The elementary diagram of a trivial, weakly minimal structure is near model complete

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Abstract

We prove that if M is any model of a trivial, weakly minimal theory, then the elementary diagram $T(M)$ eliminates quantifiers down to Boolean combinations of certain existential formulas.

1 Introduction

There has been a growing body of research in the area of 'automatic quantifier elimination' where one places strong hypotheses on a theory T and proves that the elementary diagram $T(M)$ of any model of the theory has bounded quantifier depth (see [1, 2, 4]). For example, in [1], Dolich, Raichev, and the author proved that $T(M)$ is model complete whenever T is uncountably categorical, trivial, and of Morley rank 1. In this paper we relax the hypotheses on the theory somewhat and obtain a slightly weaker result.

Throughout the paper we assume that we are given an L-theory T that is complete, *weakly minimal,* and *trivial.* See e.g., [5] for the definitions of these notions. Taken together, these hypotheses imply that the notion of forking is remarkably well behaved, and in fact is determined by algebraic closure. This remark is made explicit in Fact 2.1. As well, we fix an arbitrary model M of T and consider the elementary diagram of M in the language $L(M)$,

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which is the language L with constant symbols for each element of M added. We also fix a very large, very saturated 'monster' $\mathfrak{C}$ elementarily extending M and work inside this model.

Any weakly minimal theory is superstable. Also, any ω-stable theory of Morley rank 1 is weakly minimal, but there are trivial, weakly minimal theories with continuum many 1-types over the empty set. An example is the theory of countably many independent unary predicates.

In Section 2 we define the class $\mathcal{A}$ of quantifier free, mutually algebraic formulas $\varphi(\bar{z})$. It is immediate from the definition that if $\bar{x}\hat{\ }\bar{y}$ is a partition of $\bar{z}$ with $\lg(\bar{x}), \lg(\bar{y}) \geq 1$, then for any fixed $\bar{y}$ there are only finitely many $\bar{x}$ for which $\varphi(\bar{x}, \bar{y})$ holds. We define the wider class $\mathcal{E}$ of formulas of the form $\exists \bar{x} \varphi(\bar{x}, \bar{y})$, where $\varphi(\bar{z}) \in \mathcal{A}$ and $\bar{x}\hat{\ }\bar{y}$ form a partition of $\bar{z}$. With Theorem 4.2 we prove that every $L(M)$-formula is $T(M)$-equivalent to a Boolean combination of formulas in $\mathcal{E}$.

Most of the uses of stability in this paper are very soft, mostly relying on the definability of types. This is primarily due to Fact 2.1, demonstrating that forking is easily understood in our context. We make heavy use of Fact 2.2, stating that trivial, weakly minimal theories do not have the finite cover property, i.e., have nfcp.

For basic facts about stable theories the reader is referred to any of a number of texts, including [5]. The reader is cautioned, however, that since we are interested in the *complexity* of definitions, we most definitely cannot make the 'usual assumptions' of stability theory. In particular, it would completely destroy the context to assume that every definable set were quantifier-free definable, and we most definitely are not permitted to pass to $\mathfrak{C}^{\mathrm{eq}}$.

We close the introduction with an example. It shows that Theorem 4.2 cannot be improved to conclude model completeness. Additionally, it shows that even though forking in our context is essentially a binary relation, the notion of mutual algebraicity, which is introduced in the next section, need not be. Similar examples exist in which R is n-ary for any $n \geq 2$.

Example 1.1 *A trivial, ω-stable, Morley rank 1, 2-dimensional theory T for which $T(M)$ is not model complete for any model M of T.*

Let L consist of a single ternary relation symbol R. The theory T asserts that R is antireflexive and symmetric. That is, if $R(a, b, c)$ holds then a, b, c are distinct and R holds of any permutation of them. So R describes a collection of 'triangles.' The theory T also asserts that every element is in at most one triangle, there are infinitely many elements that are in some triangle, and infinitely many elements that are 'singletons' i.e., not in any triangles.

For any model M of T, $T(M)$ is not model complete. To see this, choose any $N_1 \succeq M$ such that $N_1 \setminus M$ contains at least one singleton a and choose $N_2 \supseteq N_1$, $N_2 \succeq M$, such that in N_2, a is part of a triangle. Then $N_1 \not\preceq N_2$, so $T(M)$ is not model complete. ∎

2 Mutually algebraic formulas

We begin by stating two facts that will be used throughout the paper. The first characterizes forking in our context. Since T is weakly minimal and trivial, hence superstable of finite U-rank, T is *perfectly trivial* in the sense of Poizat, i.e., for all $\bar{a}$, $B_1 \subseteq B$, and $C_1 \subseteq C$, if $\text{tp}(\bar{a}/B)$ does not fork over B_1 and $\text{tp}(\bar{a}/C)$ does not fork over C_1, then $\text{tp}(\bar{a}/BC)$ does not fork over B_1C_1. A proof of this is given in [3]. Another proof is given in 2.5.8 and 4.2.7 of [5]. This result makes the following Fact straightforward.

Fact 2.1 *If T is weakly minimal and trivial, then for any set B containing a model M and any tuple $\bar{a}$ disjoint from M, the type $\text{tp}(\bar{a}/B)$ does not fork over M if and only if $\bar{a} \cap \text{acl}(B) = \emptyset$.*

Proof. As noted above, T is perfectly trivial. It follows from this and forking symmetry that $\text{tp}(\bar{a}/B)$ does not fork over M if and only if $\text{tp}(a/B)$ does not fork over M for every singleton $a \in \bar{a}$. But weak minimality implies that for any singleton $a \notin M$, $\text{tp}(a/B)$ forks over M if and only if $a \in \text{acl}(B)$. ∎

The following fact has more elementary proofs, but this one was chosen for brevity.

Fact 2.2 *If T is weakly minimal and trivial, then T does not have the finite cover property.*

Proof. We first show that T is nonmultidimensional, i.e., that every nonalgebraic type is nonorthogonal to a type over the empty set (see e.g., [5]). Fix a tuple $\bar{a}$ and a set B such that $p = \text{tp}(\bar{a}/B)$ is nonalgebraic. Choose a model M of T such that $\text{tp}(B\bar{a}/M)$ does not fork over the empty set and let $\bar{c} \subseteq \bar{a}$ be the subsequence of elements that are not in $\operatorname{acl}(MB)$. Then $\bar{c}$ is nonempty and $\operatorname{tp}(\bar{c}/BM)$ does not fork over M by Fact 2.1. By transitivity of nonforking $\operatorname{tp}(\bar{c}/B)$ does not fork over the empty set and is nonorthogonal to p. Thus, T is nonmultidimensional, hence T has the nfcp. This last implication follows from Shelah's characterization of the nfcp in stable theories and, working in $\mathfrak{C}^{\mathrm{eq}}$, Remark 8.2.13 of [5]. This argument is one place where it is helpful to pass to $\mathfrak{C}^{\mathrm{eq}}$, but it is not necessary to do so. ∎

It follows from the nfcp that for any partitioned formula $\varphi(\bar{x}, \bar{y})$ there is an integer N so that for any $\bar{b}$, if $\varphi(\bar{x}, \bar{b})$ has at least N solutions, then it has infinitely many. Because of this, the expression $\exists^{<\infty}\bar{x}\varphi(\bar{x}, \bar{y})$ is first order.

For a fixed tuple $\bar{z}$ of variable symbols, a *proper partition* $\bar{z} = \bar{x}\hat{\ }\bar{y}$ satisfies $\lg(\bar{x}), \lg(\bar{y}) \geq 1$, $\bar{x} \cup \bar{y} = \bar{z}$, and $\bar{x} \cap \bar{y} = \emptyset$. We do not require $\bar{x}$ to be an initial segment of $\bar{z}$, but we will write it as if it were to simplify the notation.

Definition 2.3 An $L(M)$-formula $\varphi(\bar{z})$ is *algebraic* if $\exists^{<\infty}\bar{z}\varphi(\bar{z})$. It simplifies our notation to call inconsistent formulas algebraic. A formula $\varphi(\bar{z})$ is *mutually algebraic* if $\forall\bar{y}\exists^{<\infty}\bar{x}\varphi(\bar{x}, \bar{y})$ for all proper partitions $\bar{z} = \bar{x}\hat{\ }\bar{y}$. We call a type algebraic (mutually algebraic) if it contains an algebraic (mutually algebraic) formula. Let

- $\mathcal{A} = \{$all quantifier-free, mutually algebraic $L(M)$-formulas$\}$;
- $\mathcal{E} = \{$all $L(M)$-formulas of the form $\exists\bar{x}\theta(\bar{x}, \bar{y})$, where $\theta \in \mathcal{A}\}$ (we allow $\lg(\bar{x}) = 0$ so $\mathcal{A} \subseteq \mathcal{E}$);
- $\mathcal{A}^* = \{$all $L(M)$-formulas $T(M)$-equivalent to a Boolean combination of formulas from $\mathcal{A}\}$; and
- $\mathcal{E}^* = \{$all $L(M)$-formulas $T(M)$-equivalent to a Boolean combination of formulas from $\mathcal{E}\}$.

The reader is cautioned that these notions depend on the set of free variables displayed. In particular, the notion of mutual algebraicity is **not** preserved under the addition of dummy variables. Note that every formula $\theta(z)$ with $\lg(z) = 1$ is mutually algebraic.

The notion of mutual algebraicity was introduced in [1]. The equivalence of the definition given here with the one offered there (at least for trivial, weakly minimal theories) follows from Lemma 3.3 below.

We enumerate some closure properties of these classes. None of these properties depend on our assumptions about the theory $T(M)$.

Lemma 2.4 *1. Every algebraic formula is mutually algebraic;*

2. *$\mathcal{A}^*$ and $\mathcal{E}^*$ are closed under Boolean combinations;*

3. *If $\varphi(x,\bar{y}) \in \mathcal{A}$ and $m \in M$, then $\varphi(m,\bar{y})$ is algebraic, hence in $\mathcal{A}$;*

4. *If $\alpha(x,\bar{y}) \in \mathcal{E}$ and $m \in M$, then $\alpha(m,\bar{y}) \in \mathcal{E}$;*

5. *If $\varphi(\bar{x}) \in L(M)$ is quantifier free and $\varphi(\bar{x}) \vdash \psi(\bar{x})$ for some $\psi(\bar{x}) \in \mathcal{A}$, then $\varphi(\bar{x}) \in \mathcal{A}$;*

6. *If $\varphi(\bar{y}), \psi(\bar{z}) \in \mathcal{A}$ and $\bar{y} \cap \bar{z} \neq \emptyset$ then $(\varphi \wedge \psi)(\bar{y},\bar{z}) \in \mathcal{A}$;*

7. *If $\alpha(\bar{y}), \beta(\bar{z}) \in \mathcal{E}$ and $\bar{y} \cap \bar{z} \neq \emptyset$ then $(\alpha \wedge \beta)(\bar{y},\bar{z}) \in \mathcal{E}$;*

8. *If $\alpha(\bar{x},\bar{y}) \in \mathcal{E}$ and $r \in \omega$ then $\exists^{\geq r}\bar{x}\alpha(\bar{x},\bar{y}) \in \mathcal{E}$;*

9. *If $\alpha(\bar{x},\bar{y}) \in \mathcal{E}$ and $r \in \omega$, then $\exists^{=r}\bar{x}\alpha(\bar{x},\bar{y})$ and $\exists^{<r}\bar{x}\alpha(\bar{x},\bar{y}) \in \mathcal{E}^*$.*

Proof. The proofs of (1)–(5) are immediate. For (6), suppose $\varphi(\bar{y})$ and $\psi(\bar{z})$ are both mutually algebraic and choose a variable symbol $x \in \bar{y} \cap \bar{z}$. We first claim that $\bar{e} \subseteq \mathrm{acl}(Me)$ for any $\bar{e}$ realizing $\varphi \wedge \psi$ and any $e \in \bar{e}$. As notation, for a tuple $\bar{e}$ realizing $\varphi \wedge \psi$, let e_x denote the 'x-coordinate' of $\bar{e}$, $\bar{e}_{\bar{y}}$ denote the subsequence $\bar{e}|\bar{y}$, and $\bar{e}_{\bar{z}}$ denote $\bar{e}|\bar{z}$. Fix any $\bar{e}$ realizing $\varphi \wedge \psi$ and choose any $e \in \bar{e}$. By symmetry we may assume that $e \in \bar{e}_{\bar{y}}$. Since $\varphi(\bar{y})$ is mutually algebraic, $\bar{e}_{\bar{y}} \subseteq \mathrm{acl}(Me)$. In particular, $e_x \in \mathrm{acl}(Me)$. Also, since $\psi(\bar{z})$ is mutually algebraic, $\bar{e}_{\bar{z}} \subseteq \mathrm{acl}(Me_x)$, so $\bar{e} \subseteq \mathrm{acl}(Me)$ as claimed. It follows immediately that $\exists^{<\infty}\bar{u}(\varphi \wedge \psi)(\bar{u},\bar{v})$ for any proper partition $\bar{u}\hat{\ }\bar{v}$ of $\bar{y}\hat{\ }\bar{z}$.

(7) follows immediately from (6) provided we choose disjoint bound variables for the formulas α and β. As for (8), if α is $\exists\bar{u}\varphi$ with $\varphi \in \mathcal{A}$, then $\exists^{\geq r}\bar{x}\alpha(\bar{x},\bar{y})$ is equivalent to

$$\exists\bar{x}_0\exists\bar{u}_0\exists\bar{x}_1\exists\bar{u}_1 \ldots \exists\bar{x}_{r-1}\exists\bar{u}_{r-1}\left(\bigwedge_{i<r}\varphi(\bar{u}_i,\bar{x}_i,\bar{y}) \ \wedge \bigwedge_{i<j<r}\bar{x}_i \neq \bar{x}_j\right)$$

That this formula is in $\mathcal{E}$ follows from (7) and (5). (9) follows immediately from (8) as these formulas are Boolean combinations of $\exists^{\leq s}\bar{x}\alpha(\bar{x},\bar{y})$ for various choices of s. ∎

3 Partitions

Throughout this section T is a trivial, weakly minimal L-theory and M is any model of T.

This section is devoted to showing that any $L(M)$-formula can be 'decomposed' as a Boolean combination of mutually algebraic pieces. In Section 4 it will be used to show that any quantifier-free $L(M)$-formula is in $\mathcal{A}^*$, but the results in this section hold for an arbitrary $L(M)$-formula.

Fix an $L(M)$-formula $\theta(\bar{z})$ and a proper partition $\bar{z} = \bar{x}\hat{\ }\bar{y}$. A *$\theta, \bar{x}$-formula* is any Boolean combination of formulas of the form $\theta(\bar{x}, \overline{m})$, where $\overline{m} \in M^{\lg(\bar{y})}$. A *$\theta, \bar{x}$-type* is an ultrafilter on the Boolean algebra of $\theta, \bar{x}$-formulas, and $S_{\theta,\bar{x}}(M)$ denotes the Stone space of all $\theta, \bar{x}$-types. By stability, for any $p \in S_{\theta,\bar{x}}(M)$ the set $\{\overline{m} \in M^{\lg(\bar{y})} : \theta(\bar{x}, \overline{m}) \in p\}$ is relatively definable by an L(M)-formula, which we denote by $d_p\bar{x}\theta(\bar{x}, \bar{y})$.

The first Lemma, although very simple, plays a key role in the compactness argument used in proving Proposition 3.4.

Lemma 3.1 *If $\varphi(\bar{x})$ is mutually algebraic and $\{\bar{c}_i : i \in \omega\}$ is a set of distinct realizations of φ, then there is an infinite $I \subseteq \omega$ such that $\{\bar{c}_i : i \in I\}$ are pairwise disjoint.*

Proof. By hypothesis φ is not algebraic. Since φ is mutually algebraic, $\{i \in \omega : \bar{c}_i \cap \bigcup_{j \in F} \bar{c}_j \neq \emptyset\}$ is finite for every finite $F \subseteq \omega$. It follows that any finite set of pairwise disjoint tuples is not maximal, so an infinite, pairwise disjoint subset exists. ∎

Definition 3.2 For a formula $\theta(\bar{z})$ and a proper partition $\bar{z} = \bar{x}\hat{\ }\bar{y}$, a $\theta, \bar{x}$-formula $\gamma(\bar{x})$ *determines* $\theta(\bar{x}, \bar{y})$ *generically* if $\gamma(\bar{x})$ is mutually algebraic and for any $\bar{y}$, either $\exists^{<\infty}\bar{x}(\gamma(\bar{x}) \wedge \theta(\bar{x}, \bar{y}))$ or $\exists^{<\infty}\bar{x}(\gamma(\bar{x}) \wedge \neg\theta(\bar{x}, \bar{y}))$.

Lemma 3.3 *If $\gamma(\bar{x})$ determines $\theta(\bar{x}, \bar{y})$ generically, then*

$$d_p\bar{x}\theta(\bar{x}, \bar{y}) \leftrightarrow d_q\bar{x}\theta(\bar{x}, \bar{y})$$

for all nonalgebraic $p, q \in S_{\theta,\bar{x}}(M)$ extending γ.

Proof. It suffices to show that $d_p\bar{x}\theta(\bar{x}, \bar{y}) \leftrightarrow \exists^{\infty}\bar{x}(\gamma(\bar{x}) \wedge \theta(\bar{x}, \bar{y}))$ for any nonalgebraic $p \in S_{\theta,\bar{x}}(M)$ extending γ. To see this, fix such a p. First choose $\bar{d}$ such that $d_p\bar{x}\theta(\bar{x},\bar{d})$. Since p is nonalgebraic there are infinitely many tuples $\bar{c}$ realizing the nonforking extension of p to $M\bar{d}$. Each of these tuples realizes $\gamma(\bar{x}) \wedge \theta(\bar{x},\bar{d})$. Next, choose $\bar{d}$ such that $\neg d_p\bar{x}\theta(\bar{x},\bar{d})$ holds. Then $d_p\bar{x}\neg\theta(\bar{x},\bar{d})$ holds, so as above, there are infinitely many realizations of $\gamma(\bar{x}) \wedge \neg\theta(\bar{x},\bar{d})$. Thus $\exists^{<\infty}\bar{x}(\gamma(\bar{x}) \wedge \theta(\bar{x},\bar{d}))$. ∎

Proposition 3.4 *For any formula $\theta(\bar{z})$, any proper partition $\bar{z} = \bar{x}\hat{\ }\bar{y}$, and any mutually algebraic but not algebraic $p \in S_{\theta,\bar{x}}(M)$, there is $\gamma(\bar{x}) \in p$ determining θ generically.*

Proof. Fix a mutually algebraic $p(\bar{x}) \in S_{\theta,\bar{x}}(M)$. We first argue that $d_p\bar{x}\theta(\bar{x},\bar{y})$ implies $\exists^{<\infty}\bar{x}(\gamma_0(\bar{x}) \wedge \neg\theta(\bar{x},\bar{y}))$ for some formula $\gamma_0 \in p$. If there were no such γ_0, then by compactness there would be distinct tuples $\{\bar{c}_i : i \in \omega\}$, each realizing $p(\bar{x})$, and $\bar{d}$ in $\mathfrak{C}$ such that $\neg\theta(\bar{c}_i,\bar{d})$ holds for each i, but $d_p\bar{x}\theta(\bar{x},\bar{d})$. By applying Lemma 3.1 and reindexing, we could additionally assume that the tuples $\{\bar{c}_i : i \in \omega\}$ are pairwise disjoint. But then, another application of compactness would assert the existence of a tuple $\bar{c}^*$ realizing p such that $\neg\theta(\bar{c}^*,\bar{d})$ holds, but $\bar{c}^* \cap \mathrm{acl}(M\bar{d}) = \emptyset$. By Fact 2.1 $\mathrm{tp}(\bar{c}^*/M\bar{d})$ does not fork over M, contradicting $d_p\bar{x}\theta(\bar{x},\bar{d})$. Thus, such a formula γ_0 exists. By increasing γ_0 slightly we may assume that γ_0 is mutually algebraic as well. Arguing symmetrically, there is also a formula $\gamma_1 \in p$ such $d_p\bar{x}\neg\theta(\bar{x},\bar{y})$ implies $\exists^{<\infty}\bar{x}(\gamma_1(\bar{x}) \wedge \theta(\bar{x},\bar{y}))$. Then the formula $\gamma_0 \wedge \gamma_1$ determines θ generically. ∎

Definition 3.5 Fix a tuple $\bar{z}$ of variable symbols. A *partition $\mathcal{P}$ of $\bar{z}$* is a set $\{\bar{x}_i : i < r\}$ of nonempty subsequences of $\bar{z}$ such that each variable symbol $z \in \bar{z}$ occurs in exactly one $\bar{x}_i$. We define a partial order $\leq$ on the (finite) set of partitions of $\bar{z}$ by:

$$\mathcal{P}' \leq \mathcal{P} \text{ if and only if every } \bar{x}_i \in \mathcal{P} \text{ is the union of classes } \bar{x}'_j \in \mathcal{P}'.$$

In particular, the partition of length $\lg(\bar{z})$ is at the 'bottom' of the partial order, while the the partition $\{\bar{z}\}$ of length 1 is at the 'top.'

Definition 3.6 Fix an $L(M)$-formula $\theta(\bar{z})$. For any $\bar{c} \in \mathfrak{C}^{\lg(\bar{z})}$ and any subsequence $\bar{x}$ of $\bar{z}$, $tp_{\theta,\bar{x}}(\bar{c}/M)$ denotes the $\theta,\bar{x}$-type $\{\theta(\bar{x},\overline{m}) : \mathfrak{C} \models \theta(\bar{c}_{\bar{x}},\overline{m})\}$, where $\bar{c}_{\bar{x}}$ denotes the subsequence of $\bar{c}$ induced by $\bar{x}$.

We let $\mathrm{tp}_\theta(\bar{c}/M) = \{\mathrm{tp}_{\theta,\bar{x}}(\bar{c}/M) : \text{all subsequences } \bar{x} \text{ of } \bar{z}\}$ and let $S_\theta(M) = \{\mathrm{tp}_\theta(\bar{c}/M) : \bar{c} \in \mathfrak{C}^{\lg(\bar{z})}\}$. For any $p \in S_\theta(M)$ and any subsequence $\bar{x}$ of $\bar{z}$, $p|\bar{x} = \mathrm{tp}_{\theta,\bar{x}}(\bar{c}/M)$ for some (equivalently for every) $\bar{c}$ realizing p.

Definition 3.7 A type $p(\bar{z}) \in S_\theta(M)$ is *coordinatewise nonalgebraic* if we have $\exists^\infty z \exists \hat{z} \psi(\bar{z})$ (where $\bar{z} = z\hat{\ }\hat{z}$) for every $\psi \in p$ and every variable symbol $z \in \bar{z}$. For $\mathcal{P} = \{\bar{x}_i : i < r\}$ any partition of $\bar{z}$, a type $p(\bar{z}) \in S_\theta(M)$ is *of species* $\mathcal{P}$ if p is coordinatewise nonalgebraic and, for each $i < r$, $\bar{x}_i$ is a subsequence of $\bar{z}$, the restriction $p|\bar{x}_i$ is mutually algebraic, but $p|\bar{y}$ is not mutually algebraic for any $\bar{x} \subsetneq \bar{y} \subseteq \bar{z}$.

Lemma 3.8 *If* $\bar{c} \cap M = \emptyset$*, then* $\mathrm{tp}_\theta(\bar{c}/M)$ *is coordinatewise nonalgebraic.*

Proof. Let $q = \mathrm{tp}(\bar{c}/M) \in S(M)$ denote the 'full type' of $\bar{c}$ in the language $L(M)$. Choose a Morley sequence $\{\bar{c}_i : i \in \omega\}$ in q. Then $\bar{c}_i \cap \bar{c}_j = \emptyset$ for all $i < j < \omega$. ∎

Lemma 3.9 *Every coordinatewise nonalgebraic* $p(\bar{z}) \in S_\theta(M)$ *has a species, i.e.,* p *is of species* $\mathcal{P}$ *for some partition* $\mathcal{P}$ *of* $\bar{z}$*.*

Proof. Let $\{\bar{x}_i : i < r\}$ be all subsequences of $\bar{z}$ such that for each i, $p|\bar{x}_i$ is mutually algebraic, but $p|\bar{y}$ is not mutually algebraic for any $\bar{x}_i \subsetneq \bar{y} \subseteq \bar{z}$. We argue that $\{\bar{x}_i : i < r\}$ is a partition of $\bar{z}$. First, since p is coordinatewise nonalgebraic, $p|\langle z\rangle$ is mutually algebraic for every variable symbol $z \in \bar{z}$, so every z is contained in some $\bar{x}_i$. Second, suppose that $z \in \bar{x}_i \cap \bar{x}_j$. Choose mutually algebraic $\varphi(\bar{x}_i) \in p|\bar{x}_i$ and $\psi(\bar{x}_j) \in p|\bar{x}_j$. By Lemma 2.4(6) $\varphi \wedge \psi$ is mutually algebraic, so $\bar{x}_i = \bar{x}_j$ by the maximality of the subsequences. ∎

Definition 3.10 Fix a formula $\theta(\bar{z})$ and a partition $\mathcal{P} = \{\bar{x}_i : i < r\}$. A $\mathcal{P}$-*determining formula* $\delta(\bar{z})$ has the form $\bigwedge_{i<r} \gamma_i(\bar{x}_i)$, where $\gamma_i(\bar{x}_i)$ determines θ generically.

Lemma 3.11 *For any formula* $\theta(\bar{z})$ *and any partition* $\mathcal{P} = \{\bar{x}_i : i < r\}$*, if* δ *is a* $\mathcal{P}$*-determining formula then* $\theta(\bar{c}) \leftrightarrow \theta(\bar{d})$ *for all* $\bar{c}, \bar{d}$ *realizing* δ *such that the types* $\mathrm{tp}_\theta(\bar{c})$ *and* $\mathrm{tp}_\theta(\bar{d})$ *are both of species* $\mathcal{P}$*.*

Proof. Suppose that $\delta(\bar{z}) = \bigwedge_{i<r} \gamma_i(\bar{x}_i)$, where each $\gamma_i(\bar{x}_i)$ determines θ generically. If $\gamma_i(\bar{x}_i)$ is algebraic for any $i < r$ then the Lemma is vacuously true, so assume that each $\gamma_i(\bar{x}_i)$ is nonalgebraic. For each $i < r$ choose a nonalgebraic type $q_i \in S_{\theta,\bar{x}_i}(M)$ extending $\gamma_i(\bar{x}_i)$. We will show that $\theta(\bar{c})$ holds if and only if the *sentence* $d_{q_0}\bar{x}_0 \dots d_{q_{r-1}}\bar{x}_{r-1}\theta \in T(M)$ for every $\bar{c}$ realizing δ with $\mathrm{tp}_\theta(\bar{c}/M)$ of species $\mathcal{P}$.

By symmetry assume that the sentence $d_{q_0}\bar{x}_0 \dots d_{q_{r-1}}\bar{x}_{r-1}\theta \in T(M)$. Fix any realization $\bar{c}$ of δ for which $\mathrm{tp}_\theta(\bar{c}/M)$ is of species $\mathcal{P}$. We construct a sequence $\langle \bar{c}^i : i \leq r \rangle$ of tuples of length $\lg(\bar{z})$ as follows: Let $\bar{c}^0 = \bar{c}$. Given $\bar{c}^i$ for $i < r$, choose $\bar{c}_i^*$ (of length $\lg(\bar{x}_i)$) realizing $q_i(\bar{x}_i)$ such that $\mathrm{tp}(\bar{c}_i^*/M\bar{c}^i)$ does not fork over M and let $\bar{c}^{i+1}$ be the sequence obtained from $\bar{c}^i$ by inserting $\bar{c}_i^*$ in place of the '$\bar{x}_i$th coordinates' $\bar{c}_i^i$ of $\bar{c}^i$.

Then $\bar{c}^r$ realizes the complete type $q_0 \times \dots \times q_{r-1}$, so $\theta(\bar{c}^r)$ holds. As well, by applying Lemma 3.3 for each $i < r$ we have that $\theta(\bar{c}^i) \leftrightarrow \theta(\bar{c}^{i+1})$ holds. Thus, $\theta(\bar{c})$ holds as required. ∎

Definition 3.12 A set $X \subseteq \mathfrak{C}^n$ is *definable off M* if there is a formula $\psi(\bar{z})$ with $\lg(\bar{z}) = n$ and $X \setminus M^n = \psi(\mathfrak{C}) \setminus M^n$. Two formulas $\theta(\bar{z})$ and $\psi(\bar{z})$ are *equivalent off M* if $\theta(\mathfrak{C}) \setminus M^n = \psi(\mathfrak{C}) \setminus M^n$.

Proposition 3.13 *Fix a formula $\theta(\bar{z})$ with $\lg(\bar{z}) = n$. For every partition $\mathcal{P}$ of $\bar{z}$ we have:*

1. *There is a finite set $\{\delta_j(\bar{z}) : j < n(\mathcal{P})\}$ of $\mathcal{P}$-determining formulas such that every $p \in S_\theta(M)$ of species $\mathcal{P}$ contains some δ_j; and*
2. *The set $\{\bar{c} \in \mathfrak{C}^n : \mathrm{tp}_\theta(\bar{c}/M)$ is of species $\mathcal{P}\}$ is definable off M by a Boolean combination of $\mathcal{P}'$-determining formulas for partitions $\mathcal{P}' \leq \mathcal{P}$.*

Proof. We argue by induction on the partial ordering $\leq$. Fix a partition $\mathcal{P} = \{\bar{x}_i : i < r\}$ and assume that (1) and (2) both hold for all partitions $\mathcal{P}' < \mathcal{P}$. For each $\mathcal{P}' < \mathcal{P}$ let $\psi_{\mathcal{P}'}(\bar{z})$ be a Boolean combination of $\mathcal{P}''$-determining formulas for partitions $\mathcal{P}'' \leq \mathcal{P}'$ defining $\{\bar{c} \in \mathfrak{C}^n : \mathrm{tp}_\theta(\bar{c}/M)$ is of species $\mathcal{P}'\}$ off M. As notation, for any tuple $\bar{c}$ such that $\mathrm{tp}_\theta(\bar{c}/M)$ is of species $\mathcal{P}$, we let $\bar{c}_i$ denote the subsequence corresponding to $\bar{x}_i$. In particular, $\bar{c}_i$ realizes $p|\bar{x}_i$.

To show that (1) holds for $\mathcal{P}$, we use Lemma 3.4 for each $p \in S_\theta(M)$ of species $\mathcal{P}$ and each $i < r$ to choose $\gamma_{p,i}(\bar{x}_i)$ determining θ generically. For each $p \in S_\theta(M)$ let $\delta_p(\bar{z}) = \bigwedge_{i<r} \gamma_{p,i}(\bar{x}_i)$. If there were no such finite set as in (1), then by compactness there would be a tuple $\bar{c}$ such that

- $\bar{c} \cap M = \emptyset$;
- $\bar{c}_i \cap \mathrm{acl}(M\bar{c}_j) = \emptyset$ for all $i < j < r$;
- $\mathrm{tp}_\theta(\bar{c})$ is not of species $\mathcal{P}'$ for any $\mathcal{P}' < \mathcal{P}$, and

- $\neg\delta_p(\bar{c})$ for each $p \in S_\theta(M)$ of species $\mathcal{P}$.

Let $q = \mathrm{tp}_\theta(\bar{c}/M)$. The first condition, together with Lemmas 3.8 and 3.9, imply that q is coordinatewise nonalgebraic and has a species. The second condition, together with Fact 2.1, implies that $\{\bar{c}_i : i < r\}$ is independent over M, so the species of q is $\leq \mathcal{P}$. The third condition implies that the species of q cannot be $< \mathcal{P}$, so the species of q must be equal to $\mathcal{P}$. But then $\delta_q(\bar{c})$ holds, which contradicts the fourth condition. Thus (1) holds for $\mathcal{P}$.

As for (2), it is easily checked that

$$\psi_{\mathcal{P}}(\bar{z}) := \bigvee_{j<n(\mathcal{P})} \delta_j^{\mathcal{P}}(\bar{z}) \quad \wedge \quad \bigwedge_{\mathcal{P}'<\mathcal{P}} \neg\psi_{\mathcal{P}'}(\bar{z})$$

defines the set $\{\bar{c} \in \mathfrak{C}^n : \mathrm{tp}_\theta(\bar{c}/M) \text{ is of species } \mathcal{P}\}$ off M. ∎

Corollary 3.14 *For any formula $\theta(\bar{z})$ there is a formula $\theta^*(\bar{z})$, which is a Boolean combination of mutually algebraic $\theta, \bar{x}$-formulas, that is equivalent to θ off M.*

Proof. For each partition $\mathcal{P}$ of $\bar{z}$, choose a finite set $\{\delta_{\mathcal{P},j}(\bar{z}) : j < n(\mathcal{P})\}$ and a formula $\psi_{\mathcal{P}}(\bar{z})$ as in Proposition 3.13. For each $\mathcal{P}$ and $j < n(\mathcal{P})$ Lemma 3.11 implies that off M, the truth value of θ is invariant on $\psi_{\mathcal{P}} \wedge \delta_{\mathcal{P},j}$. For each $\mathcal{P}$, let $G(\mathcal{P}) = \{j < n(\mathcal{P}) : (\psi_{\mathcal{P}} \wedge \delta_{\mathcal{P},j}) \to \theta \text{ off } M\}$. Then off M, $\theta(\bar{z})$ is equivalent to $\bigvee_{\mathcal{P}} \left(\psi_{\mathcal{P}} \wedge \bigvee_{j \in G(\mathcal{P})} \delta_{\mathcal{P},j}\right)$. ∎

4 Quantifier elimination

Throughout this section T is a trivial, weakly minimal L-theory and M is any model of T.

Proposition 4.1 *Every quantifier-free $L(M)$-formula $\theta(\bar{z})$ is in $\mathcal{A}^*$.*

Proof. We argue by induction on $\lg(\bar{z})$. For $\lg(\bar{z}) \leq 1$ every quantifier free $\theta(z) \in \mathcal{A}$. Now fix a quantifier-free $\theta(\bar{z})$ with $\lg(\bar{z}) \geq 2$ and assume the result holds for all quantifier-free $L(M)$-formulas with fewer free variables. By Corollary 3.14 there is $\theta^*(\bar{z}) \in \mathcal{A}^*$ that is equivalent to $\theta(\bar{z})$ off M. Write

$\bar{z} = x\hat{\ }\bar{y}$ with $lg(x) = 1$. By compactness there is a finite set $F \subseteq M$ such that

$$\theta(x, \bar{y}) \leftrightarrow \bigvee_{m \in F} (x = m \wedge \theta(m, \bar{y})) \ \vee \ \bigwedge_{m \in F} (x \neq m \wedge \theta^*(x, \bar{y}))$$

By our inductive hypothesis the formula on the right hand side is in $\mathcal{A}^*$. ∎

Theorem 4.2 *Every $L(M)$-formula is $T(M)$-equivalent to a Boolean combination of formulas from $\mathcal{E}$, i.e., $\mathcal{E}^* = L(M)$.*

Proof. We argue by induction on the complexity of formulas. By Proposition 4.1 $\mathcal{E}^*$ contains every quantifier-free $T(M)$-formula and $\mathcal{E}^*$ is obviously closed under Boolean combinations. Thus, to show that $\mathcal{E}^* = L(M)$ it suffices to show that

$$\exists x \alpha(x, \bar{y}) \in \mathcal{E}^* \text{ whenever } \alpha(x, \bar{y}) \in \mathcal{E}^*. \tag{1}$$

By writing α in Disjunctive Normal Form with respect to its subformulas in $\mathcal{E}$ and noting that disjunction commutes with existential quantification, it suffices to prove (1) when $\alpha(x, \bar{y})$ has the form

$$\bigwedge_{i<k} \beta_i(x, \bar{y}_i) \wedge \bigwedge_{j<m} \neg\gamma_j(x, \bar{y}_j)$$

where each β_i and γ_j are from $\mathcal{E}$ and $\bar{y}_i$ and $\bar{y}_j$ are subsequences of $\bar{y}$. As well, we may assume that x occurs in every β_i and γ_j. Since the variable x occurs in every β_i, Lemma 2.4(7) implies that $\bigwedge_{i<k} \beta_i(x, \bar{y}_i) \in \mathcal{E}$ Thus we may assume that $k \leq 1$. We first take care of two easy cases.

Case 1. $k = 0$, i.e., $\alpha(x, \bar{y}) = \bigwedge_{j<m} \neg\gamma_j(x, \bar{y}_j)$.

In this case, since $\gamma_j(\mathfrak{C}, \bar{y}_j)$ is finite for any choice of $\bar{y}$, $\exists x \alpha(x, \bar{y})$ always holds.

Thus, we may assume that there is precisely one β, i.e., that $\alpha(x, \bar{y})$ has the form $\beta(x, \bar{y}^*) \wedge \bigwedge_{j<m} \neg\gamma_j(x, \bar{y}_j)$, where β and each γ_j are in $\mathcal{E}$ and $\bar{y}^*, \bar{y}_j \subseteq \bar{y}$.

Case 2. $\bar{y}^* = \emptyset$.

If $\beta(x)$ is algebraic then $\exists x \alpha(x, \bar{y})$ is equivalent to $\bigvee_{m \in \beta(\mathfrak{C})} \alpha(m, \bar{y})$. On the other hand, if $\beta(x)$ is nonalgebraic then $\exists x \alpha(x, \bar{y})$ always holds by the same argument as in Case 1.

Finally, we are left with the general case where $\bar{y}^* \neq \emptyset$. By replacing each γ_j by the formula $\beta \wedge \gamma_j$ (which is also in $\mathcal{E}$ by Lemma 2.4(7)) we may additionally assume that $\gamma_j(\mathfrak{C}, \bar{y}_j) \subseteq \beta(\mathfrak{C}, \bar{y}^*)$ for any choice of $\bar{y}$. Since

$\beta(x, \bar{y}^*)$ is mutually algebraic, the set $\alpha(\mathfrak{C}, \bar{y})$ is finite for any choice of $\bar{y}$. In fact, since T has nfcp, there is an integer r^* such that $\alpha(\mathfrak{C}, \bar{y})$ has size less than r^* for any choice of $\bar{y}$. Since $\gamma_j(\mathfrak{C}, \bar{y}_j) \subseteq \beta(\mathfrak{C}, \bar{y}^*)$ we have that

$$\exists x \alpha(x, \bar{y}) \leftrightarrow \bigvee_{r<r^*} \left(\exists^{=r} x \beta(x, \bar{y}^*) \wedge \exists^{<r} x \bigvee_{j<m} \gamma_j(x, \bar{y}_j) \right)$$

The formula $\exists^{=r} x \beta(x, \bar{y}^*) \in \mathcal{E}^*$ by Lemma 2.4(9), so the result will follow once we show that $\exists^{<r} x \bigvee_{j<m} \gamma_j(x, \bar{y}_j) \in \mathcal{E}^*$. To see this, note that for any sequence of integers $\langle r_S : S \subseteq m \rangle$, Lemmas 2.4(7) and 2.4(9) assert that each of the formulas $\exists^{=r_S} \bigwedge_{j \in S} \gamma_j(x, \bar{y}_j)$ is in $\mathcal{E}^*$.

We recall a classical fact, which is essentially a fact about Venn diagrams, and is easily proved by induction on m.

Fact 4.3 *If U is finite and $\{A_i : i < m\}$ are subsets of U, then the cardinality of any Boolean combination of the subsets $\{A_i\}$ is computable from the sequence $\langle r_S : S \subseteq m \rangle$, where r_S is the cardinality of $\bigcap_{i \in S} A_i$.*

Thus, the formula $\exists^{<r} x \bigvee_{j<m} \gamma_j(x, \bar{y}_j)$ is $T(M)$-equivalent to a Boolean combination of formulas of the form $\exists^{=r_S} x \bigwedge_{j \in S} \gamma_j(x, \bar{y}_j)$, each of which is in $\mathcal{E}^*$ as noted above. So $\exists x \alpha(x, \bar{y}) \in \mathcal{E}^*$ and we finish. ∎

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