A Vaught’s conjecture toolbox

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Everything begins with the work of Robert Vaught.
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A dichotomy:
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**A dichotomy:**

- If $T$ is not small, then there is a perfect set of complete types, hence $I(T, \aleph_0) = 2^{\aleph_0}$ [in fact, a perfect set of pairwise non-isomorphic models].
- If $T$ is small, then $T$ has a countable, saturated model and a prime model, which is also the unique countable atomic model.
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However... DLS can be recovered by restricting to reasonable countable fragments.
The precise definition of a fragment is not important, only that:
For all countable $\Gamma \subseteq L_{\omega_1,\omega}$ there is a reasonable countable $\Delta$ satisfying $\Gamma \subseteq \Delta \subseteq L_{\omega_1,\omega}$. 
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For all countable $\Gamma \subseteq L_{\omega_1,\omega}$ there is a reasonable countable $\Delta$ satisfying $\Gamma \subseteq \Delta \subseteq L_{\omega_1,\omega}$.

- If $\Delta$ is a reasonable countable fragment, then for any $L$-structure $M$, there is a countable $M' \preceq_\Delta M$. 
Moreover...
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**Definition (Keisler)**

Let $\Delta$ be any reasonable countable fragment of $L_{\omega_1,\omega}$.

- A set $T \subseteq \Delta$ of sentences is **consistent** if there is a model $M \models T$;
- A consistent set $T \subseteq \Delta$ is **$\Delta$-complete** if $T$ decides $\psi$ for every $\Delta$-sentence $\psi$.
- A **complete $\Delta$-$n$-type** $p(\bar{x})$ with respect to $T$ is a maximal consistent (w.r.t. $T$) set of $\Delta$-formulas with at most $(x_1 \ldots, x_n)$ free.
- A $\Delta$-complete theory $T$ is **small** if $S_n(T, \Delta)$ is countable for all $n \geq 1$.
Theorem (Keisler)

Let $\Delta$ be any reasonable countable fragment of $L_{\omega_1,\omega}$ and let $T$ be $\Delta$-complete.

- If $T$ is not small, then there is a perfect set contained in $S_n(T, \Delta)$ for some $n$ [hence a perfect set of pairwise non-isomorphic models];
- If $T$ is small, then there is a unique (up to isomorphism) $\Delta$-prime model, which is also the unique countable, $\Delta$-atomic model.
Definition (Morley)

An $L_{\omega_1,\omega}$-sentence $\Phi$ is scattered if $S_n(\Phi, \Delta)$ is countable for every (reasonable) countable fragment $\Delta$. 
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Scatteredness does not depend on our choice of ‘reasonable’.

Proposition

TFAE for a sentence $\Phi$ of $L_{\omega_1,\omega}$:

1. $\Phi$ is scattered;
2. $\text{Mod}(\Phi)$ does not contain a perfect set of pairwise non-isomorphic models.
Fix a (countable) vocabulary $L$ with at least one binary relation or function symbol.

$$X_L = \{ \text{all } L\text{-structures } M \text{ with universe } \omega \}$$

Basic open sets $U_{\varphi(m)} = \{ M \in X_L : M \models \varphi(m) \}$. 
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Basic open sets $U_{\varphi(m)} = \{ M \in X_L : M \models \varphi(m) \}$.

Then:

- $X_L$ is a standard Borel space;
- For any $\Phi \in L_{\omega_1,\omega}$, $\text{Mod}(\Phi)$ is a Borel subset of $X_L$;
- The isomorphism relation $\cong_\Phi$ is a $\Sigma^1_1$-subset of $X_L \times X_L$ ($M \cong N$ iff $\exists f(\ldots)$).
Polish space of $L$-structures

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Whether $\cong_{\Phi}$ is Borel or not will be an important distinction!
For $M, N$ countable, $M \cong N$ iff there is a back-and-forth system of finite partial functions.
Isomorphisms of countable structures

For $M, N$ countable, $M \cong N$ iff there is a back-and-forth system of finite partial functions.

Fix a countable $M$. A potential back-and-forth system $F$ is a set of finite, partial functions $f : \bar{a} \rightarrow \bar{b}$ satisfying:

- $F$ is closed under restrictions;
- If $f : \bar{a} \rightarrow \bar{b}$ is in $F$, then $qftp(\bar{a}) = qftp(\bar{b})$; and
- If $\sigma \in Aut(M)$, then each restriction $\sigma|_{\bar{a}} \in F$.

Examples: All $f : \bar{a} \rightarrow \bar{b}$ with:

- $qftp(\bar{a}) = qftp(\bar{b})$ (i.e., no additional restrictions); OR
- The first-order types $tp(\bar{a}) = tp(\bar{b})$; OR
- For any reasonable fragment $\Delta$, $tp_\Delta(\bar{a}) = tp_\Delta(\bar{b})$. 

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Fix $M$ and a potential back-and-forth system $F$. We define a sequence of equivalence relations $\sim_\alpha (\alpha < \omega_1)$ that measure how close $F$ is to being a back-and-forth system.
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- $(M, \bar{a}) \sim_0 (M, \bar{b})$ iff $f : \bar{a} \mapsto \bar{b} \in F$;
- For $\lambda$ limit, $(M, \bar{a}) \sim_\lambda (M, \bar{b})$ iff $(M, \bar{a}) \sim_\alpha (M, \bar{b})$ for all $\alpha < \lambda$;
- $(M, \bar{a}) \sim_{\alpha+1} (N, \bar{b})$ iff
  1. For all $c \in M$ there is $d \in M$ such that $(M, \bar{ac}) \sim_\alpha (M, \bar{bd})$; AND
  2. For all $d \in M$ there is $c \in M$ such that $(M, \bar{ac}) \sim_\alpha (M, \bar{bd})$. 

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Note: If $(M, \bar{a}) \sim_{\alpha+\gamma} (M, \bar{b})$ then $(M, \bar{a}) \sim_{\alpha} (M, \bar{b})$. 
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**Proposition**

TFAE for any \(M, \bar{a}, \bar{b}\) and \(F\):

1. \(\{\alpha < \omega_1 : (M, \bar{a}) \sim_{\alpha} (M, \bar{b})\}\) is uncountable;
2. For all \(\alpha < \omega_1\), \((M, \bar{a}) \sim_{\alpha} (M, \bar{b})\);
3. There is \(\sigma \in \text{Aut}(M)\) satisfying \(\sigma(\bar{a}) = \bar{b}\).
Note: If \((M, \bar{a}) \sim_{\alpha + \gamma} (M, \bar{b})\) then \((M, \bar{a}) \sim_{\alpha} (M, \bar{b})\).

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Thus: For every \(M\) and \(F\), there is a least \(\alpha^* = \alpha^* (M, F) < \omega_1\) such that for all \(\bar{a}, \bar{b}\) from \(M\),

\[(M, \bar{a}) \sim_{\alpha^*} (M, \bar{b})\] iff there is \(\sigma \in \text{Aut}(M)\) with \(\sigma(\bar{a}) = \bar{b}\).
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Thus: For every \(M\) and \(F\), there is a least \(\alpha^* = \alpha^*(M, F) < \omega_1\) such that for all \(\bar{a}, \bar{b}\) from \(M\),

\[(M, \bar{a}) \sim_{\alpha^*} (M, \bar{b})\] iff there is \(\sigma \in \text{Aut}(M)\) with \(\sigma(\bar{a}) = \bar{b}\).

When \(F\) consists of qftp-preserving partial maps, \(\alpha^*(M, F) := SH(M)\), the Scott height of \(M\).
Now suppose $\Phi \in L_{\omega_1,\omega}$ and $F$ is any of the above.

Put: $\alpha^*(\Phi, F) := \sup\{\alpha^*(M, F) : M \models \Phi\}$. We say $\Phi$ has bounded Scott heights if $\alpha^*(\Phi, F) < \omega_1$ for some/every $F$. 

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**Proposition**

$\Phi$ has bounded Scott heights if and only if $\cong_\Phi$ is Borel in $X_L \times X_L$. 
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**Proposition**

$\Phi$ has bounded Scott heights if and only if $\sim^\Phi$ is Borel in $X_L \times X_L$.

**Theorem (Morley)**

*Let $\Phi \in L_{\omega_1,\omega}$ be scattered. Then:*

- $I(\Phi, \aleph_0) \leq \aleph_1$ always; and
- $I(\Phi, \aleph_0)$ is countable if and only if $\sim^\Phi$ is Borel.
Thus: $\Phi$ is a counterexample to Vaught’s conjecture if and only if $\Phi$ is scattered, with unbounded Scott heights.
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Discussion: What happened to first-order $T$??

How can we ‘see’ $\text{Mod}(T)$ in $X_L$? Where does the compactness theorem fit in with all of this?
Thus: Φ is a counterexample to Vaught’s conjecture if and only if Φ is scattered, with unbounded Scott heights.

Discussion: What happened to first-order $T$??

How can we ‘see’ $Mod(T)$ in $X_L$? Where does the compactness theorem fit in with all of this?

Empirical fact: There are relatively few (known!) complete, first order $T$ so that $\cong_T$ is not Borel (without being Borel complete).
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How can we ‘see’ $\text{Mod}(T)$ in $X_L$? Where does the compactness theorem fit in with all of this?

Empirical fact: There are relatively few (known!) complete, first order $T$ so that $\equiv_T$ is not Borel (without being Borel complete).

$T = Th$(Binary splitting, refining eq. relations) has $\equiv_T$ non-Borel.
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**Benda’s conjecture (1965):** If $1 < I(T, \aleph_0) < \aleph_0$, must $T$ have a countable, universal, non-saturated model?
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If \( F \) is the potential back-and-forth system of complete types (i.e., \( \forall a \exists b \forall x \forall y \left( \text{tp}(a) = \text{tp}(b) \Rightarrow x = y \right) \)) then a model \( M \) is homogeneous if and only if \( \alpha^*(M, F) = 0 \).

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**Benda’s conjecture (1965):** If \( 1 < I(T, \aleph_0) < \aleph_0 \), must \( T \) have a countable, universal, non-saturated model?

**Open (1989):** If \( T \) is small and every countable universal model is saturated, must every countable weakly saturated (realize all \( n \)-types over \( \emptyset \)) model be saturated?
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Shelah/Harrington/Makkai proved Vaught’s conjecture for $\omega$-stable theories.

In December, 1986 Harrington stated that “Vaught’s conjecture for superstable theories is the major open problem in stability theory.” Newelski and Buechler have made progress on this.