. As \( M \) is mutually algebraic, every \( M \)-definable set is a boolean combination of mutually algebraic sets. So \( \mathcal{MA}^*(\overline{M}) \) contains every \( M \)-definable set and each of the sets \( B_i \). Additionally, \( \mathcal{MA}^*(\overline{M}) \) is closed under boolean combinations and projections. Thus, every \( \overline{M} \)-definable set is in \( \mathcal{MA}^*(\overline{M}) \), so \( \overline{M} \) is a mutually algebraic structure.

(2) \( \Rightarrow \) (3) Fix any \( M \models T \) and any expansion \( \overline{M} = (M, A) \) by a unary predicate. As every subset of \( M^1 \) is mutually algebraic, it follows from (2) that \( \overline{M} \) is a mutually algebraic structure, i.e., every \( \overline{M} \)-definable set is in \( \mathcal{MA}^*(\overline{M}) \). Thus, every partitioned \( \overline{M} \)-definable formula \( \varphi(\bar{x}, \bar{y}) \) has nfcp by Proposition 2.12. That is, the elementary diagram of \( \overline{M} \) and hence the theory of \( \overline{M} \) has nfcp.

(3) \( \Rightarrow \) (4) Suppose \( T \) satisfies (3). Fix any complete extension \( T' \) of \( T \) with an infinite model. As the nfcp implies stability, \( T' \) must be stable. Fix a sufficiently saturated model \( M \) of \( T' \). As \( T' \) is stable, if it were not weakly minimal then we could choose an element \( a \) and a tuple \( \bar{b} \) from \( M \) such that
tp(a/\bar{b}) forks over the empty set, but a is not algebraic over \bar{b}. Let \varphi(x,\bar{y}) be chosen so that \varphi(x,\bar{b}) \in tp(a/\bar{b}) witnesses the forking. As M is sufficiently saturated, we can find a Morley sequence \langle \bar{b}_i : i \in \omega \rangle in stp(\bar{b}) inside M. As \mathbb{T}' is stable, \langle \bar{b}_i : i \in \omega \rangle is an indiscernible set and there is a number k so that every element a* \in M is contained in at most k of the sets \mathcal{D}_i := \varphi(M, \bar{b}_i).

As each \mathcal{D}_i is infinite, we can construct a subset A of M such that each c \in A is contained in exactly one of the sets \mathcal{D}_i, and for each i, |A \cap \mathcal{D}_i| = i. Then the theory of the expansion (M, A), where the new unary predicate symbol P is interpreted as A, has the finite cover property as witnessed by the \mathcal{L}_P-formula \Psi(x, \bar{y}z) := P(x) \land \varphi(x, \bar{y}) \land x \neq z. Thus, \mathbb{T} must be weakly minimal. That \mathbb{T}' must be trivial as well follows from Lemma 3.2 and the fact that instability implies an instance of the finite cover property.

(4) \Rightarrow (1) This is the content of Theorem 4.2 of [5]. In fact, there it is shown that every M-definable formula is a boolean combination of mutually algebraic formulas of a very special form.

Corollary 3.4 Let M be any infinite structure. The mutually algebraic reduct of M described in Definition 2.9 is the maximal weakly minimal, trivial reduct of M.

Proof. The mutually algebraic reduct of M is a mutually algebraic structure, so it has a weakly minimal, trivial theory. Conversely, if any reduct of M has a weakly minimal, trivial theory, then it is a mutually algebraic structure, hence all of its definable sets are contained in MA*(M).

4 Mutually algebraic structures

Suppose that M is a mutually algebraic structure in a language L. We study models of the elementary diagram of M, or equivalently the class of elementary extensions of M. Note that if M is finite, then there are no proper elementary extensions of M, which will render all of the results that follow vacuous. Because of this, throughout this section we additionally assume that M is infinite. Thus, we may assume that M is elementarily embedded in a much larger, saturated ‘monster model’ \mathcal{C}.
By Theorem 3.3, $Th(M)$ is weakly minimal and trivial, so the quantifier elimination offered in [5] applies. Specifically, let

$$A(M) := \{ \text{all quantifier-free mutually algebraic } L(M)\text{-formulas } \alpha(\bar{z}) \}$$

and

$$E(M) = \{ \text{all } L(M)\text{-formulas of the form } \exists \bar{x} \alpha(\bar{x}, \bar{y}), \text{ where } \alpha(\bar{x}, \bar{y}) \in A(M) \}$$

and let $A^*(M)$ (respectively $E^*(M)$) denote the closure of $A(M)$ (respectively $E(M)$) under boolean combinations. Proposition 4.1 of [5] states that every quantifier-free $L(M)$-formula is equivalent to a formula in $A^*(M)$, while Theorem 4.2 states that every $L(M)$-formula is equivalent to a formula in $E^*(M)$.

As $Th(M)$ is weakly minimal, the relation ‘$a \in acl_M(B)$’ satisfies the axioms of a pre-geometry, where $acl_M(B)$ abbreviates $acl(M \cup B)$. (Algebraic closures are always computed with respect to satisfaction in $C$.) Thus, the binary relation $a \approx b$ on $C \setminus M$ defined by $a \in acl_M(\{b\})$ is an equivalence relation. The following easy Lemma is folklore.

**Lemma 4.1** Suppose $Th(M)$ is weakly minimal, $M \subseteq C$, and $A$ is any algebraically closed set satisfying $M \subseteq A \subseteq C$. Then $A$ is the universe of an elementary submodel of $C$.

**Proof.** The interpretation of any constant symbol is contained in $M$, and the fact that $A$ is algebraically closed implies that it is closed under every function symbol in the language. Thus, $A$ is the universe of a substructure of $C$. To see that this substructure is elementary, by the Tarski-Vaught criterion it suffices to show that for any $L$-formula $\varphi(x, \bar{y})$ and for any $\bar{a}$ from $A$, if $C \models \exists x \varphi(x, \bar{a})$, then there is $b \in A$ such that $C \models \varphi(b, \bar{a})$. So fix any $\varphi(x, \bar{a})$ and $b \in C$ such that $C \models \varphi(b, \bar{a})$. If $b \in A$, then we are done, so assume $b \notin A$. As $A$ is algebraically closed, this means that $tp(b/M\bar{a})$ is not algebraic. As $Th(M)$ is weakly minimal and $b$ is a singleton, this implies that $tp(b/M\bar{a})$ does not fork over $M$. But then, by symmetry and finite satisfiability of non-forking over models, there is $b^* \in M \subseteq A$ such that $C \models \varphi(b^*, \bar{a})$.

Recall that when combined with weak minimality, triviality implies that for any set $B \subseteq C$, $acl_M(B) = \bigcup_{b \in B} acl_M(\{b\})$.

**Proposition 4.2** Let $M$ be any mutually algebraic $L$-structure.
1. If $M \subseteq A \subseteq \mathcal{C}$ and $A$ is an arbitrary union of $\approx$-classes, then $A$ is an $L$-structure and $M \preceq A \preceq \mathcal{C}$; and

2. Conversely, if $M \preceq N \preceq \mathcal{C}$ and $B \subseteq N \setminus M$ is a set of $\approx$-representatives, then $N$ is the disjoint union of the sets $M$ and $\{\acl_M(\{b\}) \setminus M : b \in B\}$.

**Proof.** (1) $\Th(M)$ is weakly minimal and trivial by Theorem 3.3. By triviality, $A$ must be algebraically closed, so $A \preceq \mathcal{C}$ by Lemma 4.1. That $M \preceq A$ follows immediately from this.

(2) That the sets are disjoint follows by triviality. If there were an element $d \in N$ that was not in any of these sets, then $d$ would be $\approx$-inequivalent to every element of $B$, contradicting the maximality of $B$.

In light of the previous Proposition, it is natural to refer to the sets $\acl_M(\{b\}) \setminus M$ as the *components* of a given $N \succeq M$. Each component has size bounded by the number of $L(M)$-formulas, and one can speak of the type of a fixed enumeration of a component over $M$. The notion of a component map records this amount of data.

**Definition 4.3** Suppose that $M$ is a mutually algebraic structure and $N_1, N_2$ are both elementary extensions of $M$. A *component map* $f : N_1 \to N_2$ is a bijection such that $f|_M = id$ and for each $b \in N_1 \setminus M$,

- $f$ restricted to $\acl_M(\{b\})$ is elementary and
- $f(\acl_M(\{b\})) = \acl_M(\{f(b)\})$ setwise.

**Proposition 4.4** Suppose that $M$ is mutually algebraic and $N_1, N_2 \succeq M$. Then every component map $f : N_1 \to N_2$ is an isomorphism. Conversely, every isomorphism $f : N_1 \to N_2$ that is the identity on $M$ is a component map.

**Proof.** As every quantifier-free $L(M)$-formula is equivalent to a formula in $\mathcal{A}^*(M)$, it suffices to show that $f$ preserves every formula $\alpha(\bar{z}) \in \mathcal{A}(M)$. Choose any $\bar{a}$ from $N_1$. Without loss, by Lemma 2.4(2) and the fact that $f$ fixes $M$ pointwise, we may assume $\bar{a}$ is disjoint from $M$. There are now two cases. First, if $\bar{a} \subseteq \acl_M(\{b\})$ for some element $b$, then $f(\bar{a}) \subseteq \acl_M(\{b\})$ and $N_1 \models \alpha(\bar{a})$ if and only if $N_2 \models \alpha(f(\bar{a}))$ by the elementarity of $f$ restricted to $\acl_M(\{b\})$. Second, if $\bar{a}$ intersects at least two components, then $N_1 \models \neg \alpha(\bar{a})$
automatically. Furthermore, since \( f \) maps components onto components, \( f(\bar{a}) \) would intersect at least two components of \( N_2 \), so \( N_2 \models \neg \alpha(\bar{a}) \). Thus, \( \alpha \) is preserved in both cases, so \( f \) is an isomorphism. The converse is clear since elementary maps preserve algebraic closure.

We close with two examples of how the analog of Proposition 4.4 can fail if we work over \( \text{acl}(\emptyset) \) instead of a model.

**Example 4.5** Let \( L = \{R, S, E\} \), and let \( T \) be the theory asserting that \( E \) is an equivalence relation with exactly two classes, both infinite, and \( R \) is a binary ‘mating relation’ i.e., \( R \) is symmetric, irreflexive, and \( \forall x \exists y R(x, y) \). We further require that \( R(x, y) \rightarrow \neg E(x, y) \). Take \( S \) to be a 4-ary relation such that \( S(x, y, z, w) \) holds if and only if the four elements are distinct, and each of the relations \( R(x, y) \), \( R(z, w) \), and \( E(x, z) \) hold. Then \( T \) is complete, mutually algebraic, and \( \text{acl}(\emptyset) = \emptyset \). For any model \( N \) of \( T \), the decomposition of \( N \) into \( R \)-mated pairs is a decomposition of \( N \) into two-element ‘\( \emptyset \)-components’ i.e., sets \( A \) satisfying \( \text{acl}(\emptyset)(A) = A \). However, in contrast to Proposition 4.4, there are ‘\( \emptyset \)-component maps’ \( f : N \rightarrow N \), i.e., bijections \( f : N \rightarrow N \) whose restriction to each two-element \( \emptyset \)-component is elementary, that are not automorphisms.

The second example is from [3]. There, Baldwin, Shelah, and the author exhibit two models \( M, N \) of the theory of infinitely many, binary splitting equivalence relations that are not isomorphic in the set-theoretic universe \( V \), but there is a c.c.c. extension \( V[G] \) of \( V \) and \( M \cong N \) in \( V[G] \). This theory is also weakly minimal and trivial with \( \text{acl}(\emptyset) = \emptyset \). In fact, this theory has a prime model and every ‘component’ is a singleton. The complexity exploited by this example involves which strong types over the empty set are realized in the models \( M \) and \( N \).

**References**


