JUMPS IN SPEEDS OF HEREDITARY PROPERTIES IN FINITE RELATIONAL LANGUAGES

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ABSTRACT. Given a finite relational language $\mathcal{L}$, a *hereditary $\mathcal{L}$-property* is a class of $\mathcal{L}$-structures closed under isomorphism and substructure. The *speed of* $\mathcal{H}$ is the function which sends an integer $n \geq 1$ to the number of distinct elements in $\mathcal{H}$ with underlying set $\{1, ..., n\}$. In this paper we give a description of many new jumps in the possible speeds of a hereditary $\mathcal{L}$-property, where $\mathcal{L}$ is any finite relational language. In particular, we characterize the jumps in the polynomial and factorial ranges, and show they are essentially the same as in the case of graphs. The results in the factorial range are new for all examples requiring a language of arity greater than two, including the setting of hereditary properties of k-uniform hypergraphs for $k > 2$. Further, adapting an example of Balogh, Bollobás, and Weinreich, we show that for all $k \geq 2$, there are hereditary properties of k-uniform hypergraphs whose speeds oscillate between functions near the upper and lower bounds of the penultimate range, ruling out many natural functions as jumps in that range. Our theorems about the factorial range use model theoretic tools related to the notion of mutual algebricity.

1. INTRODUCTION

A hereditary graph property, $\mathcal{H}$, is a class of graphs which is closed under isomorphisms and induced subgraphs. The *speed* of $\mathcal{H}$ is the function which sends a positive integer n to $|\mathcal{H}_n|$, where $\mathcal{H}_n$ is the set of elements of $\mathcal{H}$ with vertex set $[n]$. Not just any function can occur as the speed of hereditary graph property. Specifically, there are discrete "jumps" in the possible speeds. Study of these jumps began with work of Scheinerman and Zito in the 90's [26], and culminated in a series of papers from the 2000's by Balogh, Bollobás, and Weinreich, which gave an almost complete picture of the jumps for hereditary graph properties. These results are summarized in the following theorem.

Theorem 1.1 ([2, 6, 7, 8, 11]). *Suppose $\mathcal{H}$ is a hereditary graph property. Then one of the following holds.*

(1) There are $k \in \mathbb{N}$ and rational polynomials $p_1(x), \dots, p_k(x)$ such that for sufficiently large n, $|\mathcal{H}_n| = \sum_{i=1}^{k} p_i(n) i^n$.
(2) There is an integer $k \geq 2$ such that $|\mathcal{H}_n| = n^{(1-\frac{1}{k}+o(1))n}$.
(3) There is an $\epsilon > 0$ such that for sufficiently large n, $\mathcal{B}_n \leq |\mathcal{H}_n| \leq 2^{n^{2-\epsilon}}$, where $\mathcal{B}_n \sim (n/\log n)^n$ denotes the n-th Bell number.
(4) There is an integer $k \geq 2$ such that $|\mathcal{H}_n| = 2^{(1-\frac{1}{k}+o(1))n^2/2}$.

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The jumps from (1) to (2) and within (2) are from [6], the jump from (2) to (3) is from [6, 8], the jump from (3) to (4) is from [2, 11], and the jumps within (4) are from [11]. Moreover, in [7], Balogh, Bollobás, and Weinreich showed that there exist hereditary graph properties whose speeds oscillate between functions near the lower and upper bound of range (3), which rules out most "natural" functions as possible jumps in that range. Further, structural characterizations of the properties in ranges (1), (2) and (4) are given in [6] (ranges (1) and (2)) and [11] (range (4)).

Despite the detailed understanding Theorem 1.1 gives us about jumps in speeds of hereditary graph properties, relatively little was known about the jumps in speeds of hereditary properties of higher arity hypergraphs. The goal of this paper is to generalize new aspects of Theorem 1.1 to the setting of hereditary properties in arbitrary finite relational languages. Specifically, we consider *hereditary $\mathcal{L}$-properties*, where, given a finite relational language $\mathcal{L}$, a hereditary $\mathcal{L}$-property is a class of $\mathcal{L}$-structures closed under isomorphisms and substructures. This notion encompasses most of the hereditary properties studied in the combinatorics literature[1], including for example, hereditary properties of posets, of linear orders, and of k-uniform hypergraphs (ordered or unordered) for any $k \geq 2$. We now summarize what was previously known about generalizing Theorem 1.1 to hereditary properties of $\mathcal{L}$-structures.

Theorem 1.2 ([1, 10, 30, 29])**.** *Suppose $\mathcal{L}$ is a finite relational language of arity $r \geq 1$ and $\mathcal{H}$ is a hereditary $\mathcal{L}$-property. Then one of the following holds.*

(i) There are constants $C, k \in \mathbb{N}^{>0}$ such that for sufficiently large n, $|\mathcal{H}_n| \leq Cn^k$.
(ii) There are constants $C, \epsilon > 0$ such that for sufficiently large n, $2^{Cn} \leq |\mathcal{H}_n| \leq 2^{n^{r-\epsilon}}$.
(iii) There is a constant $C > 0$ such that $|\mathcal{H}_n| = 2^{C\binom{n}{r}+o(n^r)}$.

The existence of a jump to (iii) was first shown for r-uniform hypergraphs in [1, 10], and later for finite relational langauges in [29]. The jump between (ii) and (iii) as stated (an improvement from [1, 10, 29]) is from [30]. The jump from (i) and (ii) for finite relational languages is from [30] (similar results were also obtained in [13]). Stronger results, including an additional jump from the exponential to the factorial range, have also been shown in special cases (see [5, 3, 4, 16, 17]). However, to our knowledge, Theorem 1.1 encompasses all that was known in general, and even in the special case of hereditary properties of r-uniform hypergraphs for $r \geq 3$ (unordered). Our focus in this paper is on the polynomial, exponential, and factorial ranges, where we obtain results analogous to those in Theorem 1.1 for arbitrary hereditary $\mathcal{L}$-properties.

Theorem 1.3. *Suppose $\mathcal{H}$ is a hereditary $\mathcal{L}$-property, where $\mathcal{L}$ is a finite relational language of arity $r \geq 1$. Then one of the following hold.*

(1) There are $k \in \mathbb{N}$ and rational polynomials $p_1(x), \dots, p_k(x)$ such that for sufficiently large n, $|\mathcal{H}_n| = \sum_{i=1}^k p_i(n) i^n$.
(2) There is an integer $k \geq 2$ such that $|\mathcal{H}_n| = n^{(1-\frac{1}{k}-o(1))n}$.
(3) $|\mathcal{H}_n| \geq n^{n(1-o(1))}$.

[1]Notable exceptions include hereditary properties of permutations and non-uniform hypergraphs.

The most interesting and difficult parts of Theorem 1.3 are the jumps within range (2), which were not previously known for any hereditary property in a language of arity larger than two. Combining Theorem 1.2 with Theorem 1.4 yields the following overall result about jumps in speeds of hereditary $\mathcal{L}$-properties.

Theorem 1.4. *Suppose $\mathcal{H}$ is a hereditary $\mathcal{L}$-property, where $\mathcal{L}$ is a finite relational language of arity $r \geq 1$. Then one of the following hold.*

(1) There are $k \in \mathbb{N}$ and rational polynomials $p_1(x), \ldots, p_k(x)$ such that for sufficiently large n, $|\mathcal{H}_n| = \sum_{i=1}^{k} p_i(n) i^n$.
(2) There is an integer $k \geq 2$ such that $|\mathcal{H}_n| = n^{(1-\frac{1}{k}-o(1))n}$.
(3) There is $\epsilon > 0$ such that $n^{n(1-o(1))} \leq |\mathcal{H}_n| \leq 2^{n^{r-\epsilon}}$.
(4) There is a constant $C > 0$ such that $|\mathcal{H}_n| = 2^{Cn^r + o(n^r)}$.

We also generalize results of [7] to show there are properties whose speeds oscillate between functions near the extremes of the penultimate range (case (3) of Theorem 1.4).

Theorem 1.5. *For all integers $r \geq 2$, and real numbers $c \geq 1/(r-1)$ and $\epsilon > 1/\ell$, there is a hereditary property of r-uniform hypergraphs $\mathcal{H}$ such that for arbitrarily large n, $|\mathcal{H}_n| = n^{cn(r-1)(1-o(1))}$ and for arbitrarily large n, $|\mathcal{H}_n| \geq 2^{n^{r-\epsilon}}$.*

While there still remain many open problems about the penultimate range, Theorem 1.5 shows that, for instance, there are no jumps of the form n^{kn} for $k > 1$ or 2^{n^k} for $1 < k < r$.

Together, Theorems 1.4 and 1.5 give us a much more complete picture of the possible speeds of hereditary properties in arbitrary finite relational languages. In particular, our results show the possibilities are very close to those for hereditary graph properties from Theorem 1.1. Our proof of Theorem 1.4 also gives structural characterizations of the properties in cases (1) and (2), and we will give explicit characterizations of the minimal properties with each speed in range (1). Characterizations of the minimal properties in range (2) are more complicated and will appear in forthcoming work of the authors.

The proofs in this paper owe much to the original proofs from the graphs setting, especially those appearing in [6, 7]. However, a wider departure was required to deal with the jumps in the factorial range, namely case (2) of Theorem 1.3. Our proofs here required a new theorem related to the model theoretic notion of *mutual algebricity*, first defined by Laskowski in [20]. A mutually algebraic property can be thought of as a generalization of a hereditary graph property of bounded degree graphs. The most important new ingredient in this paper is Theorem 4.8, proved by the authors in [18], which says a hereditary $\mathcal{L}$-property $\mathcal{H}$ is mutually algebraic if and only if no $\mathcal{M} \in \mathcal{H}$ can "distinguish" infinitely many infinite sets of disjoint tuples (see Section 4 for details). We will also make frequent use of the compactness theorem, which allows us to work with infinite elements of hereditary properties rather than large finite ones.

The effectiveness of model theoretic tools in the context of this paper can be attributed to the fact that any hereditary $\mathcal{L}$-property $\mathcal{H}$ can be viewed as the class of models of a universal, first-order theory $T_{\mathcal{H}}$. The speed of $\mathcal{H}$ is then the same as the function sending n to the number of distinct quantifier-free types in n variables appearing in any model of $T_{\mathcal{H}}$.

Problems about counting types have been fundamental to model theory for many years (see e.g. [28]). From this perspective it is not surprising that tools from model theory turn out to be useful for solving problems about speeds of hereditary $\mathcal{L}$-properties. Further, variations of this kind of problem have previously been investigated in model theory (see for example, [12, 21, 22]). We will point out direct connections this line of work throughout the paper.

We end this introduction with a some problems which remain open around this topic. First, Theorem 1.1 describes precisely the speeds occurring within the fastest growth rate (case (4)). A similar analysis in the hypergraph setting would amount to understanding the possible Turán densities of hereditary hypergraph properties, a notoriously difficult question which we have made no attempt to address in this paper (see e.g. [24]).

There are many questions remaining around the penultimate range. For instance, in the graph case, Theorem 1.1 gives a precise lower bound for the penultimate range, namely the n-th Bell number $\mathcal{B}_n$. This is accomplished in [8] by characterizing the minimal graph properties in this range, which they show are the properties consisting of disjoint unions of cliques $\mathcal{H}_{cl}$, or disjoint unions of anti-cliques, $\mathcal{H}_{\overline{cl}}$. A general analogue of this kind of result would be very interesting.

Problem 1.6. *Given a finite relational language $\mathcal{L}$, characterize the minimal hereditary $\mathcal{L}$-properties in the penultimate range.*

It is easy to see the answer must be more complicated in general than the graph case. Indeed, let $\mathcal{L} = \{R(x, y)\}$ and consider the hereditary $\mathcal{L}$-property $\mathcal{H}$ consisting of all transitive tournaments. Then $|\mathcal{H}_n| = n! < \mathcal{B}_n$ falls into the penultimate range. Consequently, while $\mathcal{H}_{cl}$, $\mathcal{H}_{\overline{cl}}$ are the only minimal hereditary *graph* properties in the penultimate range, there are other hereditary $\mathcal{L}$-properties in this range with strictly smaller speed.

Theorem 1.5 rules out many possible jumps in the penultimate range, however, there does not yet exist a satisfying formalization of the idea that there can be no more "reasonable" jumps in this range (see [7] for a thorough discussion of this). Finally, while Theorem 1.5 shows there are properties whose speeds oscillate infinitely often between functions near the upper and lower bounds, there also exist properties whose speeds lies in the penultimate range, and for which wide oscillation is not possible (for an example of this, see [9]). This leads to the following question.

Question 1.7. *Suppose $\mathcal{L}$ is a finite relational language. Are there jumps within the penultimate range among restricted classes of hereditary $\mathcal{L}$-properties (for instance among those which can be defined using finitely many forbidden configurations)?*

1.1. **Notation and outline.** We now give an outline of the paper. In Section 2 we deal with the polynomial/exponential case, i.e. case (1) of Theorem 1.4. Specifically, we define the class of *basic* hereditary properties, and show the speed of any basic $\mathcal{H}$ has the form appearing in case (1) of Theorem 1.4. In Section 3 we prove counting dichotomies for a restricted class of properties called *totally bounded properties*, which generalize bounded degree graph properties. In Section 4 we define mutually algebraic properties. We show non-mutually algebraic properties fall into cases (4) or (5), then prove counting dichotomies for mutually algebraic properties by showing they can be seen as "unions" of finitely many totally bounded properties, after an appropriate change in language. In Section 6, we generalize an example from [7] to show that for all $r \geq 2$, there are hereditary properties of r-uniform hypergraphs whose speeds oscillate between functions near the upper and lower bounds of the penultimate range

We spend the rest of this subsection fixing notation and definitions. We have attempted to include sufficient information here so that the reader with a only a basic knowledge of first-order logic could read this paper.

Suppose $\ell \geq 1$ is an integer, X is a set, and $\overline{x} = (x_1, \ldots, x_\ell) \in X^\ell$. Then $[\ell] = \{1, \ldots, \ell\}$, $\cup\overline{x} = \{x_1, \ldots, x_\ell\}$, and $|\overline{x}| = \ell$. We will sometimes abuse notation and write $\overline{x}$ instead of $\cup\overline{x}$ when it is clear from context what is meant. Given $\overline{x} = (x_1, \ldots, x_\ell)$ and $I \subseteq [\ell]$, $\overline{x}_I$ is the tuple $(x_i : i \in I)$. Given a sequence of variables $(z_1, \ldots, z_s)$, we write $\overline{z} = \overline{x} \wedge \overline{y}$ to mean there is a partition $I \cup J$ of $[s]$ into nonempty sets such that $\overline{x} = \overline{z}_I$ and $\overline{y} = \overline{z}_J$. In this case, we call $\overline{x} \wedge \overline{y}$ a *proper partition of* $\overline{z}$. Set

$$X^{\underline{\ell}} = \{(x_1, \ldots, x_\ell) \in X^\ell : x_i \neq x_j \text{ for each } i \neq j\} \quad \text{and} \quad \binom{X}{\ell} = \{Y \subseteq X : |Y| = \ell\}.$$

Given $u, v \in \mathbb{N}^{>0}$, a permutation $\sigma : [u] \to [u]$, and a set $\Sigma \subseteq [v]^u$ let

$$\sigma(\Sigma) := \{(v_{\sigma(1)}, \ldots, v_{\sigma(u)}) : (v_1, \ldots, v_u) \in \Sigma\}.$$

We say Σ is *invariant under* σ when $\sigma(\Sigma) = \Sigma$.

Suppose $\mathcal{L}$ is a first-order language consisting of finitely many relation and constant symbols. We let $|\mathcal{L}|$ denote the total number of constants and relations in $\mathcal{L}$. By convention, the *arity* of $\mathcal{L}$ is 0 if $\mathcal{L}$ consists of only constant symobls, and is otherwise the largest arity of a relation in $\mathcal{L}$. Given an $\mathcal{L}$-formula φ and a tuple of variables $\overline{x}$, we write $\varphi(\overline{x})$ to denote that the free variables of φ are all in the set $\cup\overline{x}$. Similarly, if p is a set of formulas, we write $p(\overline{x})$ to mean every formula in p has free variables in the set $\cup\overline{x}$. We will use script letters for $\mathcal{L}$-structures and the corresponding non-script letters for their underlying set. So for instance if $\mathcal{M}$ is an $\mathcal{L}$-structure, M denotes the underlying set of $\mathcal{M}$.

Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure. Given a formula $\varphi(x_1, \ldots, x_s)$,

$$\varphi^{\mathcal{M}} = \{(m_1, \ldots, m_s) \in M^s : \mathcal{M} \models \varphi(m_1, \ldots, m_s)\}.$$

A *forumla with parameters from* $\mathcal{M}$ is a an expression of the form $\varphi(\overline{x}, \overline{a})$ where $\varphi(\overline{x}, \overline{y})$ is a formula and $\overline{a} \in M^{|\overline{y}|}$. The *set of realizations of* $\varphi(\overline{x}, \overline{a})$ *in* $\mathcal{M}$ is

$$\varphi(\mathcal{M}, \overline{a}) := \{\overline{m} \in M^{|\overline{x}|} : \mathcal{M} \models \varphi(\overline{m}, \overline{a})\}.$$

Given $A \subseteq M$, a set $B \subseteq M^{|\overline{x}|}$ is *defined by* $\varphi(\overline{x}, \overline{y})$ *over* A if there is $\overline{a} \in A^{|\overline{y}|}$ such that $B = \varphi(\mathcal{M}; \overline{a})$. If $\Delta(x_1, \ldots, x_s)$ is a set of formulas (possibly with parameters from M), a *realization of* Δ *in* $\mathcal{M}$ is a tuple $\overline{m} \in M^s$ such that $\mathcal{M} \models \varphi(\overline{m})$ for all $\varphi(x_1, \ldots, x_s) \in \Delta$.

If C is the set of constants of $\mathcal{L}$, $C^{\mathcal{M}}$ is the set of interpretations of C in $\mathcal{M}$. If $f : M \to N$ is a bijection, then $f(\mathcal{M})$ is the $\mathcal{L}$-structure with domain N such that $c^{f(\mathcal{M})} = f(c^{\mathcal{M}})$ for each $c \in C$, and for each relation $R \in \mathcal{L}$, $R^{f(\mathcal{M})} = f(R^{\mathcal{M}})$. An *isomorphism from* $\mathcal{M}$ *to*

$\mathcal{N}$ is a bijection $f : M \to N$ such that $c^{\mathcal{N}} = f(c^{\mathcal{M}})$ for each $c \in C$, and for each relation $R \in \mathcal{L}$, $R^{\mathcal{N}} = f(R^{\mathcal{M}})$. An *automorphism* of $\mathcal{M}$ is an isomorphism from $\mathcal{M}$ to $\mathcal{M}$.

Given $X \subseteq M$ containing $C^{\mathcal{M}}$, $\mathcal{M}[X]$ is the $\mathcal{L}$-structure with domain X such that for all $c \in C$, $c^{\mathcal{M}[X]} = c^{\mathcal{M}}$ and for all relations $R(x_1, \ldots, x_s) \in \mathcal{L}$, $R^{\mathcal{M}[X]} = R^{\mathcal{M}} \cap X^s$. An $\mathcal{L}$-structure $\mathcal{N}$ is an *$\mathcal{L}$-substructure of* $\mathcal{M}$, denoted $\mathcal{N} \subseteq_{\mathcal{L}} \mathcal{M}$ if and only if $\mathcal{N} = \mathcal{M}[X]$ for some $X \subseteq M$. The *atomic formulas of* $\mathcal{L}$ are the formulas of the form $R(t_1, \ldots, t_n)$ where $R(x_1, \ldots, x_n)$ is a relation symbol of $\mathcal{L}$ and each t_i is either a variable or a constant from $\mathcal{L}$. It will be convenient in Section 2 to work with the following set of formulas.

Definition 1.8.

$$\Delta_{neq} := \{\varphi(x_1, \ldots, x_s) \wedge \bigwedge_{1 \leq i \neq j \leq s} x_i \neq x_j : \varphi(x_1, \ldots, x_s) \text{ is an atomic } \mathcal{L}\text{-formula}\}.$$

For example, if $R(x, y)$ is a relation of $\mathcal{L}$, then both $\tau(x_1, x_2) = R(x_1, x_2) \wedge x_1 \neq x_2$ and $\varphi(x) = R(x, x)$ and are in Δ_{neq}. Observe that for any $\mathcal{L}$-structure $\mathcal{M}$ and $\tau(x_1, \ldots, x_s) \in \Delta_{neq}$, $\tau^{\mathcal{M}} \subseteq M^{\underline{s}}$. Further, $\mathcal{M}$ is completely determined by knowing $\tau^{\mathcal{M}}$ for each $\tau \in \Delta_{neq}$. Specifically, if $\mathcal{N}$ is an $\mathcal{L}$-structure satisfying $\tau^{\mathcal{N}} = \tau^{\mathcal{M}}$ for all $\tau \in \Delta_{neq}$, then $\mathcal{M} = \mathcal{N}$.

A *hereditary $\mathcal{L}$-property* is a collection of $\mathcal{L}$-structures which is closed under isomorphism and $\mathcal{L}$-substructures. A hereditary property is often defined in the literature as a class of *finite* structures closed under isomorphisms and substructures, however, for the purposes of this paper it is convenient to allow infinite structures as well. Every hereditary $\mathcal{L}$-property $\mathcal{H}$ can be axiomatized using a (usually incomplete) universal theory, which we will denote by $T_{\mathcal{H}}$. In other words, there is a set of sentences, $T_{\mathcal{H}}$, which has the property that for any $\mathcal{L}$-structure, $\mathcal{M}$, $\mathcal{M} \in \mathcal{H}$ if and only if $\mathcal{M} \models T_{\mathcal{H}}$.

A hereditary $\mathcal{L}$-property $\mathcal{H}$ is *trivial* if there are only finitely many non-isomorphic $\mathcal{M} \in \mathcal{H}$. Equivalently, $\mathcal{H}$ is trivial if there is $N \in \mathbb{N}$ such that $\mathcal{H}_n = \emptyset$ for all $n \geq N$. Since we are interested in the size of $\mathcal{H}_n$ for large n, we will be exclusively concerned with non-trivial hereditary $\mathcal{L}$-properties in this paper. Any non-trivial hereditary $\mathcal{L}$-property must contain infinite elements by the compactness theorem.

Definition 1.9. Given an $\mathcal{L}$-structure $\mathcal{M}$, the *universal theory of* $\mathcal{M}$, $Th_{\forall}(\mathcal{M})$ is the set of sentences true in $\mathcal{M}$ which are of the form $\forall x_1 \ldots \forall x_n \varphi(x_1, \ldots, x_n)$, where $\varphi(x_1, \ldots, x_n)$ is a quantifier-free formula.

The *age of* $\mathcal{M}$, denoted *age*$(\mathcal{M})$, is the class of models of $Th_{\forall}(\mathcal{M})$.

The age of $\mathcal{M}$ is always a hereditary $\mathcal{L}$-property (but not every hereditary $\mathcal{L}$-property is the age of a single structure). When $\mathcal{H} = age(\mathcal{M})$, then for all $n \in \mathbb{N}$, $\mathcal{H}_n$ is the set of $\mathcal{L}$-structures with domain $[n]$ isomorphic to a substructure of $\mathcal{M}$. In general, $\mathcal{N} \in age(\mathcal{M})$ if and only if $\mathcal{N}$ is isomorphic to a substructure of an elementary extension of $\mathcal{M}$.

2. Case 1: Polynomial/Exponential Growth

In this section we give a sufficient condition for a hereditary property to have speed of the special form appearing in case (1) of Theorem 1.4 (we will see later it is in fact necessary and sufficient). Throughout this section, **$\mathcal{L}$ is finite language consisting of**

relations and/or constants, and $r \geq 0$ is the arity of $\mathcal{L}$. We will use the following natural relation defined on any $\mathcal{L}$-structure.

Definition 2.1. Given an $\mathcal{L}$-structure $\mathcal{M}$ and $a, b \in M$, define $a \sim b$ if and only if for every atomic formula $R(x_1, \ldots, x_s)$ and $m_2, \ldots, m_s \in M \setminus \{a, b\}$,

$$\mathcal{M} \models \Big(R(a, b, m_3, \ldots, m_s) \leftrightarrow R(b, a, m_3, \ldots, m_s)\Big) \wedge \Big(R(a, m_2, \ldots, m_s) \leftrightarrow R(b, m_2, \ldots, m_s)\Big).$$

In model theory terms, $a \sim b$ if and only if $\text{qftp}^{\mathcal{M}}(ab/(M \setminus \{a, b\})) = \text{qftp}^{\mathcal{M}}(ba/(M \setminus \{a, b\}))$.

Example 2.2. *Suppose $\mathcal{M} = (M, E)$ is a directed r-uniform hypergraph with vertex set M and edge set $E \subseteq M^{\underline{r}}$. Given $a, b \in M$ an $1 \leq i < j \leq r$, let*

$$N_{ij}(a, b) = \{(c_1, \ldots, c_{r-2}) \in M^{r-2} : (c_1, \ldots, c_{i-1}, a, c_i, \ldots, c_j, b, c_{j+1}, \ldots, c_{r-2}) \in E\} \text{ and}$$
$$N_i(a, b) = \{(c_1, \ldots, c_{r-1}) \in (M \setminus \{b\})^{r-1} : (c_1, \ldots, c_{i-1}, a, c_i, \ldots, c_r) \in E\}.$$

Considering $\mathcal{M}$ as an $\mathcal{L} = \{R(x_1, \ldots, x_r)\}$ structure in the usual way yields that for all $a, b \in \mathcal{M}$, $a \sim b$ holds if and only if for all $1 \leq i < j \leq r$, $N_{ij}(a, b) = N_{ij}(b, a)$ and $N_i(a, b) = N_i(b, a)$

It is easy to check that for any $\mathcal{L}$-structure $\mathcal{M}$, $\sim$ is an equivalence relation on $\mathcal{M}$.

Definition 2.3. A hereditary $\mathcal{L}$-property $\mathcal{H}$ is *basic* if there is $k \in \mathbb{N}$ such that every $\mathcal{M} \in \mathcal{H}$ has at most k distinct $\sim$-classes.

The main theorem of this section shows that the speeds of basic properties have the form appearing in case 1 of Theorem 1.4.

Theorem 2.4. *Suppose $\mathcal{H}$ is a basic hereditary $\mathcal{L}$-property. Then there is $k \in \mathbb{N}$ and rational polynomials $p_1(x), \ldots, p_k(x)$ such that for sufficiently large n, $|\mathcal{H}_n| = \sum_{i=1}^{k} p_i(n) i^n$.*

We will see later that something even stronger holds, namely that $\mathcal{H}$ is basic *if and only if* its speed has the form appearing in case (1) of Theorem 1.4 (see Corollary 5.1). **For the rest of this section, $\mathcal{H}$ is a fixed non-trivial, basic hereditary $\mathcal{L}$-property.**

We end this introductory subsection with a historical note. The equivalence relation of Definition 2.1 also makes an appearance in [15] (see section 2 there). In that paper, the authors show that for a countably infinite $\mathcal{L}$-structure $\mathcal{M}$, several properties are equivalent to $\mathcal{M}$ having finitely many $\sim$-classes. As a direct consequence, we obtain equivalent formulations of basic properties. Specifically, in the terminology of [15], a hereditary $\mathcal{L}$-property $\mathcal{H}$ is basic if and only if every coutable model of $\mathcal{T}_{\mathcal{H}}$ is *finitely partitioned*, if and only if every countable model of $\mathcal{T}_{\mathcal{H}}$ is *absolutely ubiquitous*.

2.1. **Infinite models as templates.** Our proof of Theorem 2.4 can be seen as a generalization of the proof of Theorem 20 in [6]. One idea used in our proof of Theorem 2.4 is to view countably infinite $\mathcal{M} \in \mathcal{H}$ as "templates" for finite elements of $\mathcal{H}$. In this subsection we fix notation to make this idea precise, and show that the set of finite structures compatible with a fixed template can be described using first order sentences. The main advantage of this approach is that it allows us to leverage the compactness theorem in the next subsection.

Fix a countably infinite $\mathcal{M} \in \mathcal{H}$. We make a series of definitions related to $\mathcal{M}$. First, by our assumption on $\mathcal{H}$, $\mathcal{M}$ has finitely many $\sim$-classes. Fix an enumeration of them, say $A_1, \ldots, A_k$, satisfying $0 < |A_1| \leq \ldots \leq |A_k|$, and call this the *canonical decomposition* of $\mathcal{M}$. It is straightforward to check that for each $\tau(x_1, \ldots, x_s) \in \Delta_{neq}$, there is $\Sigma_\tau^{\mathcal{M}} \subseteq [k]^s$ such that

$$\tau^{\mathcal{M}} = \bigcup \{(A_{i_1} \times \ldots \times A_{i_s}) \cap M^{\underline{s}} : (i_1, \ldots, i_s) \in \Sigma_\tau^{\mathcal{M}}\}.$$

Let $t = \max\{i \in [k] : A_i \text{ is finite}\}$ and set $K = \max\{r, |A_t|\}$. Given any set X, let $\Omega^{\mathcal{M}}(X)$ denote the set of ordered partitions $(X_1, \ldots, X_k)$ of X satisfying $|X_i| = |A_i|$ for each $i \in [t]$, $\min\{|X_i| : t < i \leq k\} > K$. Note $(A_1, \ldots, A_k) \in \Omega^{\mathcal{M}}(M)$.

Definition 2.5. An $\mathcal{L}$-structure $\mathcal{N}$ is *compatible* with $\mathcal{M}$ if there is $(B_1, \ldots, B_k) \in \Omega^{\mathcal{M}}(N)$ such that for each $\tau(x_1, \ldots, x_s) \in \Delta_{neq}$, $\tau^{\mathcal{N}} = \bigcup\{(B_{i_1} \times \ldots \times B_{i_s}) \cap N^{\underline{s}} : (i_1, \ldots, i_s) \in \Sigma_\tau^{\mathcal{M}}\}$.

Observe that if $\mathcal{N}$ is compatible with $\mathcal{M}$, witnessed by $(B_1, \ldots, B_k) \in \Omega^{\mathcal{M}}(N)$, then $\{B_1, \ldots, B_k\}$ are the $\sim$-classes of $\mathcal{N}$. It is straightforward to see that if $\mathcal{N}$ is finite and compatible with $\mathcal{M}$, then $\mathcal{N}$ is isomorphic to a substructure of $\mathcal{M}$, and is thus in $\mathcal{H}$. For this reason we think of $\mathcal{M}$ as forming a "template" for the structures $\mathcal{N}$ which are compatible with $\mathcal{M}$. For any $\mathcal{N} \models \theta_{\mathcal{M}}$, there is a sufficiently saturated elementary extension $\mathcal{M} \prec \mathcal{M}'$ such that $\mathcal{N}$ is isomorphic to a substructure of $\mathcal{M}'$. Thus $\{\mathcal{N} \in \mathcal{H} : \mathcal{N} \models \theta_{\mathcal{M}}\} \subseteq age(\mathcal{M})$. We now show that being compatible with $\mathcal{M}$ can be defined using a first-order sentence. We leave it to the reader to check that there is a formula $\varphi(x, y)$ (with quantifiers) such that for any $\mathcal{L}$-structure $\mathcal{G}$ and $a, b \in G$, $a \sim b$ if and only if $\mathcal{G} \models \varphi(a, b)$. We will abuse notation and write $x \sim y$ for this formula.

Lemma 2.6. *There is a sentence $\theta_{\mathcal{M}}$ such that for any $\mathcal{L}$-structure $\mathcal{N}$, $\mathcal{N} \models \theta_{\mathcal{M}}$ if and only if $\mathcal{N}$ is compatible with $\mathcal{M}$.*

Proof. Given $\tau(x_1, \ldots, x_s) \in \Delta_{neq}$, let $\varphi_{\tau,\mathcal{M}}(z_1, \ldots, z_k)$ be the following formula.

$$\forall x_1 \ldots \forall x_s \Big(\tau(x_1, \ldots, x_s) \leftrightarrow \Big(\Big(\bigwedge_{1 \leq i \neq j \leq s} x_i \neq x_j\Big) \wedge \Big(\bigvee_{(i_1, \ldots, i_s) \in \Sigma_\tau^{\mathcal{M}}} \Big(\bigwedge_{j=1}^{s} x_j \sim z_{i_j}\Big)\Big)\Big)\Big).$$

Note that for any $(a_1, \ldots, a_k) \in A_1 \times \ldots \times A_k$ and $\tau \in \Delta_{neq}$, $\mathcal{M} \models \varphi_{\tau,\mathcal{M}}(a_1, \ldots, a_k)$. Define $\theta_{\mathcal{M}}$ to be the following $\mathcal{L}$-sentence, where $n_i = |A_i|$ for each $i \in [t]$.

$$\exists z_1 \ldots \exists z_k \Big(\Big(\bigwedge_{i \in [t]} \exists^{=n_i} x(x \sim z_i)\Big) \wedge \Big(\bigwedge_{i \in [k] \setminus [t]} \exists^{>K} x(x \sim z_i)\Big) \wedge \Big(\bigwedge_{\tau \in \Delta_{neq}} \varphi_{\tau,\mathcal{M}}(z_1, \ldots, z_k)\Big)\Big).$$

We leave it to the reader to verify that for any $\mathcal{L}$-structure $\mathcal{N}$, $\mathcal{N} \models \theta_{\mathcal{M}}$ if and only if $\mathcal{N}$ is compatible with $\mathcal{M}$. □

2.2. **Proof of Theorem 2.4.** In this subsection we prove Theorem 2.4 by showing the speed of $\mathcal{H}$ is asymptotically equal to a sum of the form $\sum_{i=1}^k p_i(n) i^n$ for some rational polynomials $p_1, \ldots, p_k$. Our strategy is as follows. First, we compute the the number of $\mathcal{G} \in \mathcal{H}_n$ compatible with a single fixed $\mathcal{M} \in \mathcal{H}$ (Proposition 2.10). We then use the compactness theorem to show there are finitely many $\mathcal{M} \in \mathcal{H}$ which serve as templates for all sufficiently large elements of $\mathcal{H}$. This will then allow us to compute the speed of $\mathcal{H}$.

The first goal of the section is to prove Proposition 2.10, which shows the number of elements of $\mathcal{H}_n$ which are compatible with a fixed $\mathcal{M} \in \mathcal{H}$ is equal to $C|\Omega^{\mathcal{M}}([n])|$ for some constant C depending on $\mathcal{M}$. We give a brief outline of the argument here. Given a fixed $\mathcal{M} \in \mathcal{H}$ and $n \in \mathbb{N}$, every element of $\mathcal{H}_n$ which is compatible with $\mathcal{M}$ can be constructed by choosing an element of $\mathcal{P} \in \Omega^{\mathcal{M}}([n])$, then choosing the realizations of each $\tau \in \Delta_{neq}$ as prescribed by the set $\Sigma_\tau^{\mathcal{M}}$. This gives an upper bound of $|\Omega^{\mathcal{M}}([n])|$. The constant factor then arises from considering double counting. Dealing with the double counting is the motivation for the next definition.

Definition 2.7. Suppose $\mathcal{M} \in \mathcal{H}$ is countably infinite and $A_1, \ldots, A_k$ is its canonical decomposition. Define $Aut^*(\mathcal{M})$ to be the set of permutations $\sigma : [k] \rightarrow [k]$ with the property that there is an automorphism f of $\mathcal{M}$ satisfying $f(A_i) = A_{\sigma(i)}$ for each $i \in [k]$.

Proposition 2.10 will show that the number of element of $\mathcal{H}_n$ compatible with a fixed $\mathcal{M}$ is $|\Omega^{\mathcal{M}}([n])|/|Aut^*(\mathcal{M})|$. We need the next two lemmas for this.

Lemma 2.8. *Let $k \in \mathbb{N}^{>0}$, and let $\sigma : [k] \rightarrow [k]$ be a permutation. Assume $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{H}$ are countably infinite, both have k distinct $\sim$-classes, and $\mathcal{M}_2$ has at least as many finite $\sim$-classes as $\mathcal{M}_1$. If $\sigma(\Omega^{\mathcal{M}_2}(X)) \cap \Omega^{\mathcal{M}_1}(X) \neq \emptyset$ for some set X, then $\sigma(\Omega^{\mathcal{M}_2}(Y)) \subseteq \Omega^{\mathcal{M}_1}(Y)$ for all sets Y.*

Proof. Let $n_1 \leq \ldots \leq n_{t_1}$ and $m_1 \leq \ldots \leq m_{t_2}$ be the sizes of the sizes of the finite $\sim$-classes of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Note by assumption, $t_1 \leq t_2$. Set $K_1 = \max\{r, n_{t_1}\}$. By assumption, there is $(X_1, \ldots, X_k) \in \Omega^{\mathcal{M}_2}(X)$ such that $(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \in \Omega^{\mathcal{M}_1}(X)$. By definition, we must have that for each $i \in [t_1]$, $n_i = m_{\sigma(i)}$, and for all $i > t_1$, $m_{\sigma(i)} > K_1$. Since $t_1 \leq t_2$, this implies $\sigma([t_1]) = [t_1]$, and for all $i \in [t_1]$, $n_i = m_i$. Consequently, for all $j > t_1$, $m_j > K_1$. Clearly this implies that for any set Y, if $(Y_1, \ldots, Y_k) \in \Omega^{\mathcal{M}_2}(Y)$, then $(Y_{\sigma(1)}, \ldots, Y_{\sigma(k)}) \in \Omega^{\mathcal{M}_1}(Y)$, i.e. $\sigma(\Omega^{\mathcal{M}_2}(Y)) \subseteq \Omega^{\mathcal{M}_1}(Y)$.

□

The next Lemma gives us useful information about elements of $Aut^*(\mathcal{M})$.

Lemma 2.9. *Suppose $\mathcal{M} \in \mathcal{H}$ is countably infinite, $A_1, \ldots, A_k$ is its canonical decomposition, and $\sigma : [k] \rightarrow [k]$ is a permutation. Then $\sigma \in Aut^*(\mathcal{M})$ if and only if for all $\tau(x_1, \ldots, x_s) \in \Delta_{neq}$, $\Sigma_\tau^{\mathcal{M}}$ and $\Omega^{\mathcal{M}}(M)$ are invariant under σ.*

Proof. Let $t = \max\{i \in [k] : A_i \text{ is finite}\}$. First, suppose $\sigma \in Aut^*(\mathcal{M})$. Then there is an automorphism f of $\mathcal{M}$ such that for each $i \in [k]$, $f(A_i) = A_{\sigma(i)}$. This implies that for each $\tau(x_1, \ldots, x_s) \in \Delta_{neq}$,

$$\tau^{\mathcal{M}} = \bigcup_{(i_1,\ldots,i_s) \in \Sigma_\tau^{\mathcal{M}}} (A_{i_1} \times \ldots \times A_{i_s}) \cap M^{\underline{s}} = \bigcup_{(i_1,\ldots,i_s) \in \Sigma_\tau^{\mathcal{M}}} (A_{\sigma(i_1)} \times \ldots \times A_{\sigma(i_s)}) \cap M^{\underline{s}},$$

which implies $\sigma(\Sigma_\tau^{\mathcal{M}}) = \Sigma_\tau^{\mathcal{M}}$, i.e. $\Sigma_\tau^{\mathcal{M}}$ is invariant under σ. Further, since f is a bijection, $n_i = n_{\sigma(i)}$ for each $i \in [t]$, and $\sigma([t]) = [t]$. Therefore $\Omega^{\mathcal{M}}(M)$ is invariant under σ.

Suppose conversely that for all $\tau(x_1,\ldots,x_s)\in\Delta_{neq}$, $\Omega^{\mathcal{M}}(M)$ and $\Sigma^{\mathcal{M}}_{\tau}$ are invariant under σ. Since $\Omega^{\mathcal{M}}(M)$ is invariant under σ, $(A_{\sigma(1)},\ldots,A_{\sigma(k)})\in\Omega^{\mathcal{M}}(M)$. Therefore, for each $i\in[t]$, $|A_i|=|A_{\sigma(i)}|$, and for each $i\in[k]\setminus[t]$, $|A_i|=|A_{\sigma(i)}|=\aleph_0$. Thus there is a bijection $f:M\to M$ satisfying $f(A_i)=A_{\sigma(i)}$ for each $i\in[k]$.

Now fix $\tau(x_1,\ldots,x_s)\in\Delta_{neq}$ and $(a_1,\ldots,a_s)\in M^{\underline{s}}$. Suppose $\mathcal{M}\models\tau(a_1,\ldots,a_s)$. Then $(a_1,\ldots,a_s)\in A_{i_1}\times\ldots\times A_{i_s}$ for some $(i_1,\ldots,i_s)\in\Sigma^{\mathcal{M}}_{\tau}$. By definition of f, $(f(a_1),\ldots,f(a_s))\in A_{\sigma(i_1)}\times\ldots\times A_{\sigma(i_s)}$. Since $\sigma(\Sigma^{\mathcal{M}}_{\tau})=\Sigma^{\mathcal{M}}_{\tau}$, $(\sigma(i_1),\ldots,\sigma(i_s))\in\Sigma^{\mathcal{M}}_{\tau}$, so by definition of $\Sigma^{\mathcal{M}}_{\tau}$, $\mathcal{M}\models\tau(f(a_1),\ldots,f(a_s))$. This shows that $\mathcal{M}\models\tau(f(a_1),\ldots,f(a_s))$ if and only if $\mathcal{M}\models\tau(a_1,\ldots,a_s)$ (the "only if" part comes from the same argument applied to $\neg\tau$). Thus f is an automorphism of $\mathcal{M}$ and consequently $\sigma\in Aut^*(\mathcal{M})$. □

Proposition 2.10. *For any countably infinite $\mathcal{M}\in\mathcal{H}$ and sufficiently large $n\in\mathbb{N}$,*

$$|\{\mathcal{N}\in\mathcal{H}_n:\mathcal{N}\models\theta_{\mathcal{M}}\}|=|\Omega^{\mathcal{M}}([n])|/|Aut^*(\mathcal{M})|.$$

Proof. Fix a countably infinite $\mathcal{M}\in\mathcal{H}$ and a large $n\in\mathbb{N}$. Given $\mathcal{P}=(X_1,\ldots,X_k)\in\Omega^{\mathcal{M}}([n])$, let $\mathcal{N}_{\mathcal{P}}$ be the structure with domain $[n]$ satsifying, for each $\tau(x_1,\ldots,x_s)\in\Delta_{neq}$,

$$\tau^{\mathcal{N}_{\mathcal{P}}}=\bigcup\{(X_{i_1}\times\ldots\times X_{i_s})\cap[n]^{\underline{s}}:(i_1,\ldots,i_s)\in\Sigma^{\mathcal{M}}_{\tau}\}.$$

By definition, $\mathcal{G}\in\mathcal{H}_n$ is compatible with $\mathcal{M}$ if and only if $\mathcal{G}=\mathcal{N}_{\mathcal{P}}$ for some $\mathcal{P}\in\Omega^{\mathcal{M}}([n])$. Thus if we define $\Phi(\mathcal{P})=\mathcal{N}_{\mathcal{P}}$ for each $\mathcal{P}\in\Omega^{\mathcal{M}}([n])$, then Φ is a function $\Phi:\Omega^{\mathcal{M}}([n])\to\mathcal{H}_n$ satisfying $Im(\Phi)=\{\mathcal{N}\in\mathcal{H}_n:\mathcal{N}\models\theta_{\mathcal{M}}\}$. It suffices to show that for all $\mathcal{G}\in Im(\Phi)$, $|\Phi^{-1}(\mathcal{G})|=|Aut^*(\mathcal{M})|$, since then

$$|Im(\Phi)|=|\{\mathcal{N}\in\mathcal{H}_n:\mathcal{N}\models\theta_{\mathcal{M}}\}|=|\Omega^{\mathcal{M}}([n])|/|Aut^*(\mathcal{M})|.$$

Fix $\mathcal{G}\in Im(\Phi)$. By definition, there is a $\mathcal{P}\in\Omega^{\mathcal{M}}([n])$ so that $\mathcal{G}=\mathcal{N}_{\mathcal{P}}$. We show

$$\Phi^{-1}(\mathcal{G})=\{\sigma(\mathcal{P}):\sigma\in Aut^*(\mathcal{M})\}. \tag{1}$$

Suppose $\sigma\in Aut^*(\mathcal{M})$. By Lemma 2.9, $\Omega^{\mathcal{M}}(M)\cap\sigma(\Omega^{\mathcal{M}}(M))\neq\emptyset$, so Lemma 2.8 implies $\sigma(\Omega^{\mathcal{M}}([n]))\subseteq\Omega^{\mathcal{M}}([n])$. Thus $\sigma(\mathcal{P})\in\Omega^{\mathcal{M}}([n])$. Then, also by Lemma 2.9, $\sigma(\Sigma^{\mathcal{M}}_{\tau})=\Sigma^{\mathcal{M}}_{\tau}$ for each $\tau\in\Delta_{neq}$, which implies $\tau^{\mathcal{N}_{\mathcal{P}}}=\tau^{\mathcal{N}_{\sigma(\mathcal{P})}}$. Thus $\Phi(\sigma(\mathcal{P}))=\mathcal{N}_{\sigma(\mathcal{P})}=\mathcal{N}_{\mathcal{P}}=\mathcal{G}$, so $\sigma(\mathcal{P})\in\Phi^{-1}(\mathcal{G})$.

On the other hand, suppose $\mathcal{Q}\in\Phi^{-1}(\mathcal{G})$. Note $\mathcal{G}=\mathcal{N}_{\mathcal{P}}=\mathcal{N}_{\mathcal{Q}}$ implies there is a permutation $\eta:[k]\to[k]$ such that $\mathcal{Q}=\eta(\mathcal{P})$. Then $\mathcal{Q}\in\Omega^{\mathcal{M}}([n])\cap\eta(\Omega^{\mathcal{M}}([n]))$ and $\mathcal{P}\in\Omega^{\mathcal{M}}([n])\cap\eta^{-1}(\Omega^{\mathcal{M}}([n]))$ imply by Lemma 2.8 that $\Omega^{\mathcal{M}}(M)=\eta(\Omega^{\mathcal{M}}(M))$. Further, $\mathcal{N}_{\mathcal{P}}=\mathcal{N}_{\mathcal{Q}}$ implies that for each $\tau(x_1,\ldots,x_s)\in\Delta_{neq}$, $\eta(\Sigma^{\mathcal{M}}_{\tau})=\Sigma^{\mathcal{M}}_{\tau}$. Thus by Lemma 2.9, $\eta\in Aut^*(\mathcal{M})$. Since $Aut^*(\mathcal{M})$ is clearly closed under inverses and $\mathcal{Q}=\eta^{-1}(\mathcal{P})$, $\mathcal{Q}\in\{\sigma(\mathcal{P}):\sigma\in Aut^*(\mathcal{M})\}$. Thus we have shown (1) and $|\Phi^{-1}(\mathcal{G})|=|Aut^*(\mathcal{M})|$. □

Lemma 2.11 below, proved in [6], will be used to compute $|\Omega^{\mathcal{M}}([n])|$ for a fixed $\mathcal{M}\in\mathcal{H}$.

Lemma 2.11 (Lemma 19 in [6]). *Suppose $\ell,s,c_1,\ldots,c_t$ are integers. If $c_1,\ldots,c_t\leq s$, then there are rational polynomials $p_1,\ldots,p_\ell$ such that the following holds for all $n\geq\ell s+c$,* *where* $c = \sum_{i=1}^{t} c_i$.

$$\sum_{n_1,\ldots,n_\ell > s, \sum_{i=1}^{\ell} n_i = n-c} \binom{n}{n_1,\ldots,n_\ell,c_1,\ldots,c_t} = \sum_{i=1}^{\ell} p_i(n) i^n. \tag{2}$$

Further, if $\ell = 1$*, then* p_1 *has degree* c*.*

Note that for any $\mathcal{M} \in \mathcal{H}$, $|\Omega^{\mathcal{M}}([n])|$ is by definition of the form appearing in the left hand side of Lemma 2.11. We use this along with Proposition 2.10 to compute the number of $\mathcal{G} \in \mathcal{H}_n$ compatible with a fixed $\mathcal{M} \in \mathcal{H}$.

Corollary 2.12. *Suppose* $\mathcal{M} \in \mathcal{H}$ *is countably infinite with* ℓ *infinite* $\sim$*-classes. Then there are rational polynomials* $p_1, \ldots, p_\ell$ *such that for all sufficiently large* $n \in \mathbb{N}$*,*

$$|\{\mathcal{N} \in \mathcal{H}_n : \mathcal{N} \models \theta_{\mathcal{M}}\}| = \sum_{i=1}^{\ell} p_i(n) i^n.$$

Further, when $\ell = 1$*, the degree of* $p_1(x)$ *is equal to the number of elements of* $\mathcal{M}$ *in a finite* $\sim$*-class.*

Proof. Let $c_1 \leq \ldots \leq c_t$ be the sizes of the finite $\sim$-classes of $\mathcal{M}$. Set $c = \sum_{i \in I} c_i$, and $K = \max\{r, c_t\}$. By definition, if $\ell = k - t$, then

$$|\Omega^{\mathcal{M}}([n])| = \sum_{n_1,\ldots,n_\ell > K, \sum_{i=1}^{\ell} n_i = n-c} \binom{n}{n_1,\ldots,n_\ell,c_1,\ldots,c_t}. \tag{3}$$

By Lemma 2.11, there are rational polynomials $q_1, \ldots, q_\ell$ such that for large enough n, $|\Omega^{\mathcal{M}}([n])| = \sum_{i=1}^{\ell} q_i(n) i^n$. Further, if $\ell = 1$, then $q_1(x)$ has degree c. Combining this with Proposition 2.10, we obtain that for sufficiently large n,

$$|\{\mathcal{N} \in \mathcal{H}_n : \mathcal{N} \models \theta_{\mathcal{M}}\}| = |\Omega^{\mathcal{M}}([n])|/|Aut^*(\mathcal{M})| = \Big(\sum_{i=1}^{\ell} q_i(n) i^n\Big)/|Aut^*(\mathcal{M})| = \sum_{i=1}^{\ell} p_i(n) i^n,$$

where each $p_i(x)$ is the rational polynomial obtained by dividing the coefficients of $q_i(x)$ by the integer $|Aut^*(\mathcal{M})|$. □

We now prove our final lemma, which reduces the problem of counting the number of $\mathcal{G} \in \mathcal{H}_n$ compatible with finitely many templates to the problem of counting the number compatible with a single template.

Lemma 2.13. *Suppose* $\ell \geq 1$ *is an integer,* $\mathcal{M}_1, \ldots, \mathcal{M}_\ell \in \mathcal{H}$ *are countably infinite, and* $\theta_{\mathcal{M}_1} \wedge \ldots \wedge \theta_{\mathcal{M}_\ell}$ *is satisfiable. Then there is* $i \in [\ell]$ *such that* $\theta_{\mathcal{M}_1} \wedge \ldots \wedge \theta_{\mathcal{M}_\ell} \equiv \theta_{\mathcal{M}_i}$*.*

Proof. By induction it suffices to do the proof for $\ell = 2$. Suppose $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{H}$ are countably infinite and $\theta_{\mathcal{M}_1} \wedge \theta_{\mathcal{M}_2}$ is satsifiable. Clearly this implies $\mathcal{M}_1$ and $\mathcal{M}_2$ must have the same number of $\sim$-classes, say this number is $k \geq 1$. Without loss of generality, assume $\mathcal{M}_2$ has at least as many finite $\sim$-classes as $\mathcal{M}_1$.

By assumption, there is an $\mathcal{L}$-structure $\mathcal{B}$ satisfying both $\theta_{\mathcal{M}_1}$ and $\theta_{\mathcal{M}_2}$. Since $\mathcal{B} \models \theta_{\mathcal{M}_2}$, there is $(B_1, \ldots, B_k) \in \Omega^{\mathcal{M}_2}(B)$ so that for each $\tau(x_1, \ldots, x_s) \in \Delta_{neq}$,

$$\tau^{\mathcal{B}} = \bigcup \{(B_{j_1} \times \ldots \times B_{j_s}) \cap B^{\underline{s}} : (j_1, \ldots, j_s) \in \Sigma_\tau^{\mathcal{M}_2}\}. \tag{4}$$

Since $\mathcal{B} \models \theta_{\mathcal{M}_1}$, there is a permutation $\sigma : [k] \to [k]$ so that $(B_{\sigma(1)}, \ldots, B_{\sigma(k)}) \in \Omega^{\mathcal{M}_1}(B)$ and for all $\tau(x_1, \ldots, x_s) \in \Delta_{neq}$,

$$\tau^{\mathcal{B}} = \bigcup \{(B_{\sigma(j_1)} \times \ldots \times B_{\sigma(j_s)}) \cap B^{\underline{s}} : (j_1, \ldots, j_s) \in \Sigma_\tau^{\mathcal{M}_1}\}. \tag{5}$$

Observe (4) and (5) imply $\sigma(\Sigma_\tau^{\mathcal{M}_1}) = \Sigma_\tau^{\mathcal{M}_2}$ for all $\tau \in \Delta_{neq}$. Further, $\sigma(\Omega^{\mathcal{M}_2}(B)) \cap \Omega^{\mathcal{M}_1}(B) \neq \emptyset$, so Lemma 2.8 implies that $\sigma(\Omega^{\mathcal{M}_2}(Y)) \subseteq \Omega^{\mathcal{M}_1}(Y)$ for any set Y.

We now show $\theta_{\mathcal{M}_1} \wedge \theta_{\mathcal{M}_2} \equiv \theta_{\mathcal{M}_2}$. Clearly it suffices to show $\theta_{\mathcal{M}_2} \models \theta_{\mathcal{M}_1}$. Fix $\mathcal{N} \models \theta_{\mathcal{M}_2}$. Then there is $(N_1, \ldots, N_k) \in \Omega^{\mathcal{M}_2}(N)$ so that for each $\tau(x_1, \ldots, x_s) \in \Delta_{neq}$

$$\tau^{\mathcal{N}} = \bigcup \{(N_{i_1} \times \ldots \times N_{i_s}) \cap N^{\underline{s}} : (i_1, \ldots, i_s) \in \Sigma_\tau^{\mathcal{M}_2}\}. \tag{6}$$

Since $\sigma(\Omega^{\mathcal{M}_2}(N)) \subseteq \Omega^{\mathcal{M}_1}(N)$, we have $(N_{\sigma(1)}, \ldots, N_{\sigma(k)}) \in \Omega^{\mathcal{M}_1}(N)$. Further, for each $\tau(x_1, \ldots, x_s) \in \Delta_{neq}$, $\sigma(\Sigma_\tau^{\mathcal{M}_1}) = \Sigma_\tau^{\mathcal{M}_2}$, so (6) implies that

$$\tau^{\mathcal{N}} = \bigcup \{(N_{\sigma(i_1)} \times \ldots \times N_{\sigma(i_s)}) \cap N^{\underline{s}} : (i_1, \ldots, i_s) \in \Sigma_\tau^{\mathcal{M}_1}\}.$$

Thus by definition, $\mathcal{N} \models \theta_{\mathcal{M}_1}$. □

We now prove the main result of this subsection. The proof uses the compactness theorem and the preceding lemma to show the speed of $\mathcal{H}$ is a linear combination of finitely many functions of the form appearing in Corollary 2.12.

Theorem 2.14. *There are $k \in \mathbb{N}$ and rational polynomials $p_1(x), \ldots, p_k(x)$ such that for sufficiently large n, $|\mathcal{H}_n| = \sum_{i=1}^k p_i(n) i^n$.*

Proof. Clearly the following set of sentences is inconsistent.

$$T_{\mathcal{H}} \cup \{\neg \theta_{\mathcal{M}} : \mathcal{M} \models T_{\mathcal{H}} \text{ is countably infinite}\} \cup \{\exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j : n \geq 1\}.$$

Thus by compactness, there are finitely many $\theta_{\mathcal{M}_1}, \ldots, \theta_{\mathcal{M}_k}$ such that for sufficiently large n, any element of $\mathcal{H}_n$ must satisfy $\bigvee_{i=1}^k \theta_{\mathcal{M}_i}$. Combining this with the inclusion/exclusion principle yields that for large n,

$$\begin{aligned} |\mathcal{H}_n| &= |\{\mathcal{M} \in \mathcal{H}_n : \mathcal{M} \models \theta_{\mathcal{M}_1} \vee \ldots \vee \theta_{\mathcal{M}_k}\}| \\ &= \sum_{u=1}^k (-1)^{u+1} \Big(\sum_{1 \leq i_1 < \ldots < i_u \leq k} |\{\mathcal{M} \in \mathcal{H}_n : \mathcal{M} \models \theta_{\mathcal{M}_{i_1}} \wedge \ldots \wedge \theta_{\mathcal{M}_{i_u}}\}| \Big). \end{aligned}$$

Apply Lemmas 2.10 and 2.13 to finish the proof. □

Theorem 2.14 proves Theorem 2.4, since $\mathcal{H}$ was an arbitrary basic non-trivial hereditary $\mathcal{L}$-property. We have actually shown more about basic properties, which we sum up in Corollary 2.15 below. We leave the proof to the reader, as it follows from the proof of Theorem 2.14 and the fact that for any $\mathcal{M} \in \mathcal{H}$, $\{\mathcal{N} \in \mathcal{H} : \mathcal{N} \models \theta_{\mathcal{M}}\} \subseteq age(\mathcal{M}) \subseteq \mathcal{H}$.

Corollary 2.15. *Suppose $\mathcal{H}$ is a non-trivial basic hereditary $\mathcal{L}$-property. Then there is a trivial hereditary $\mathcal{L}$-property $\mathcal{F}$ and finitely many countably infinite basic $\mathcal{L}$-structures $\mathcal{M}_1, \ldots, \mathcal{M}_m$ such that $\mathcal{H} = \mathcal{F} \cup \bigcup_{i=1}^m age(\mathcal{M}_i)$.*

Moreover the following hold, where for each $1 \leq i \leq m$, m_i is the number of elements of $\mathcal{M}_i$ in a finite $\sim$-class and ℓ_i is the number of infinite $\sim$-classes of $\mathcal{M}_i$.

1. *If $\ell = \max\{\ell_i : i \in [m]\} = 1$ then for large n, $|\mathcal{H}_n| = p(n)$ where $p(n)$ is a rational polynomial of degree $c := \max\{m_i : i \in [m]\}$.*
2. *If $\ell = \max\{\ell_i : i \in [m]\} \geq 2$, then for large n, $|\mathcal{H}_n| = \sum_{i=1}^{\ell} p_i(n) i^n$ where each $p_i(n)$ is a rational polynomial.*

The dichotomy between cases (1) and (2) in Corollary 2.15 also made an appearance in [15]. Specifically, their terminology, the results of [15] show case (1) in Corollary 2.15 holds if and only if every model $\mathcal{M} \models T_{\mathcal{H}}$ is absolutely $|M|$-ubiquitous.

3. Totally bounded properties

In this section we prove results about a very restricted class of properties, namely those which are *totally bounded*. The main result of this section is Theorem 3.9 which tells us about the speeds of totally bounded properties. Totally bounded hereditary $\mathcal{L}$-properties behave much like hereditary graph properties with uniformly bounded degree, and our proofs in this section largely follow the corresponding proofs for these kinds of graphs (see Lemmas 24 and 25 in [6]). The results of this subsection will be used in Section 4 to prove counting dichotomies for more general classes.

In this section $\mathcal{L}$ consists of finitely many relations and/or constants and $r \geq 0$ is the arity of $\mathcal{L}$. Throughout, C denotes the set of constants of $\mathcal{L}$.

Definition 3.1. An $\mathcal{L}$-structure $\mathcal{M}$ is *totally k-bounded* if for every relation $R(x_1, \ldots, x_s)$ in $\mathcal{L}$, and every partition $[s] = I \cup J$ into nonempty sets I and J,

$$\mathcal{M} \models \forall \overline{x}_I \exists^{<k} \overline{x}_J R(x_1, \ldots, x_s).$$

A hereditary $\mathcal{L}$-property $\mathcal{H}$ is *totally bounded* if there is an integer k such that every $\mathcal{M} \in \mathcal{H}$ is totally k-bounded.

For example, if $k \in \mathbb{N}$ and $\mathcal{H}$ is a hereditary graph property, then $\mathcal{H}$ is totally k-bounded if and only if all of the graphs in $\mathcal{H}$ have maximum degree less than k.

We begin by considering a generalization of the notion of a connected component in a graph. Given an $\mathcal{L}$-structure $\mathcal{M}$ and $a, b \in M$, a *path from a to b* is a finite sequence $\overline{a}_1, \ldots, \overline{a}_k$ of tuples of elements of M such that the following hold.

1. For each $1 \leq i \leq k-1$, $\overline{a}_i \cap \overline{a}_{i+1} \neq \emptyset$.
2. $\mathcal{M} \models \psi_1(\overline{a}_1) \wedge \ldots \wedge \psi_k(\overline{a}_k)$ for some relations $\psi_1(\overline{x}_1), \ldots, \psi_k(\overline{x}_k)$ from $\mathcal{L}$.

Given $a, b \in M$, a *path from a to b* is a path $\bar{a}_1, \ldots, \bar{a}_k$ such that $a \in \bar{a}_1$ and $b \in \bar{a}_k$. We say a subset $A \subseteq M$ is *connected* if for all $a \neq b \in A$, there is a path from a to b which is contained in A. When $\mathcal{M}$ is a graph, then these are just the usual graph theoretic notions of a path and of connected sets.

Definition 3.2. Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure, and $A \subseteq M$. We say that A is a *component in $\mathcal{M}$* if it is a maximal connected set, i.e. it is connected and for all $a, c \in M$, if there is a path from a to c and $a \in A$, then $c \in A$.

We would like to point out that the notion of components and $\sim$-classes are different. For example, if $\mathcal{M}$ is an infinite graph with no edges, then $\mathcal{M}$ has infinitely many components (since each vertex is in a component of size 1), but only one $\sim$-class. On the other hand, if $\mathcal{M}$ is an infinite path, then it it has only one component, but infinitely many $\sim$-classes.

Given a hereditary $\mathcal{L}$-property $\mathcal{H}$ and $m \in \mathbb{N}^{>0}$, we say $\mathcal{H}$ has *infinitely many components of size m* if there is $\mathcal{M} \in \mathcal{H}$ such that $\mathcal{M}$ has infinitely many distinct components of size m. We say $\mathcal{H}$ has *infinitely many components* if it there is $\mathcal{M} \in \mathcal{H}$ with infinitely many components. Otherwise we say $\mathcal{H}$ has finitely many components. We say $\mathcal{H}$ has *finite components* if there is $K \in \mathbb{N}$ such that for every $\mathcal{M} \in \mathcal{H}$, every component of $\mathcal{M}$ has size at most K. Otherwise $\mathcal{H}$ has *infinite components.* Note that if $\mathcal{H}$ is non-trivial and has finite components, then it must have infinitely many components.

Our first goal is to prove two lemmas about hereditary $\mathcal{L}$-properties with restrictions placed on their components. Specifically, Lemma 3.4 will give us lower bounds for any hereditary $\mathcal{L}$-property with infinitely many components of a fixed finite size, and Lemma 3.6 will characterize the speeds of hereditary $\mathcal{L}$-properties with finite components. We require the following lemma, which will be used several times throughout the paper.

Lemma 3.3. *Suppose $k, n \in \mathbb{N}^{>0}$ and $k \ll n$. Then the number of ways to partition $[k\lfloor n/k \rfloor]$ into $\lfloor n/k \rfloor$ parts of size k is at least $n^{n(1-1/k-o(1))}$.*

Proof. Set $m = k\lfloor n/k \rfloor$ and let $f(n)$ be the number of ways to partition $[m]$ into $\ell := \lfloor n/k \rfloor$ parts, each of size k. Clearly $f(n) \geq \binom{m}{k,\ldots,k}\frac{1}{\ell!}$. Using Stirling's approximation and the definitions of ℓ and m, we obtain the following.

$$\binom{m}{k, \ldots, k}\frac{1}{\ell!} \geq \frac{m!}{(k!)^{\lfloor \frac{n}{k} \rfloor}\lfloor \frac{n}{k} \rfloor!} \geq \frac{\sqrt{2\pi}m^{m+\frac{1}{2}}e^{-m}}{(k!)^{\lfloor \frac{n}{k} \rfloor}(\frac{n}{k})^{\frac{n}{k}+\frac{1}{2}}e^{1-\frac{n}{k}}} = m^{m(1-o(1))}n^{-\frac{n}{k}(1+o(1))} = n^{(n(1-\frac{1}{k}-o(1))},$$

where the last equality is because $n - m < k$ and $n \gg k$. Thus $f(n) \geq n^{n(1-1/k-o(1))}$. □

Lemma 3.4. *Suppose $\mathcal{H}$ is a hereditary $\mathcal{L}$-property and $k \geq 2$ is an integer. Assume $\mathcal{H}$ has infinitely many components of size k. Then $|\mathcal{H}_n| \geq n^{(1-1/k-o(1))n}$.*

Proof. Suppose $\mathcal{H}$ has infinitely many components of size k. By definition, there is $\mathcal{M} \in \mathcal{H}$ with infinitely many distinct components, each of size k. Let $D = C^{\mathcal{M}}$. Since $\mathcal{L}$ is finite, we can find distinct components $\{A_i : i \in \mathbb{N}\}$, each of which has size k and is disjoint from D, (since D is finite, and each element of D is in at most one component).

Fix $n \gg |D|$ and set $\ell = \lfloor (n - |D|)/k \rfloor$. Now choose $B \subseteq A_{\ell+1}$ of size $n - |D| - k\ell$ and set $A = A_1 \sqcup \ldots \sqcup A_\ell \sqcup B \sqcup D$. Observe $|B| < k$, $|A| = n$ and for every bijection

$f : A \to [n]$, $f(\mathcal{M}[A]) \in \mathcal{H}_n$. Note that the only components of size k in $f(\mathcal{M}[A])$ which are also disjoint from $f(D)$ are $f(A_1), \dots, f(A_\ell)$.

Fix $f_0 : D \cup B \to [n]$. Suppose f and f' are bijections from A to $[n]$ extending f_0 with $\{f(A_1), \dots, f(A_\ell)\} \neq \{f'(A_1), \dots, f'(A_\ell)\}$. Then clearly $f(\mathcal{M}[A]) \neq f'(\mathcal{M}[A])$, since these structures disagree on what are the components of size k disjoint from $f_0(D)$. Therefore $|\mathcal{H}_n|$ is at least the number of distinct ways to choose ℓ disjoint sets of size k in $[n_0]$ where $n_0 = n - |B| - |D|$. By Lemma 3.3, this is at least $n_0^{n_0(1-1/k-o(1))}$. Since $|B| < k$, $|D|$ is constant, and n is large, this shows $|\mathcal{H}_n| \geq n^{(1-1/k-o(1))n}$. □

We now consider the case where $\mathcal{H}$ has finite components. We will use the following fact, which is a consequence of the inequality of arithmetic and geometric means.

Fact 3.5. *Suppose $a_1, \dots, a_t \in \mathbb{N}^{>0}$. Then $a_1! \dots a_t! \geq (((\sum_{i=1}^t a_i)/t)!)^t$.*

Lemma 3.6. *Suppose $\mathcal{H}$ has finite components and k is the largest integer such that $\mathcal{H}$ contains infinitely many components of size k. If $k = 1$ then $\mathcal{H}$ is basic. If $k \geq 2$ then $|\mathcal{H}_n| = n^{(1-1/k-o(1))n}$.*

Proof. Suppose first that $k = 1$. Then there is a fixed integer w such that for any $\mathcal{M} \in \mathcal{H}$, all but w elements of M are in a component of size 1. Fix $\mathcal{M} \in \mathcal{H}$. Let $D = C^{\mathcal{M}}$, and let $X \subseteq M$ be the set of elements contained in a component of size greater than 1. Observe that for all $a \neq b \in M \setminus (X \cup D)$, $a \sim b$ if and only if for every relation $R(x_1, \dots, x_s)$ of $\mathcal{L}$, $\mathcal{M} \models R(a, \dots, a) \leftrightarrow R(b, \dots, b)$. Consequently, the number of distinct $\sim$-classes of $\mathcal{M}$ is at most $|X| + |D| + 2^{|\mathcal{L}|} \leq w + |C| + 2^{|\mathcal{L}|}$. Since this bound does not depend on $\mathcal{M}$, this shows $\mathcal{H}$ is basic.

Suppose now $k \geq 2$. That $|\mathcal{H}_n| \geq n^{(1-1/k-o(1))n}$ is immediate from Lemma 3.4. We now show that $|\mathcal{H}_n| \leq n^{(1-1/k+o(1))n}$. By choice of k, there is an integer w such that for all $\mathcal{M} \in \mathcal{H}$, all but w elements of M are contained in a component of size at most k in $\mathcal{M}$. Let $c = |C|$, and let $d = \sum_{i=1}^k |\mathcal{H}_i|$. By convention, let $\mathcal{H}_0$ consist of the empty structure, so $|\mathcal{H}_0| = 1$. Suppose now n is large. Then we can construct every $\mathcal{G} \in \mathcal{H}_n$ as follows.

1. Choose $D = C^{\mathcal{G}}$, the interpretations of the constants in $\mathcal{G}$. There are at most $\sum_{i=0}^c \binom{n}{i} c^i \leq (c+1)c^c n^c$ ways to do this.
2. Choose a set $A \subseteq [n]$ of size at most w and choose $\mathcal{G}[A]$. The number of ways to do this is at most $\sum_{i=0}^w \binom{n}{i} |\mathcal{H}_i| \leq (w+1)n^w |\mathcal{L}|^{w^r}$.
3. Choose a sequence of natural numbers $(b_1, \dots, b_n)$ (some of which may be 0) such that each $b_i \leq k$ and $\sum_{i=1}^n b_i = |[n] \setminus A|$. Then partition $[n] \setminus A$ into parts $B_1, \dots, B_n$ of sizes $b_1, \dots, b_n$, respectively (note some of the B_i may be empty). The number of ways to do this step is at most $\sum_{\{(b_1,\dots,b_n) \in [n]^n : b_i \leq k, \sum b_i = n\}} \binom{n}{b_1,\dots,b_n} \leq k^n n^n$.
4. Choose a sequence $(\mathcal{G}_1, \dots, \mathcal{G}_n)$ such that for each $1 \leq i \leq n$, $\mathcal{G}_i \in \mathcal{H}_{b_i}$. Then make $\mathcal{G}[B_i]$ isomorphic to $\mathcal{G}_i$ via the order-preserving bijection from $[b_i]$ to B_i. There are at most $(d+1)^n$ ways to do this.
5. For all relations $R(\overline{x})$ in $\mathcal{L}$ and $\overline{a} \in [n]^{|\overline{x}|} \setminus ((A^{|\overline{x}|} \cup \bigcup_{i=1}^n B_i^{|\overline{x}|})$, let $\mathcal{G} \models \neg R(\overline{a})$. There is only one way to do this given our previous choices.

This yields the following upper bound (recall $c, w, |\mathcal{L}|, d$ are all constants).

$$|\mathcal{H}_n| \le (c+1)c^c n^c |\mathcal{L}|^{w^r}(w+1)n^w k^n n^n (d+1)^n = n^{n(1+o(1))}. \tag{7}$$

We now consider how many times each $\mathcal{G} \in \mathcal{H}_n$ was counted. Fix $\mathcal{G} \in \mathcal{H}_n$, and assume $\mathcal{G}$ is constructed from the sets $D = C^{\mathcal{G}}$ (step 1), A (step 2), and the sequences $(b_1, \ldots, b_n)$, $(B_1, \ldots, B_n)$, and $(\mathcal{G}_1, \ldots, \mathcal{G}_n)$ (steps 3, 4 and 5 respectively). Let $\mathcal{N}_1, \ldots, \mathcal{N}_d$ enumerate all the distinct elements of $\cup_{i=1}^k \mathcal{H}_i$, and for each $i \in [d]$, let $J_i = \{j \in [n] : \mathcal{G}_j = \mathcal{N}_i\}$ and set $a_i = |J_i|$. Then for any permutation $\sigma : [n] \to [n]$ satisfying $\sigma(J_i) = J_i$ for each $i \in [d]$, $\mathcal{G}$ is also generated by making the same choices in steps 1 and 2, while in steps 3-5 choosing the sequences $(b_1, \ldots, b_n)$, $(B_{\sigma(1)}, \ldots, B_{\sigma(n)})$, and $(\mathcal{G}_1, \ldots, \mathcal{G}_n)$. So $\mathcal{G}$ is counted at least $a_1! \cdots a_d!$ times. Observe $n - w \le n - |A| = \sum_{i=1}^d |N_i| a_i \le k \sum_{i=1}^d a_i$. Combining this inequality with Fact 3.5, we obtain

$$a_1! \cdots a_d! \ge \Big(\Big(\Big(\sum_{i=1}^d a_i\Big)/d\Big)!\Big)^d \ge \Big(\Big(\frac{n-w}{kd}\Big)!\Big)^d \ge \Big(\frac{n-w}{kd}\Big)^{\frac{n-w}{k}} \ge n^{n/k - o(n)},$$

where the last inequality is because n is large and w, c, k, d are a constant. Therefore each $\mathcal{G}$ is counted at least $n^{n/k - o(1)}$ many times. Combining this with (7) yields that $|\mathcal{H}_n| \le n^{n(1+o(1))} n^{-n/k + o(n)} = n^{n(1 - \frac{1}{k} + o(1))}$. □

Note that by Lemma 3.6, we now understand the speed of hereditary properties with finite components. The next two lemmas will help us understand the case of a totally bounded property with infinite components. Specifically, they show that given a totally bounded $\mathcal{M}$, if $\mathcal{M}$ has an infinite component, then by deleting elements, we can find a substructure of $\mathcal{M}$ with infinitely many components of size k for arbitrarily large finite k. In the proof of Theorem 3.9 we will combine this with Lemma 3.4 to show that if a totally bounded hereditary $\mathcal{L}$-property $\mathcal{H}$ has infinite components, then $|\mathcal{H}_n| \ge n^{n(1-o(1))}$.

Lemma 3.7. *Suppose $t \ge 1$ is an integer, and $\mathcal{L}$ has maximum arity $r \ge 2$. Assume $\mathcal{M}$ is an $\mathcal{L}$-structure which contains an infinite component A. Then there is $t \le t' \le tr$ and a connected set $A' \subseteq A$ with $|A'| = t'$.*

Proof. Fix $a \in A$. Since A is connected and infinite, there is a relation $\psi_1(\overline{x}_1)$, and $\overline{a}_1 \in A^{|\overline{x}_1|}$ such that $\psi_1(\overline{a}_1)$ and $a \in \overline{a}_1$. Suppose $1 \le i < t$ and we have chosen $\overline{a}_1, \ldots, \overline{a}_i$ such that $\overline{a}_1 \cup \ldots \cup \overline{a}_i \subseteq A$ is connected and $i \le |\overline{a}_1 \cup \ldots \cup \overline{a}_i| \le ir$. Since A is infinite and connected, there is a relation $\psi_{i+1}(\overline{x}_{i+1})$, and $\overline{a}_{i+1} \in A^{|\overline{x}_{i+1}|}$ such that $M \models \psi_{i+1}(\overline{a}_{i+1})$, $\overline{a}_{i+1} \cap (\overline{a}_1 \cup \ldots \cup \overline{a}_i) \ne \emptyset$, and $\overline{a}_{i+1} \setminus (\overline{a}_1 \cup \ldots \cup \overline{a}_i) \ne \emptyset$. Note $i+1 \le |\overline{a}_1 \cup \ldots \cup \overline{a}_{i+1}| \le (i+1)r$. After t steps, $A' := \overline{a}_1 \cup \ldots \cup \overline{a}_t$ will be connected with $t \le |A'| \le tr$. □

Lemma 3.8. *Suppose $k, t \ge 1$ are integers, and $\mathcal{L}$ has maximum arity $r \ge 2$. Assume $\mathcal{M}$ is a totally k-bounded $\mathcal{L}$-structure, and $\mathcal{M}$ contains an infinite component. Then there is $t \le t' \le tr$ and a substructure $\mathcal{M}' \subseteq_{\mathcal{L}} \mathcal{M}$ so that $\mathcal{M}'$ contains infinitely many distinct components of size t'.*

Proof. Given $a \in M$, define

$$N(a) := \{e \in M : \text{there is } \psi \text{ a relation of } \mathcal{L} \text{ and } \overline{a} \in \psi^{\mathcal{M}} \text{ such that } \{e, a\} \subseteq \overline{a}\}.$$

Given $X \subseteq M$, set $N(X) = \bigcup_{a\in X} N(a)$. Observe that since $\mathcal{M}$ is totally k-bounded, we have that for all $a \in M$, $|N(a)| \leq rk|\mathcal{L}|$. Thus for any $X \subseteq M$, $|N(X)| \leq rk|\mathcal{L}||X|$. Note that for any finite, nonempty, $X \subseteq M$ and component $Y \subseteq M$, the number of components in $\mathcal{M}[Y \setminus X]$ is at most $|N(X)|$.

Let $D = C^{\mathcal{M}}$. By above, $|N(N(D))| \leq r^2k^2|\mathcal{L}|^2|D|$. Therefore $\mathcal{M}[A \setminus N(D)]$ has at most $r^2k^2|\mathcal{L}|^2|D|$ distinct components, so one of them is infinite, call this A_0. Note A_0 is a component in $M[D \cup A_0]$.

Suppose now $1 \leq i$ and assume by induction we have chosen an infinite, connected set A_i and distinct sets $B_0, \ldots, B_{i-1} \subseteq M \setminus A_i$ such that for each $0 \leq j \leq i-1$, $t \leq |B_j| \leq tr$ and B_j is a component in $\mathcal{M}[D \cup B_0 \cup \ldots B_{i-1} \cup A_i]$. By Lemma 3.7, there is connected set $B_i \subseteq A_i$ with $t \leq |B_i| \leq tr$. By above, $|N(N(B_i))| \leq r^2k^2|\mathcal{L}|^2|B_i| \leq r^3k^2|\mathcal{L}|^2t$, thus $M[A_i \setminus N(B_i)]$ has at most $r^3k^2|\mathcal{L}|^2t$ distinct components. Thus there is one which is infinite, call it A_{i+1}. Then for each $0 \leq j \leq i$, B_j is a component of $\mathcal{M}[D \cup B_0 \cup \ldots B_i \cup A_{i+1}]$. In this way we obtain distinct sets $\{B_i : i \in \mathbb{N}\}$, so that for each i, B_i is a component of $\mathcal{M}[D \cup \bigcup_{i\in\mathbb{N}} B_i]$ and $t \leq |B_i| \leq tr$. Clearly there is some $t \leq t' \leq tr$ and an infinite set $I \subseteq \mathbb{N}$ such that for all $i \in I$, $|B_i| = t'$. Then we may take $\mathcal{M}' = \mathcal{M}[D \cup \bigcup_{i\in I} B_i]$ as the desired substructure. □

We can now prove our counting theorem for totally bounded properties.

Theorem 3.9. *Suppose $\mathcal{H}$ is totally bounded. Then either $\mathcal{H}$ is basic, $|\mathcal{H}_n| \geq n^{n(1-o(1))n}$, or for some integer $k \geq 2$, $|\mathcal{H}_n| = n^{n(1-1/k-o(1))}$.*

Proof. Clearly if $\mathcal{L}$ has maximum arity $r \leq 1$, then $\mathcal{H}$ is basic. So assume $\mathcal{L}$ has maximum arity $r \geq 2$. Suppose $\mathcal{H}$ has infinite components. Then there is $\mathcal{M} \in \mathcal{H}$ with an infinite component. Lemma 3.8 implies that for all t, there is $t \leq t' \leq tr$ such that $\mathcal{H}$ has infinitely many components of size t'. By Lemma 3.4, $|\mathcal{H}_n| \geq n^{n(1-1/tr-o(1))}$ for all $t \geq 1$, consequently $|\mathcal{H}_n| \geq n^{n(1-o(1))}$.

Assume now that $\mathcal{H}$ has finite components. Then there is an integer m such that for every $\mathcal{M} \in \mathcal{H}$, every component of $\mathcal{M}$ has size at most m and there is a maximal $m' \leq m$ such that $\mathcal{H}$ contains infinitely many components of size m'. By Lemma 3.6, if $m' = 1$ then $\mathcal{H}$ is basic. Otherwise $m' \geq 2$, and Lemma 3.6 implies $|\mathcal{H}_n| = n^{n(1-1/m'-o(1))}$. □

Note Theorem 3.9 shows that if $\mathcal{H}$ is totally bounded, then $|\mathcal{H}_n| \geq n^{n(1-o(1))}$ if and only if $\mathcal{H}$ has infinite components. Given $\ell \geq 2$, a hereditary $\mathcal{L}$-property $\mathcal{H}$ is *factorial of degree* ℓ if $|\mathcal{H}_n| = n^{n(1-1/\ell-o(1))}$. The proof of Theorem 3.9 shows that if $\mathcal{H}$ is totally bounded and factorial of degree ℓ, then ℓ is the largest integer so that $\mathcal{H}$ has infinitely many components of size ℓ. In fact we can show something stronger.

Corollary 3.10. *Suppose $\mathcal{H}$ is a totally bounded hereditary $\mathcal{L}$-property with finite components and $\ell \geq 2$ is an integer. The there are finitely many countably infinite $\mathcal{L}$-structures $\mathcal{M}_1, \ldots, \mathcal{M}_m$, each of which is totally bounded with finite components, and a trivial property $\mathcal{F}$ such that $\mathcal{H} = \mathcal{F} \cup \bigcup_{i=1}^m age(\mathcal{M}_i)$.*

Further, $\mathcal{H}$ is factorial of degree ℓ if and only if ℓ is the largest integer such that for some $i \in [m]$, $\mathcal{M}_i$ has infinitely many components of size ℓ.

Proof. Since $\mathcal{H}$ has finite components there are integers $t \geq 1$ and $w \geq 0$ such that there exists $\mathcal{M} \in \mathcal{H}$ with infinitely many components of size t, but for all $\mathcal{M}' \in \mathcal{H}$, there are at most w elements of $\mathcal{M}'$ in a component of size strictly larger than t. Since $\mathcal{L}$ is finite, this implies there are only finitely many non-isomorphic $\mathcal{L}$-structures with domain $\mathbb{N}$ in $\mathcal{H}$, say $\mathcal{M}_1, \ldots, \mathcal{M}_m$. Since each $\mathcal{M}_i$ is totally bounded and has all but w elements in a component of size at most t, it is straightforward to see that $Th_\forall(\mathcal{M}_i)$ is finitely axiomatizable, in fact by a single sentence, say ψ_i. Then the following set of sentences is inconsistent.

$$T_\mathcal{H} \cup \{\neg\psi_i : i \in [m]\} \cup \{\exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i \neq j \leq n} x_i \neq x_j : n \geq 1\}.$$

By compactness, there is K such that for all $\mathcal{M} \in \mathcal{H}$ of size at least K, $\mathcal{M} \models \psi_i$ for some $i \in [m]$, i.e. $\mathcal{M} \in age(\mathcal{M}_i)$. Let $\mathcal{F}$ be the property consisting of the elements in $\mathcal{H}$ of size at most K. Then $\mathcal{H} = \mathcal{F} \cup \bigcup_{i=1}^m age(\mathcal{M}_i)$. For all $\ell \geq 2$, the proof of Theorem 3.9 shows that $\mathcal{H}$ is factorial of degree ℓ if and only if ℓ is the largest integer such that one of the $\mathcal{M}_i$ has infinitely many components of size ℓ. □

4. A dividing line: mutual algebricity

This section contains the remaining ingredients needed for Theorem 1.3. We proceed by splitting everything into two cases based on the dividing line of mutual algebricity (see Subsection 4.1 for precise definitions). The idea is that non-mutually-algebraic properties have "bad behavior" implying a relatively fast speed, while mutually algebraic properties are "well behaved," allowing a detailed analysis of their structure and speeds.

Specifically, in Subsection 4.2, we show that for any finite relational language $\mathcal{L}$, if $\mathcal{H}$ is a non-mutually algebraic hereditary $\mathcal{L}$-property $\mathcal{H}$, then $|\mathcal{H}_n| \geq n^{n(1-o(1))}$. This will rely crucially on a model theoretic result, Theorem 4.8, proved by the authors for this purpose in [18].

In Subsection 4.3, we consider the case where $\mathcal{H}$ is mutually algebraic. We use structural implications of this assumption to prove the remaining counting dichotomies. Namely, either $|\mathcal{H}_n| \geq n^{n(1-o(1))}$, $|\mathcal{H}_n| = n^{n(1-1/k-o(1))}$ for some integer $k \geq 2$, or $\mathcal{H}$ is basic, in which case by Section 2, the speed of $\mathcal{H}$ is asymptotically equal to a sum of the form $\sum_{i=1}^k p_i(n)i^n$ for finitely many rational polynomials $p_1, \ldots, p_k$. The proofs in Section 4.3 rely on Section 3 along with the fact that a mutually algebraic property is always controlled by finitely many totally bounded properties (see Subsection 4.2 for details).

Our strategy can be seen as a generalization of the strategy employed by Balogh, Bollobás, and Weinreich in the graph case [6]. Specifically, one direction of Theorem 4.8 (the direction we use) is a generalization of Lemma 27 of [6]. However, this result is significantly harder when dealing with relations of arity larger than 2, and the proof of both directions of Theorem 4.8 required ideas from stability theory.

4.1. Preliminaries.

In this subsection we give the relevant background on mutually algebraic properties. For this subsection, $\mathcal{L}$ is a finite language consisting of relations and/or constant symbols. We begin with the basic definitions, first introduced in [20].

Definition 4.1. Given a formula $\varphi(\overline{x}) = \varphi(x_1, \dots, x_s)$ and an $\mathcal{L}$-structure $\mathcal{M}$, $\varphi(\overline{x})$ is *k-mutually algebraic in* $\mathcal{M}$ if for every partition $[s] = I \cup J$ into nonempty sets I and J,

$$\mathcal{M} \models \forall \overline{x}_I \exists^{<k} \overline{x}_J \varphi(x_1, \dots, x_s).$$

Note that an $\mathcal{L}$-structure $\mathcal{M}$ is totally k-bounded if and only if all relations of $\mathcal{L}$ are k-mutually algebraic in $\mathcal{M}$.

Definition 4.2. An $\mathcal{L}$-structure $\mathcal{M}$ is *mutually algebraic* if, for every formula $\psi(\overline{x})$ there is a finite set $\Delta = \Delta(\overline{x}; \overline{y})$ of $\mathcal{L}$-formulas, an integer k, and parameters $\overline{a} \in M^{|\overline{y}|}$ such that the following hold.

1. For every $\varphi(\overline{x}, \overline{y}) \in \Delta$, $\varphi(\overline{x}, \overline{a})$ is k-mutually algebraic in $\mathcal{M}$.
2. There is a formula $\theta(\overline{x}; \overline{y})$ which is a boolean combination of elements of Δ, such that $\mathcal{M} \models \forall \overline{x}(\psi(\overline{x}) \leftrightarrow \theta(\overline{x}; \overline{a}))$.

In the definition above, there is no bound on the quantifier complexity of either φ or of the formulas in Δ. However, detecting whether or not a structure is mutually algebraic can be seen by looking at quantifier free formulas. Additionally, as we are working in a finite language $\mathcal{L}$ without function symbols, we have the following characterization.

Lemma 4.3. An $\mathcal{L}$-structure $\mathcal{M}$ is mutually algebraic if and only if there is a finite set $\Delta = \Delta(x_1, \dots, x_r; \overline{y})$ of quantifier-free $\mathcal{L}$-formulas, an integer k, and parameters $\overline{a} \in M^{|\overline{y}|}$ such that the following hold.

1. For every $\varphi(x_1, \dots, x_s, \overline{y}) \in \Delta$, $\varphi(x_1, \dots, x_s, \overline{a})$ is k-mutually algebraic in $\mathcal{M}$.
2. For each atomic formula $R(x_1, \dots, x_s)$ of $\mathcal{L}$, there is a formula $\theta(x_1, \dots, x_s; \overline{y})$ which is a boolean combination of elements of Δ, such that

$$\mathcal{M} \models \forall x_1 \dots \forall x_s (R(x_1, \dots, x_s) \leftrightarrow \theta(x_1, \dots, x_s; \overline{a})).$$

Proof. First, assume $\mathcal{M}$ is mutually algebraic. By Proposition 4.1 of [19], every atomic formula is equivalent to a boolean combination of **quantifier-free** mutually algebraic formulas. As there are only finitely many atomic $R \in \mathcal{L}$, we obtain a large enough finite Δ and a uniform finite k. Conversely, let $MA^*(\mathcal{M})$ denote the set of formulas $\theta(\overline{x}, \overline{a})$, where $\overline{a}$ is from $\mathcal{M}$, that are equivalent to boolean combinations of mutually algebraic formulas. By assumption, every atomic $R \in \mathcal{L}$ is in $MA^*(\mathcal{M})$. As $MA^*(\mathcal{M})$ is clearly closed under boolean combinations and is closed under existential quantification by Proposition 2.7 of [20], it follows that $\mathcal{M}$ is mutually algebraic.

Observe that every totally bounded $\mathcal{L}$-structure is automatically mutually algebraic (just take Δ in Lemma 4.3 to be the set of all atomic formulas). Thus mutual algebricity can be thought of as a weakening of total-bounded-ness.

Definition 4.4. We say a (possibly incomplete) theory T is *mutually algebraic* if every $\mathcal{M} \models T$ is mutually algebraic. A hereditary $\mathcal{L}$-property $\mathcal{H}$ is *mutually algebraic* if $T_{\mathcal{H}}$ is mutually algebraic (equivalently, every $\mathcal{M} \in \mathcal{H}$ is mutually algebraic).

Crucial to this section is Theorem 4.8 below, which gives an equivalent formulation of mutual algebricity. We require some definitions to state Theorem 4.8. Given an $\mathcal{L}$-structure

$\mathcal{M}$ and an integer $m \geq 1$, an *m-array* in $\mathcal{M}$ is a sequence $\{\bar{d}_i : i < m\}$ of pairwise disjoint tuples in M^k for some integer $k \geq 1$. Given a formula $R(\bar{z})$, a set $A \subseteq M$, and a proper partition $\bar{z} = \bar{x} \wedge \bar{y}$, an *$R(\bar{x}; \bar{y})$-formula over A* is any formula of the following forms.

(1) $R(\bar{x}, \bar{a})$ or $\neg R(\bar{x}, \bar{a})$ for some $\bar{a} \in A^{|\bar{y}|}$, or
(2) $x_i = x_j, x_i \neq x_j, x_i = a$ or $x_i \neq a$ where $x_i, x_j \in \bar{x}$ and $a \in A$.

Definition 4.5. Assume $\mathcal{M}$ is an $\mathcal{L}$-structure, $A \subseteq M$ is finite, $R(\bar{z})$ is an atomic formula, and $\bar{z} = \bar{x} \wedge \bar{y}$ is a proper partition. An *$R(\bar{x}; \bar{y})$-type over A* is a maximal set of $R(\bar{x}; \bar{y})$-formulas over A which has a nonempty set of realizations in $\mathcal{M}$.

Let $S^{\mathcal{M}}_{R,\bar{x}}(A)$ denote the set of all $R(\bar{x}; \bar{y})$-types over A.

We would like to point out to the model theorist that all our type spaces consist of sets *quantifier-free types over finite parameter sets.* In the notation of Definition 4.5, since A is finite, $S^{\mathcal{M}}_{R,\bar{x}}(A)$ is also finite.

Definition 4.6. Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure, $R(\bar{z})$ is an atomic formula and $\bar{z} = \bar{x} \wedge \bar{y}$ is a proper partition. Fix a finite $A \subseteq M$, an integer m and $p \in S^{\mathcal{M}}_{R,\bar{x}}(A)$. We say *$p$ supports an m-array* if the set of realizations of p in $\mathcal{M}$ contains an m-array.

Let $N^{\mathcal{M}}_{\bar{x},R,m}(A)$ be the number of $p \in S^{\mathcal{M}}_{R,\bar{x}}(A)$ that support an m-array.

Observe that in the notation of Definition 4.6, if $p \in S^{\mathcal{M}}_{R,\bar{x}}(A)$ supports an m-array and $m \gg |A|$, then all realizations of p in $\mathcal{M}$ are disjoint from A.

Definition 4.7. We say a (possibly incomplete) theory T *has uniformly bounded arrays* if for every atomic $\mathcal{L}$-formula $R(\bar{z})$, there is an integer m such that for every nonempty $\bar{x} \subsetneq \bar{z}$, there is an integer N such that the following holds. For all $\mathcal{M} \models T$ and every finite $A \subseteq M$, $N^{\mathcal{M}}_{R,\bar{x},m}(A) \leq N$.

We say a hereditary $\mathcal{L}$-property $\mathcal{H}$ *has uniformly bounded arrays* if $T_{\mathcal{H}}$ does. If $\mathcal{H}$ does not have uniformly bounded arrays, we say $\mathcal{H}$ *has unbounded arrays.*

We now state the crucial equivalent formulation of mutual algebricity. This theorem was proved by the authors in [18].

Theorem 4.8 (Corollary 7.5 in [18])**.** *Suppose $\mathcal{L}$ is a finite relational language. Then a (possibly incomplete) theory T is mutually algebraic if and only if T has uniformly bounded arrays. Consequently, for any hereditary $\mathcal{L}$-property, $\mathcal{H}$ is mutually algebraic if and only if $\mathcal{H}$ has uniformly bounded arrays.*

In this paper we use only one direction of Theorem 4.8, namely, that having uniformly bounded arrays implies mutual algebricity (this direction generalizes Lemma 27 of [6]). The way we use this is as follows. If $\mathcal{H}$ is not mutually algebraic then by Theorem 4.8, $\mathcal{H}$ has unbounded arrays, which we will show implies $|\mathcal{H}_n| \geq n^{n(1-o(1))}$ (see Proposition 4.10). On the other hand, if $\mathcal{H}$ is mutually algebraic, we use the formula-wise condition from Lemma 4.3 to analyze the structure of elements of $\mathcal{H}$. More specifically, when $\mathcal{H}$ is mutually algebraic, its speed will be determined by the speeds of finitely many totally bounded properties (in slightly different languages), which we know how to analyze by Section 3.

4.2. **The lower bound.** For the rest of the section, $\mathcal{L}$ consists of finitely many relations, of maximum arity $r \geq 1$. Note that in this context, an atomic $\mathcal{L}$-formula is just a relation of $\mathcal{L}$.

In this subsection, we show that if a hereditary $\mathcal{L}$-property $\mathcal{H}$ is not mutually algebraic, then $|\mathcal{H}_n| \geq n^{(1-o(1))n}$. This relies on our equivalent characterization of mutually algebraic theories contained in Theorem 4.8. We begin with two lemmas. Given an $\mathcal{L}$-structure $\mathcal{M}$, $A' \subseteq A \subseteq M$, and set of formulas p with parameters from A, let $p|_{A'}$ denote the elements of p which have parameters only from A'.

Lemma 4.9. *Suppose $R(\overline{z})$ is an atomic formula, $\overline{z} = \overline{x} \wedge \overline{y}$ is a proper partition, $\mathcal{M}$ is an $\mathcal{L}$-structure, and $A \subseteq M$ is finite. Suppose $p_1, \ldots, p_N$ are distinct elements of $S^{\mathcal{M}}_{R,\overline{x}}(A)$, and $\mathbb{B}_1, \ldots, \mathbb{B}_N$ are m arrays such that for each $i \in [m]$, the element of $\mathbb{B}_i$ realize p_i. Then there is $A' \subseteq A$ such that $|A'| \leq N|\overline{y}|$ and such that for each $1 \leq i \neq j \leq N$, $p_i|_{A'} \neq p_j|_{A'}$.*

Proof. Choose $\varphi_1(\overline{x}, \overline{a}_1)$ an $R(\overline{x}; \overline{y})$-formula over A such that $\varphi_1(\overline{x}, \overline{a}_1) \in p_1$ and $\varphi_1(\overline{x}, \overline{a}_1) \notin p_2$. Let $A_1 = \cup \overline{a}_1$ and note that $|\{p_1|_{A_1}, \ldots, p_N|_{A_1}\}| \geq 2$. Suppose by induction that $1 \leq s < N$ and we have chosen $A_s \subseteq A$ such that $|A_s| \leq s|\overline{y}|$ and such that $|\{p_1|_{A_s}, \ldots, p_N|_{A_s}\}| \geq s$. If $|\{p_1|_{A_s}, \ldots, p_N|_{A_s}\}| \geq N$, then set $A' = A_s$ and we are done. So assume this is not the case. Then there is $i \neq j$ such that $p_i|_{A_s} = p_j|_{A_s}$. Since $p_i \neq p_j$ there is an $R(\overline{x}; \overline{y})$-formula over A, say $\varphi_{s+1}(\overline{x}, \overline{a}_{s+1})$, such that $\varphi_{s+1}(\overline{x}, \overline{a}_{s+1}) \in p_i$ and $\varphi_{s+1}(\overline{x}, \overline{a}_{s+1}) \notin p_j$. Let $A_{s+1} = A_s \cup (\cup \overline{a}_{s+1})$. Then $|A_{s+1}| \leq (s+1)|\overline{y}|$ and $|\{p_1|_{A_s}, \ldots, p_N|_{A_s}\}| \geq s+1$. After at most N steps, when obtain $A' \subseteq A$ such that $|A'| \leq N|\overline{y}|$ and $|\{p_1|_{A'}, \ldots, p_N|_{A'}\}| = N$. □

Given variables $\overline{x}$, an $\mathcal{L}$-structure $\mathcal{M}$ and $A \subseteq M$, an *equality type in $\overline{x}$ over A* is a maximal subset $p(\overline{x}) \subseteq \{x_i = x_j, x_i = a, x_i \neq x_j, x_i \neq a : x_i, x_j \in \overline{x}, a \in A\}$ such that $p(\overline{x})$ is realized in $\mathcal{M}$. Let $S^{\mathcal{M}}_{\overline{x},=}(A)$ denote the set of equality types in $\overline{x}$ over A.

Proposition 4.10. *For any hereditary $\mathcal{L}$-property, if $\mathcal{H}$ is not mutually algebraic, then $|\mathcal{H}_n| \geq n^{(1-o(1))n}$.*

Proof. By Theorem 4.8, $\mathcal{H}$ does not have uniformly bounded arrays. Thus by definition, there is an atomic formula $R(\overline{z})$ such that for every $m \in \mathbb{N}$ there is a nonempty $\overline{x} \subsetneq \overline{z}$ such that for all $N \in \mathbb{N}$, there is $\mathcal{M} \in \mathcal{H}$ and a finite $A \subseteq M$ with $N^{\mathcal{M}}_{\overline{x},R,m}(A) > N$. Clearly there is a nonempty $\overline{x} \subsetneq \overline{z}$ so that for arbitrarily large m, it holds that for all $N \in \mathbb{N}$ there is $\mathcal{M} \in \mathcal{H}$ and finite $A \subseteq M$ with $N^{\mathcal{M}}_{\overline{x},R,m}(A) > N$. Let $\overline{y}$ be such that $\overline{z} = \overline{x} \wedge \overline{y}$, and let $|\overline{x}| = s$. Note by assumption, $|\overline{y}| \geq 1$.

Fix an integer $t \geq 1$. We show $|\mathcal{H}_n| \geq n^{(1-\frac{r}{r+t}-o(1))n}$. Choose $m \gg N \gg n \gg tr$. By assumption there is $\mathcal{M} \in \mathcal{H}$ and $A \subseteq M$ such that $N^{\mathcal{M}}_{\overline{x},R,m}(A) > N$. By Lemma 4.9, we may assume $|A| \leq N|\overline{y}|$. Let $p_1, \ldots, p_N$ be distinct elements of $N^{\mathcal{M}}_{\overline{x},R,m}(A)$, and let $\mathbb{B}_1, \ldots, \mathbb{B}_N \subseteq M^{|\overline{x}|}$ be m-arrays such that for each $1 \leq i \leq N$, the elements of $\mathbb{B}_i$ realize p_i. Since $m \gg Nr \geq |A|$, the remark following Definition 4.6 implies the elements of $\mathbb{B}_1, \ldots, \mathbb{B}_N$ are all disjoint from A. Since $m \gg N \gg t$ we may choose, for each $1 \leq i \leq N$, a subset $\mathbb{B}'_i \subseteq \mathbb{B}_i$ such that $|\mathbb{B}'_i| = t$ and such that for each $1 \leq i \neq j \leq N$, the underlying sets of $\mathbb{B}'_i$ and $\mathbb{B}'_j$ are disjoint from one another. For each $1 \leq i \leq N$, let $q_i \in S^{\mathcal{M}}_{\overline{x},=}(A)$ be the restriction of p_i to formulas in the language of equality. Note that each q_i says $A \cap \overline{x} = \emptyset$

and $|\{q \in S^{\mathcal{M}}_{\overline{x},=}(A) : q \text{ says } \overline{x} \cap A = \emptyset\}| \leq s^2$, so at least N/s^2 of the q_i are the same. Thus, since $N \gg n$, we may assume $q_1 = \ldots = q_n$ (possibly after reindexing). Observe this implies there is $1 \leq k^* \leq s$ that for any $1 \leq i \leq n$ and $\mathbb{B} \subseteq \bigcup_{i=1}^n \mathbb{B}'_i$, the underlying set of $\mathbb{B}$ has size $|\mathbb{B}|k^*$. Set $L = \lfloor \frac{n}{tk^*+|\overline{y}|} \rfloor$.

By Lemma 4.9, there is $A' \subseteq A$ with $|A'| \leq L|\overline{y}|$ such that for each $1 \leq i \neq j \leq L$, $p_i|_{A'} \neq p_j|_{A'}$. For each $i \in [L]$, let B'_i be the underlying set of $\mathbb{B}'_i$ and set $B = \bigcup_{i=1}^L B'_i$. Note $B \cap A' = \emptyset$ and $|B| = Lk^*t$. Since $|A'| \leq L|\overline{y}|$, we have $|A' \cup B| \leq n$. Fix a set $X \subseteq M \setminus (A' \cup B)$ so that $|A' \cup B \cup X| = n$, and let $\mathcal{G} := \mathcal{M}[A' \cup B \cup X]$. Note every bijection $f : A' \cup B \cup X \to [n]$ corresponds to an element of $\mathcal{H}_n$, namely the element $\mathcal{G}_f := f(\mathcal{G}) \in \mathcal{H}_n$. Observe that for each $1 \leq i \neq j \leq L$, $p_i|_{A'} \neq p_j|_{A'}$ while $q_i|_{A'} = q_j|_{A'}$, so there must be $\overline{a}_{ij} \in A^{|\overline{y}|}$ and $\psi_{ij}(\overline{x}, \overline{a}_{ij}) \in \{R(\overline{x}; \overline{a}_{ij}), \neg R(\overline{x}; \overline{a}_{ij})\}$ such that $\mathbb{B}'_i \subseteq \psi_{ij}(\mathcal{G}; \overline{a}_{ij})$ and $\mathbb{B}'_j \cap \psi_{ij}(\mathcal{G}; \overline{a}_{ij}) = \emptyset$. Because the $\mathbb{B}'_i$ are totally disjoint from one another, for each i, the set B'_i is defined in $\mathcal{G}$ by the following quantifier-free formula with parameters in $A' \cup X$, which we denote by $\theta_i(w; A' \cup X)$:

$$\bigwedge_{d \in A' \cup X} w \neq d \wedge \Big(\bigvee_{1 \leq u \leq s} \Big(\exists x_1 \ldots \widehat{\exists x_u} \ldots \exists x_s \bigwedge_{j \in [L] \setminus \{i\}} \psi_{ij}(x_1, \ldots, x_{u-1}, w, x_{u+1}, \ldots, x_s, \overline{a}_{ij}) \Big) \Big).$$

Fix an injection $f_0 : A' \cup X \to [n]$. Suppose f_1, f_2 are bijections from $A' \cup B \cup X$ to $[n]$ which extend f_0, but such that $f_1(B'_i) \neq f_2(B'_i)$ for some $1 \leq i \leq L$. Then $\theta_i(\mathcal{G}_{f_1}; f_0(A'C)) \neq \theta_i(\mathcal{G}_{f_2}; f_0(A' \cup X))$ implies $\mathcal{G}_{f_1} \neq \mathcal{G}_{f_2}$. Thus $|\mathcal{H}_n|$ is at least the number of ways to partition $[n] \setminus f_0(A' \cup C)$ into L ordered parts, each of size tk^*.

Set $m = |[n] \setminus f_0(A' \cup X)| = Lk^*t$. Then $|\mathcal{H}_n|$ is at least the number of ways of partitioning $[m]$ into L ordered parts, each of size k^*t. Since $L = m/k^*t$, Lemma 3.3 implies the number of ways to choose the unordered partition is at least $m^{m(1-\frac{1}{k^*t}-o(1))}$, so the number of ways to choose the ordered partition is at least $L! m^{m(1-\frac{1}{k^*t}+o(1))}$. Using this, the definitions of m, L, and Stirling's approximation yields the following.

$$\begin{aligned} |\mathcal{H}_n| &\geq n^{L(1-o(1))} n^{Lk^*t(1-\frac{1}{k^*t}+o(1))} = n^{Lkt^*(1-o(1))} = n^{\frac{kt^*n}{kt^*+|\overline{y}|}(1-o(1))} \\ &= n^{n(1-\frac{|\overline{y}|}{k^*t+|\overline{y}|}-o(1))} \geq n^{n(1-\frac{r}{r+k^*t}-o(1))} \geq n^{n(1-\frac{r}{r+t}-o(1))}, \end{aligned}$$

where the last inequalities are because $|\overline{y}| \leq r$ and $k^*t \geq t$. We have now shown that for all $t \in \mathbb{N}$, $|\mathcal{H}_n| \geq n^{n(1-\frac{r}{r+t}-o(1))}$. Thus $|\mathcal{H}_n| \geq n^{(1-o(1))n}$. □

4.3. **Counting dichotomies for mutually algebraic properties.** In this section we analyze the possible speeds of mutually algebraic hereditary $\mathcal{L}$-properties. The main idea is that any mutually algebraic property is essentially controlled by finitely many totally bounded properties, although these totally bounded properties will be in different languages from $\mathcal{L}$. We begin with a few definitions.

Definition 4.11. Suppose $\mathcal{L}'$ is a finite language. A *relational interpretation* $\alpha : \mathcal{L} \to \mathcal{L}'$ is a function which sends every relation symbol $R(\overline{x})$ of $\mathcal{L}$ to a Boolean combination of relations from $\mathcal{L}'$ in the variables $\overline{x}$.

Every relational interpretation $\alpha : \mathcal{L} \to \mathcal{L}'$ induces a map $\overline{\alpha}$ sending $\mathcal{L}'$-structures to $\mathcal{L}$-structures. In particular, given an $\mathcal{L}'$-structure $\mathcal{N}'$, let $\overline{\alpha}(\mathcal{N}')$ be the $\mathcal{L}$-structure with domain N' and satisfying $R^{\overline{\alpha}(\mathcal{N}')} = \alpha(R)^{\mathcal{N}'}$ for each relation R of $\mathcal{L}$. Note that if $\mathcal{M}'$ and $\mathcal{N}'$ are $\mathcal{L}'$-structures and $\mathcal{M}' \subseteq_{\mathcal{L}'} \mathcal{N}'$, then $\overline{\alpha}(\mathcal{N}') \subseteq_{\mathcal{L}} \overline{\alpha}(\mathcal{M}')$.

Definition 4.12. Suppose $\mathcal{L}'$ is a finite language, $\mathcal{H}'$ is a hereditary $\mathcal{L}'$-property, $\mathcal{H}$ is a hereditary $\mathcal{L}$-property, and $\alpha : \mathcal{L} \to \mathcal{L}'$ is a relational interpretation. We say $\overline{\alpha}$ is a *relational interpretation of* $\mathcal{H}'$ *in* $\mathcal{H}$ if for all $\mathcal{N}' \in \mathcal{H}'$, $\overline{\alpha}(\mathcal{N}') \in \mathcal{H}$ and further, for all $\mathcal{M}' \in \mathcal{H}'$ with the same universe as $\mathcal{N}'$, $\overline{\alpha}(\mathcal{N}') = \overline{\alpha}(\mathcal{M}')$ if and only $R^{\mathcal{N}'} = R^{\mathcal{M}'}$ for all relations R from $\mathcal{L}'$ (note $\mathcal{M}'$ and $\mathcal{N}'$ may still differ on interpretations of constant or function symbols).

In this section, we are most interested in relational interpretations where $\mathcal{H}'$ is a totally bounded hereditary property in a finite language without function symbols. Our next definition formalizes what it means for a hereditary $\mathcal{L}$-property to be controlled by finitely many totally bounded properties.

Definition 4.13. Suppose $\mathcal{H}$ is a hereditary $\mathcal{L}$-property. A *totally bounded cover of* $\mathcal{H}$ is a finite set $\{(\mathcal{L}_i, \mathcal{H}_i, \alpha_i) : 1 \leq i \leq m\}$ such that the following hold.

1. For each $1 \leq i \leq m$, $\mathcal{L}_i$ is a finite language consisting of relation and constant symbols and $\mathcal{H}_i$ is a totally bounded hereditary $\mathcal{L}_i$-property.
2. For each $1 \leq i \leq m$, $\alpha_i : \mathcal{L} \to \mathcal{L}_i$ is a relational interpretation, and $\overline{\alpha_i}$ is a relational interpretation of $\mathcal{H}_i$ in $\mathcal{H}$.
3. $\mathcal{H} = \mathcal{F} \cup \bigcup_{i=1}^m \overline{\alpha_i}(\mathcal{H}_i)$ for some trivial hereditary $\mathcal{L}$-property $\mathcal{F}$.

Note that if $\{(\mathcal{L}_i, \mathcal{H}_i, \alpha_i) : 1 \leq i \leq m\}$ is a totally bounded cover of a hereditary $\mathcal{L}$-property $\mathcal{H}$, then for all sufficiently large n, there are some (not necessarily unique) $i \in [m]$ and $\mathcal{M}' \in \mathcal{H}_i$ such that $\mathcal{M} = \overline{\alpha}_i(\mathcal{M}')$. One goal of this section is to show that any mutually algebraic hereditary $\mathcal{L}$-property has a totally bounded cover. Knowing a hereditary $\mathcal{L}$-property $\mathcal{H}$ has a totally bounded cover is useful because, as the next lemma shows, the speed of such an $\mathcal{H}$ is approximately determined by the speeds of its covering properties.

Lemma 4.14. *Suppose $\mathcal{H}$ is a hereditary $\mathcal{L}$-property, $m \in \mathbb{N}^{>0}$ and $\{(\mathcal{L}_i, \mathcal{H}_i, \alpha_i) : i \in [m]\}$ is a totally bounded cover of $\mathcal{H}$. If $c = \max\{|\mathcal{L}_j| : j \in [m]\}$ then for all sufficiently large n,*

$$n^{-c} \max\{|(\mathcal{H}_j)_n| : j \in [m]\} \leq |\mathcal{H}_n| \leq \sum_{j \in [m]} |(\mathcal{H}_j)_n|. \tag{8}$$

Proof. Since $\mathcal{H} = \mathcal{F} \cup \bigcup_{i=1}^m \overline{\alpha}_i(\mathcal{H}_i)$ and $\mathcal{F}$ is trivial, we have that for sufficiently large n, $|\mathcal{H}_n| \leq \sum_{i=1}^m |\overline{\alpha}_i(\mathcal{H}_i)_n| \leq \sum_{i=1}^m |(\mathcal{H}_i)_n|$. This proves the right hand inequality of (8). We now show the left hand inequality holds. Fix $i \in [m]$ and $\mathcal{M} \in Im(\overline{\alpha}_i)$. Since $\overline{\alpha}_i$ is a relational interpretation of $\mathcal{H}_i$ in $\mathcal{H}$, we know by definition that for all $\mathcal{N}, \mathcal{N}' \in \overline{\alpha}_i^{-1}(\mathcal{M})$, $R^{\mathcal{N}} = R^{\mathcal{N}'}$ for all relations R from $\mathcal{L}_i$. Therefore $|\overline{\alpha}_i^{-1}(\mathcal{M})|$ is at most the number of different interpretations for the constants from $\mathcal{L}_i$ in $|M|$, which is at most n^c. Thus $|\mathcal{H}_n| \geq |(\mathcal{H}_i)_n|/n^c$ for each $i \in [m]$, so the left hand inequality holds. ☐

We now show that a mutually algebraic hereditary $\mathcal{L}$-property has a totally bounded cover.

Proposition 4.15. *Suppose $\mathcal{H}$ is a mutually algebraic hereditary $\mathcal{L}$-property. Then $\mathcal{H}$ has a totally bounded cover.*

Proof. Set $\overline{x} = (x_1, \ldots, x_r)$. Let $\{T_i : i < \lambda\}$ enumerate the distinct completions of $T_{\mathcal{H}}$ which have infinite models, and for each $i < \lambda$, fix some $\mathcal{M}_i \models T_i$. Fix $i < \lambda$. By Lemma 4.3, there is a finite set of quantifier-free $\mathcal{L}$-formulas $\Delta_i(\overline{x}; \overline{y}_i)$, an integer k_i, and $\overline{a}_i \in M_i^{|\overline{y}_i|}$ such that the following holds: every element of $\Delta_i(\overline{x}; \overline{a}_i)$ is k_i-mutually algebraic in $\mathcal{M}_i$, and every relation $E(x_1, \ldots, x_s)$ of $\mathcal{L}$ is equivalent in $\mathcal{M}_i$ to a formula $\psi^i_E(x_1, \ldots, x_s; \overline{a}_i)$ which is a boolean combination of formulas from $\Delta_i(\overline{x}; \overline{a}_i)$. Let $\theta_i(\overline{y}_i)$ be the $\mathcal{L}$-formula which says every element of $\Delta_i(\overline{x}; \overline{y}_i)$ is k_i-mutually algebraic (with respect to the variables contained in $\overline{x}$), and which says that for each relation $E(x_1, \ldots, x_s)$ of $\mathcal{L}$, $\forall x_1, \ldots, x_s (E(x_1, \ldots, x_s) \leftrightarrow \psi^i_E(x_1, \ldots, x_s, \overline{y}_i))$. Observe $\theta_i(\overline{y}_i)$ can be chosen to be a universal formula. Now let τ^i be the sentence $\exists \overline{y}_i \theta_i(\overline{y}_i)$, and set

$$H(i) = \{\mathcal{M} \in \mathcal{H} : \mathcal{M} \models \tau^i\}.$$

Consider the following set of $\mathcal{L}$-sentences.

$$\Sigma = T_{\mathcal{H}} \cup \{\exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i \neq j \leq n} x_i \neq x_j : n \geq 1\} \cup \{\neg \tau^i : i < \lambda\}.$$

Clearly Σ is inconsistent, so it is finitely inconsistent. Therefore, there are integers $k, K \geq 1$ and $j_1, \ldots, j_k < \lambda$ such that for all $\mathcal{M} \in \mathcal{H}$ of size at least K, $\mathcal{M} \models \tau^{j_1} \vee \ldots \vee \tau^{j_k}$. Let $\mathcal{F}$ be the property whose isomorphism types are those in $\bigcup_{n=1}^{K} \mathcal{H}_n$. Then $\mathcal{H} = \mathcal{F} \cup \bigcup_{i=1}^{k} H(j_i)$. By reindexing, we may assume $(j_1, \ldots, j_k) = (1, \ldots, k)$, so that $\mathcal{H} = \mathcal{F} \cup \bigcup_{j=1}^{k} H(j)$.

Fix $j \in [k]$. Let $\mathcal{L}_j$ be a new language consisting of constant symbols $\overline{c}_j = (c_y : y \in \overline{y}_j)$, and a new relation symbol $R_\varphi(x_1, \ldots, x_s)$ for each $\varphi(x_1, \ldots, x_s) \in \bigcup_{j=1}^{k} \Delta_j(\overline{x}; \overline{y}_j)$. For each $\mathcal{M} \in H(j)$, choose some $\overline{b}_{\mathcal{M}} \in M^{|\overline{y}_j|}$ such that $\mathcal{M} \models \theta_j(\overline{b}_{\mathcal{M}})$, and define $\Phi_j(\mathcal{M})$ to be the $\mathcal{L}_j$-structure with domain M, where the constants $\overline{c}_j$ are interpreted as $\overline{b}_{\mathcal{M}}$ and where for each $\varphi \in \Delta_j$, $R_\varphi^{\Phi_j(\mathcal{M})} = \varphi(\mathcal{M}, \overline{b}_{\mathcal{M}})$. Let $\mathcal{H}(j)$ be obtained by closing $\{\Phi_j(\mathcal{M}) : \mathcal{M} \in H(j)\}$ under isomorphism and substructure. By definition, $\mathcal{H}(j)$ is a totally bounded hereditary $\mathcal{L}_j$-property.

Given an $\mathcal{L}$-formula ψ which is a boolean combination of formulas $\varphi_1(\overline{x}_1), \ldots, \varphi_m(\overline{x}_m)$ from $\Delta_j(\overline{x}; \overline{y}_j)$, let R_ψ denote the $\mathcal{L}_j$-formula obtained by taking the corresponding Boolean combination of $R_{\varphi_1}(\overline{x}_1), \ldots, R_{\varphi_m}(\overline{x}_m)$. Now define $\alpha_j : \mathcal{L} \to \mathcal{L}_j$ to be the relational interpretation where $\alpha_j(E) = R_{\psi^j_E}$ for each relation E from $\mathcal{L}$. We now show that $\overline{\alpha}_j$ is a relational interpretation of $\mathcal{H}_j$ in $\mathcal{H}$, and that moreover, $Im(\overline{\alpha_j}) = H(j)$.

It is straightforward to check that for all $\mathcal{M} \in H(j)$, $\mathcal{M} = \overline{\alpha}_j(\Phi_j(\mathcal{M}))$, and therefore $H(j) \subseteq Im(\overline{\alpha}_j)$. We now check $Im(\overline{\alpha}_j) \subseteq H(j)$. Fix $\mathcal{N} \in \mathcal{H}(j)$. By definition of $\mathcal{H}(j)$, there is $\mathcal{M} \in H(j)$ such that $\mathcal{N}$ is isomorphic to an $\mathcal{L}_j$-substructure of $\Phi_j(\mathcal{M})$, say $\mathcal{U} \subseteq_{\mathcal{L}_j} \Phi_j(\mathcal{M})$. Note that $\overline{c}_j^{\mathcal{U}} = \overline{b}_{\mathcal{M}} \subseteq U$. It is straightforward to check $\overline{\alpha}_j(\mathcal{U})$ must be an $\mathcal{L}$-substructure of $\overline{\alpha}_j(\Phi_j(\mathcal{M})) = \mathcal{M}$. Since $\mathcal{M} \models \theta_j(\overline{b}_{\mathcal{M}})$, $\overline{b}_{\mathcal{M}}$ is contained in the domain of $\overline{\alpha}_j(\mathcal{U})$, and $\theta_j(\overline{y}_j)$ is universal, we must have $\overline{\alpha}_j(\mathcal{U}) \models \theta_j(\overline{b}_{\mathcal{M}})$. Therefore

$\overline{\alpha}_j(\mathcal{U}) \in H(j)$. Clearly $\overline{\alpha}_j(\mathcal{N})$ is isomorphic to $\overline{\alpha}_j(\mathcal{U})$, so since $H(j)$ is closed under isomorphism, $\overline{\alpha}_j(\mathcal{N}) \in H(j)$. This shows $\overline{\alpha}_j$ is a surjective map from $\mathcal{H}(j)$ onto $H(j)$. Given $\mathcal{N}, \mathcal{N}' \in \mathcal{H}_j$, we leave it to the reader to verify that $\overline{\alpha}(\mathcal{N}) = \overline{\alpha}(\mathcal{N}')$ holds if and only if $R_\varphi^{\mathcal{N}} = R_\varphi^{\mathcal{N}'}$ for all relations R_φ of $\mathcal{L}_j$, and so $\overline{\alpha}_j$ is a relational interpretation from $\mathcal{H}_j$ into $\mathcal{H}$. It follows that $\{(\mathcal{L}_j, \mathcal{H}(j), \alpha_j : 1 \leq j \leq m\}$ is a totally bounded cover of $\mathcal{H}$. □

We now combine Theorem 4.15, Lemma 4.14 and the results of Section 3 to prove counting dichotomies for the speed of a mutually algebraic property, characterized in terms of the speeds of totally bounded properties covering it.

Theorem 4.16. *Suppose $\mathcal{H}$ is mutually algebraic. Then either $|\mathcal{H}_n| \geq n^{(1-o(1))n}$, $|\mathcal{H}_n| = n^{(1-1/\ell-o(1))n}$ for some $\ell \geq 2$, or $\mathcal{H}$ is basic.*

More specifically, there is $m \in \mathbb{N}$ and $\{(\mathcal{L}_i, \mathcal{H}_i, \alpha_i) : i \in [m]\}$ a totally bounded cover of $\mathcal{H}$ such that the following hold.

(a) For some $j \in [m]$, $\mathcal{H}_j$ has infinite components. In this case, $|\mathcal{H}_n| \geq n^{n(1-o(1))n}$.
(b) For every $j \in [m]$, $\mathcal{H}_j$ has finite components, and $\ell \geq 2$ is maximal such that for some $j \in [k]$, $\mathcal{H}_j$ has infinitely many components of size ℓ. In this case, $|\mathcal{H}_n| = n^{n(1-1/\ell-o(1))}$.
(c) For every $j \in [m]$, $\mathcal{H}_j$ is basic. In this case $\mathcal{H}$ is basic.

Proof. By Proposition 4.15 there is $m \in \mathbb{N}$ and $\{(\mathcal{L}_i, \mathcal{H}_i, \alpha_i) : i \in [m]\}$ a totally bounded cover of $\mathcal{H}$. By definition, there is a trivial property $\mathcal{F}$ such that $\mathcal{H} = \mathcal{F} \cup \bigcup_{i=1}^m \overline{\alpha}_i(\mathcal{H}_i)$. Let K be the maximum size of an element of $\mathcal{F}$. Suppose first that for each $j \in [m]$, $\mathcal{H}_j$ is basic. Then there is $K_1 \in \mathbb{N}$ such that for each $j \in [m]$, every element of $\mathcal{H}_j$ has at most K_1 distinct $\sim$-classes. For each $j \in [m]$ and $\mathcal{M} \in \mathcal{H}_j$, since $\overline{\alpha}_j$ is a relational interpretation of $\mathcal{H}_j$ in $\mathcal{H}$, the number of $\sim$-classes of $\overline{\alpha}_j(\mathcal{M})$ is at most the number of $\sim$-classes of $\mathcal{M}$. Since $\mathcal{H} \subseteq \mathcal{F} \cup \bigcup_{j\in[m]} \overline{\alpha}_j(\mathcal{H}_j)$, this shows every element of $\mathcal{H}$ has at most $\max\{K, K_1\}$ distinct $\sim$-classes, so $\mathcal{H}$ is basic.

Suppose now that for some $j \in [m]$, $\mathcal{H}_j$ has infinite components. Then by Theorem 3.9, $|(\mathcal{H}_j)_n| \geq n^{n(1-o(1))}$, so by (8) $|\mathcal{H}_n| \geq n^{n(1-o(1))}$.

We are left with the case where $J := \{j \in [m] : \mathcal{H}_j \text{ is not basic}\} \neq \emptyset$ and for each $j \in [m]$, $\mathcal{H}_j$ has finite components. By Theorem 3.9, we know that for each $j \in J$, there is an integer $w(j) \geq 2$ such that $|(\mathcal{H}_j)_n| = n^{n(1-1/w(j)-o(1))}$ and $w(j)$ is the maximum integer such that some element of $\mathcal{H}_j$ has infinitely many components of size $w(j)$. Thus if $\ell = \max\{w(j) : j \in [m]\}$, these observations and Lemma 4.14 imply

$$n^{-\max\{|\mathcal{L}_j|: j\in[m]\}} n^{n(1-1/\ell-o(1))} \leq |\mathcal{H}_n| \leq |J| n^{n(1-1/\ell+o(1))},$$

which implies $|\mathcal{H}_n| = n^{n(1-1/\ell+o(1))}$ since $|J| \leq m$, $\max\{|\mathcal{L}_j| : j \in [m]\}$ are constants. □

Note that Theorem 4.16 together with Corollary 3.10 give us a strong structural understanding of the properties in the factorial range, although we are required to consider properties in other languages. In forthcoming work, the authors consider characterizations of these properties in terms of the original language, in analogy to the structural characterizations of the factorial range for graph properties from [6]. This work also shows the gap between the factorial and penultimate range is directly related to *cellularity*, a notion with several interesting model theoretic formulations (see for instance [21, 22, 27]).

5. Proof of Theorem 1.3 and Minimal Properties in Range 1

In this section we bring together what we have shown to prove Theorem 1.3. We then characterize the minimal properties of each speed in range 1.

Proof of Theorem 1.3 Assume $\mathcal{L}$ is a finite relational language of arity $r \geq 1$. If $\mathcal{H}$ is basic, then by Theorem 2.4 there are finitely many rational polynomials $p_1, \ldots, p_k$ such that for all sufficiently large n, $|\mathcal{H}_n| = \sum_{i=1}^{k} p_i(x)i^n$, so case (1) holds. So assume $\mathcal{H}$ is not basic. Observe that by definition, this implies $r \geq 2$.

If $\mathcal{H}$ is also not mutually algebraic, then Theorem 4.10 implies $n^{n(1-o(1))} \leq |\mathcal{H}_n|$ and case (3) holds. We are left with the case when $\mathcal{H}$ is mutually algebraic and not basic. By Theorem 4.16, either $|\mathcal{H}_n| \geq n^{n(1-o(1))}$ (so case (3) holds), or there is an integer $k \geq 2$ such that $|\mathcal{H}_n| = n^{n(1-1/k-o(1))}$ (case (2) holds). □

An immediate corollary of the proof of Theorem 1.3 is the converse of Theorem 2.4.

Corollary 5.1. *$\mathcal{H}$ is basic if and only if there is $k \geq 1$ and rational polynomials $p_1, \ldots, p_k$ such that for sufficiently large n, $|\mathcal{H}_n| = \sum_{i=1}^{k} p_i(x)i^n$.*

Now that we have Corollary 5.1, we can characterize the minimal properties of each speed in range (1). Suppose $\mathcal{H}$ is a hereditary $\mathcal{L}$-property. A *strict subproperty of* $\mathcal{H}$ is any hereditary $\mathcal{L}$-property $\mathcal{H}'$ satisfying $\mathcal{H}' \subsetneq \mathcal{H}$. We say $\mathcal{H}$ is *polynomial* if asymptotically $|\mathcal{H}_n| = p(n)$ for some rational polynomial $p(x)$. In this case, the *degree of* $\mathcal{H}$ is the degree of $p(x)$. We say $\mathcal{H}$ is *exponential* if its speed is asymptotically equal to a sum of the form $\sum_{i=1}^{\ell} p_i(n)i^n$, where the p_i are rational polynomials and $\ell \geq 2$. In this case, the *degree of* $\mathcal{H}$ is ℓ. A polynomial hereditary $\mathcal{L}$-property $\mathcal{H}$ is *minimal* if every strict subproperty of $\mathcal{H}$ is polynomial of strictly smaller degree. An exponential hereditary $\mathcal{L}$-property $\mathcal{H}$ is *minimal* if every strict subproperty of $\mathcal{H}$ is either exponential of strictly smaller degree or polynomial.

Theorem 5.2. *Suppose $\mathcal{H}$ is a non-trivial hereditary $\mathcal{L}$-property.*

(1) $\mathcal{H}$ is a minimal polynomial property of degree $k \geq 0$ if and only if $\mathcal{H} = age(\mathcal{M})$ for some countably infinite $\mathcal{L}$-structure with one infinite $\sim$-class and exactly k elements contained in finite $\sim$-classes.

(2) $\mathcal{H}$ is a minimal exponential property of degree $\ell \geq 2$ if and only if $\mathcal{H} = age(\mathcal{M})$ for some countably infinite $\mathcal{L}$-structure with ℓ infinite $\sim$-classes and no finite $\sim$-classes.

Proof. Fix a hereditary $\mathcal{L}$-property $\mathcal{H}$ which is polynomial of degree $k \geq 0$ (respectively exponential of degree $\ell \geq 2$) and minimal. By Corollary 5.1, $\mathcal{H}$ is basic. By Corollary 2.15 there are finitely many countably infinite $\mathcal{L}$-structures $\mathcal{M}_1, \ldots, \mathcal{M}_m$, each with finitely many $\sim$-classes, such that $\mathcal{H} = \mathcal{F} \cup \bigcup_{i=1}^{m} age(\mathcal{M}_i)$, where $\mathcal{F}$ is a trivial hereditary $\mathcal{L}$-property. Since $\mathcal{H}$ is minimal, we may assume $\mathcal{F} = \emptyset$ (since deleting $\mathcal{F}$ does not change the asymptotic speed of $\mathcal{H}$). Further, since $age(\mathcal{M}_i)$ is a basic hereditary $\mathcal{L}$-property for each $i \in [m]$, Corollary 5.1 and $\mathcal{H} = \bigcup_{i=1}^{m} age(\mathcal{M}_i)$ implies there is $1 \leq i \leq m$ such that $age(\mathcal{M}_i)$ is also polynomial of degree k (respectively exponential of degree ℓ). By minimality, $\mathcal{H} = age(\mathcal{M}_i)$. Since $\mathcal{H} = age(\mathcal{M}_i)$ is polynomial of degree k (respectively exponential of degree $\ell \geq 2$), Corollary 2.15 implies $\mathcal{M}_i$ has one infinite $\sim$-class and exactly k elements in finite $\sim$-classes (respectively ℓ infinite $\sim$-classes). We now show further, that in the case when $\mathcal{H}$ is exponential of degree $\ell \geq 2$, $\mathcal{M}_i$ has no finite $\sim$-classes. Indeed, suppose it did. Let $\mathcal{M}_i'$ be the substructure of $\mathcal{M}_i$ obtained by deleting the finite $\sim$-classes. Then $age(\mathcal{M}_i')$ is a strict subproperty of $\mathcal{H}$ which is exponential of degree ℓ by Lemma 2.10, a contradiction. This takes care of the forward directions of both (1) and (2).

Suppose for the converse that $\mathcal{M}$ is a countably infinite $\mathcal{L}$-structure with one infinite $\sim$-class and exactly $k \geq 0$ elements contained in a finite $\sim$-classes (respectively with ℓ infinite $\sim$-classes and no finite $\sim$-classes). By Corollary 2.15, $age(\mathcal{M})$ is polynomial of degree k (respectively exponential of degree ℓ). Suppose by contradiction there is is a strict subproperty $\mathcal{H}'$ of $age(\mathcal{M})$ which is also polynomial of degree k (respectively exponential of degree ℓ).

Corollary 2.15 implies there are finitely many countably infinite $\mathcal{L}$-structures $\mathcal{M}_1, \ldots, \mathcal{M}_m$, each with finitely many $\sim$-classes, such that $\mathcal{H}' = \mathcal{F} \cup \bigcup_{i=1}^m age(\mathcal{M}_i)$, where $\mathcal{F}$ is a trivial hereditary $\mathcal{L}$-property. Again, there must be some $i \in [m]$ so that $age(\mathcal{M}_i)$ is itself polynomial of degree k (respectively exponential of degree ℓ). By Corollary 2.15, $age(\mathcal{M}_i)$ has one infinite $\sim$-class and exactly k elements contained in a finite $\sim$-classes (respectively with ℓ infinite $\sim$-classes an no finite ones). Since $\mathcal{M}_i \in age(\mathcal{M})$ and both $\mathcal{M}, \mathcal{M}_i$ are countably infinite, a standard argument shows that $\mathcal{M}$ has a substructure isomorphic to $\mathcal{M}_i$. This along with what we have shown about the structure of $\mathcal{M}$ and $\mathcal{M}_i$ imply that $\mathcal{M}_i$ must in fact be isomorphic to $\mathcal{M}$, contradicting that $age(\mathcal{M}_i) \subsetneq age(\mathcal{M})$.

□

6. PENULTIMATE RANGE

In this section we show that for each $r \geq 2$ there is a hereditary property of r-uniform hypergraphs whose speed oscillates between speeds close the lower and upper bounds of the penultimate range (case (3) of Theorem 1.4). Our example is a straightforward generalization of one used in the graph case (see [7]). We include the full proofs for completeness.

Throughout this section $r \geq 2$ is an integer and $\mathcal{G}$ is the class of finite r-uniform hypergraphs. We will use different notational conventions in this section, as it requires no logic. For this section, a *property* $\mathcal{P}$ means a class of finite r-uniform hypergraphs closed under isomorphism. The *speed* of a property $\mathcal{P}$ is the function $n \mapsto |\mathcal{P}_n|$. We denote elements of $\mathcal{G}$ as pairs $G = (V, E)$ where V is the set of vertices of G and $E \subseteq \binom{V}{r}$ is the set of edges. In this notation, we let $v(G) = |V|$ and $e(G) = |E|$. Given $U \subseteq V$, $G[U]$ is the hypergraph $(U, E \cap \binom{U}{r})$. Given a hypergraph $H = (U, E')$, we write $H \subseteq G$ to denote $U \subseteq V$ and $H = G[U]$. We begin by defining properties which we will use throughout the section.

Definition 6.1. *For $c \in \mathbb{R}^{\geq 0}$, define*

$$\mathcal{Q}^c = \{G \in \mathcal{G} : e(H) \leq cv(H) \text{ for all } H \subseteq G\} \text{ and } \mathcal{S}^c = \{G \in \mathcal{G} : e(G) \leq cv(G)\}.$$

Suppose $G = (V, E)$ is a finite r-uniform hypergraph. The *density of* G is $\rho(G) = e(G)/v(G)$, and we say G is *strictly balanced* if for all $V' \subsetneq V$, $\rho(G[V']) < \rho(G)$. The following theorem, proved by Matushkin in [23], is a generalization to hypergraphs of results about strictly balanced graphs (see [14, 25]).

Theorem 6.2 (Matushkin [23]). *Suppose $r \geq 2$ is an integer and $c \in \mathbb{Q}^{\geq 0}$. There exists a strictly balanced r-uniform hypergraph with density c if an only if $c \geq \frac{1}{r-1}$ or $c = \frac{k}{1+k(r-1)}$ for some integer $k \geq 1$.*

Given an vertex set V and a partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ of V, a *matching compatible with* $\mathcal{P}$ is a set $E \subseteq \binom{V}{r}$ such that for every $e \in E$ and $1 \leq i \leq k$, $|e \cap P_i| \leq 1$. We say $\mathcal{P}$ is an *equipartition* of V if $||P_i| - |P_j|| \leq 1$ for all $1 \leq i, j \leq k$. The following is a straightforward generalization of Theorem 16 in [26].

Proposition 6.3. *For any constant $c \geq \frac{1}{r-1}$, $|\mathcal{Q}_n^c| = |\mathcal{S}_n^c| = n^{(r-1)(c+o(1))n}$.*

Proof. For the upper bound bound, note that by definition, for large n, both $|\mathcal{Q}_n^c|$ and $|\mathcal{S}_n^c|$ are bounded above by the following.

$$\sum_{j=0}^{\lceil cn \rceil} \binom{\binom{n}{r}}{j} \leq (cn+2)\binom{n^r}{cn+1} \leq (cn+2)\Big(\frac{n^r e}{cn+1}\Big)^{cn+1} = n^{(r-1)cn(1+o(1))}.$$

For the lower bound, it suffices to show $|\mathcal{Q}_n^c \cap \mathcal{S}_n^c| \geq n^{(r-1)cn-o(n)}$. Assume first c is rational. Then by Theorem 6.2, we can choose a finite, strictly balanced r-uniform hypergraph $H = (V, E)$ with density c. Without loss of generality, say $V = [t]$ for some $t \in \mathbb{N}^{>0}$. Note $|E| = ct$ and $H \in \mathcal{Q}^c \cap \mathcal{S}^c$. Suppose $n \gg t$ is sufficiently large, and choose an equipartition $\mathcal{P} = \{W_1, \ldots, W_t\}$ of $[n]$. For each $e \in E$, choose E_e to be a maximal matching in $[n]$ compatible with $\mathcal{P}$ and satisfying $E_e \subseteq \bigcup_{x \in e} W_x$. Note the number of ways to choose E_e is at least $(\lfloor n/t \rfloor !)^{r-1}$.

We claim $G := ([n], \bigcup_{e \in E} E_e) \in \mathcal{Q}_n^c \cap \mathcal{S}_n^c$. Fix $X \subseteq [n]$. We show $e(G[X]) \leq c|X|$. For each $1 \leq i \leq t$, let $X_i = X \cap W_i$ and $n_i = |X_i|$. Now for each $1 \leq \ell \leq n$, let $V_\ell = \{i \in [t] : n_i \geq \ell\}$ and let $H_\ell = H[V_\ell]$ (note some of the V_ℓ will be empty). Observe that $|X| = \sum_{i=1}^n v(H_i)$ and $e(G[X]) = \sum_{i=1}^n e(H_i)$. Since H is strictly balanced of density c, we know that for each $i \in [n]$, $e(H_i) \leq cv(H_i)$. Combining these observations yields that

$$e(G[X]) = \sum_{i=1}^n e(H_i) \leq c \sum_{i=1}^n v(H_i) = c|X|.$$

Thus $G \in \mathcal{Q}_n^c \cap \mathcal{S}_n^c$. Clearly distinct choices for the set $\{E_e : e \in E\}$ yield distinct elements $([n], \bigcup_{e \in E} E_e)$ of $\mathcal{Q}_n^c \cap \mathcal{S}_n^c$. Since for each $e \in E$, the number of ways to choose E_e is at least $(\lfloor n/t \rfloor !)^{r-1}$, this shows that

$$|\mathcal{Q}_n^c \cap \mathcal{S}_n^c| \geq \Big(\Big(\lfloor n/t \rfloor !\Big)^{r-1}\Big)^{|E|} = \Big(\lfloor n/t \rfloor !\Big)^{(r-1)ct} = n^{(r-1)cn-o(n)}.$$

Assume now c is irrational. Note that for all $c' \leq c$, $\mathcal{Q}^{c'} \cap \mathcal{S}^{c'} \subseteq \mathcal{Q}^c \cap \mathcal{S}^c$. Thus by the calculations above, for all $\frac{1}{r-1} \leq c' < c$, where $c' \in \mathbb{Q}$, we have $|\mathcal{Q}_n^c \cap \mathcal{S}_n^c| \geq n^{(r-1)(c'-o(1))n}$. Clearly this implies $|\mathcal{Q}_n^c \cap \mathcal{S}_n^c| \geq n^{(r-1)(c-o(1))n}$. □

Definition 6.4. *Given an increasing (possibly finite) sequence $\nu = (\nu_1, \nu_2, \dots)$ of natural numbers, let*

$$\mathcal{P}^{\nu,c} = \{G \in \mathcal{G} : \text{ if } H \subseteq G \text{ and } v(H) = \nu_i \text{ for some } i, \text{ then } e(H) \leq c\nu_i\}.$$

We now consider the speed of $\mathcal{P}^{\nu,c}$ in the case where $\sup \nu < \infty$.

Lemma 6.5. *Let $c \geq \frac{1}{r-1}$, $\epsilon > 1/c$, and $\nu = (\nu_i)_{i\in\mathbb{N}}$ a sequence of natural numbers with $k = \sup\{\nu_i : i \in \mathbb{N}\} \in \mathbb{N} \cup \{\infty\}$. Then*

1. *$|\mathcal{P}^{\nu,c}_n| \geq n^{(r-1)(c+o(1))n}$ and $|\mathcal{P}^{\nu,c}_n| = n^{(r-1)(c+o(1))n}$ whenever $n = \nu_i$ for some $i \in \mathbb{N}$,*
2. *if $k < \infty$ and n is sufficiently large, then $|\mathcal{P}^n_{\nu,c}| \geq 2^{n^{r-\epsilon}}$.*

Proof. Note $\mathcal{Q}^c \subseteq \mathcal{P}^{c,\nu}$ so by Proposition 6.3, $|\mathcal{P}^{c,\nu}_n| \geq |\mathcal{Q}^c_n| \geq n^{(r-1)(c+o(1))n}$. When $n = \nu_i$ for some $i \in \mathbb{N}$, then by definition, $\mathcal{Q}^c_n \subseteq \mathcal{P}^{\nu,c}_n \subseteq \mathcal{S}^c_n$. Consequently Proposition 6.3 implies $|\mathcal{P}^{c,\nu}_n| = n^{(c(r-1)+o(1))n}$. This shows (1) holds.

Assume now $k < \infty$ and let $(k) = (1, \dots, k)$. Note $\mathcal{P}^{(k),c} \subseteq \mathcal{P}^{\nu,c}$, so it suffices to show that for large enough n, $|\mathcal{P}^{(k),c}_n| \geq 2^{n^{2-\epsilon}}$. Choose δ satisfying $\epsilon > \delta > 1/c$ and let $p = n^{-\delta}$. We consider the probability $G_{n,p} \notin \mathcal{P}^{(k),c}_n$, i.e. the probability that there is $H \subseteq G_{n,p}$ with $v(H) \leq k$ and $e(H) > cv(H)$. Given $1 \leq j \leq k$ and $S \in \binom{[n]}{j}$, let $X_S : G_{n,p} \to \mathbb{N}$ be the random variable defined by $X_S(G) = 1$ if $e(G[S]) > cj$. Note $\mathbb{P}(X_S = 1) \leq \sum_{i=\lceil cj\rceil}^{\binom{j}{r}} \binom{\binom{j}{r}}{i} p^i (1-p)^{\binom{j}{r}-i} \leq C_j p^{cj}$ for some constant C_j depending only on j, r and c. Therefore we have the following.

$$\mathbb{P}(G_{n,p} \notin \mathcal{P}^{(k),c}_n) \leq \sum_{j=1}^{k} \sum_{S\in\binom{[n]}{j}} \mathbb{P}(X_S = 1) \leq \sum_{j=1}^{k} \binom{n}{j} C_j p^{cj} \leq \sum_{i=1}^{k} C_j (n^{1-\delta c})^j.$$

Since $\delta c > 1$, we have $1 - \delta c < 0$ and thus $\mathbb{P}(G_{n,p} \notin \mathcal{P}^{(k),c}_n) \to 0$ as $n \to \infty$. Note that for any n, $\mathbb{P}(G_{n,p} \notin \mathcal{P}^{(k),c}_n) \leq \mathbb{P}(G_{n+1,p} \notin \mathcal{P}^{(k),c}_{n+1})$. Thus we may choose n_0 so that $\mathbb{P}(G_{n,p} \notin \mathcal{P}^{(k),c}_n) < 1/3$ for all $n \geq n_0$.

Fix any $0 < \epsilon_0 < 1/6$. If n is sufficiently large, $\mathbb{P}(e(G_{n,p}) < pN/2) \leq 1/2 + \epsilon_0$, where $N = \binom{n}{r}$. Combining this with the above, we have shown that for all sufficiently large n,

$$\mathbb{P}(G_{n,p} \in \mathcal{P}^{(k),c}_n \text{ and } e(G_{n,p}) \geq pN/2) \geq 1/2 - \epsilon_0 - 1/3 > 0.$$

Thus for all sufficiently large n, there exists $G = ([n], E) \in \mathcal{P}^{(k),c}_n$ such that $e(G) \geq pN/2$. Since $([n], E') \in \mathcal{P}^{(k),c}_n$ for all $E' \subseteq E$, we have that

$$|\mathcal{P}^{(k),c}_n| \geq 2^{pN/2} = 2^{n^{-\delta}\binom{n}{r}/2} = 2^{n^{r-\epsilon}},$$

where the last inequality is because n is sufficiently large, and $\epsilon < \delta$. □

We now show that for any $c \geq 1/(r-1)$ and $\epsilon > 1/c$, there is a property whose speed oscillates between $n^{nc(r-1)(1-o(1))}$ and $2^{n^{r-\epsilon}}$.

Theorem 6.6. *Let $c \geq \frac{1}{r-1}$ and $\epsilon > 1/c$. There exists sequences $\nu = (\nu_i)_{i\in\mathbb{N}}$ and $\mu = (\mu_i)_{i\in\mathbb{N}}$ where $\mu_i = \nu_i - 1$ for all $i \in \mathbb{N}$ such that the following hold.*

(1) $|\mathcal{P}^{\nu,c}_n| = n^{(r-1)(c+o(1))n}$ *whenever* $n = \nu_i$,
(2) $|\mathcal{P}^{\nu,c}_n| \geq 2^{n^{r-\epsilon}}$ *whenever* $n = \mu_i$,
(3) $n^{(r-1)(c+o(1))n} \leq |\mathcal{P}^{\nu,c}| < 2^{n^{r-\epsilon}}$ *if* $n \neq \mu_i$.

Proof. Set $\nu_0 = r + 1$. Assume now $k \geq 0$ and suppose by induction we have chosen $\nu_0, \ldots, \nu_k$. Let $\nu = (\nu_1, \ldots, \nu_k)$ and note by Lemma 6.5, $|\mathcal{P}^{\nu,c}_n| \geq 2^{n^{r-\epsilon}}$ for large enough n. Choose $\mu_k > \nu_k$ minimal so that $|\mathcal{P}^{\nu,c}_{\mu_k}| \geq 2^{\mu_{k+1}^{r-\epsilon}}$ and set $\nu_{k+1} = \mu_k + 1$. □

Proof of Theorem 1.5 Let $\mathcal{L} = \{R(x_1, \ldots, x_r)\}$. Given $c \geq \frac{1}{r-1}$ and $\epsilon > 1/c$, let $\nu = (\nu_i)_{i\in\mathbb{N}}$ and $\mu = (\mu_i)_{i\in\mathbb{N}}$ be sequences as in Theorem 6.6. Let $T_{\nu,c}$ be the universal theory of $\mathcal{P}^{\nu,c}$ and observe that class of models of $T_{\nu,c}$ is a hereditary $\mathcal{L}$-property $\mathcal{H}$ such that for arbitrarily large n, $|\mathcal{H}_n| = n^{(r-1)(c+o(1))n}$ and for arbitrarily large n, $|\mathcal{H}_n| \geq 2^{n^{r-\epsilon}}$. □

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