Borel complexity of complete, first order theories
(status report)

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Recall:

- \( X_L = \{ \text{all } L\text{-structures with universe } \omega \} \).
- \( S_\infty \) induces the logic action on \( X_L \).
- From Sam’s talk: A Borel subset \( Y \subseteq X_L \) is invariant under this action iff \( Y = \text{Mod}(\Phi) \) for some \( \Phi \in L_{\omega_1,\omega} \).

**Theorem (Friedman-Stanley)**

*With respect to Borel reducibility, among all pairs \( (\text{Mod}(\Phi), \equiv_\Phi) \), there is a maximum Borel degree.*
Definition
We say $\equiv_\Phi$ is **Borel complete** if it is Borel equivalent to this maximum degree.

Examples: (Friedman-Stanley) The following classes of structures $(\text{Mod}(\Phi), \equiv_\Phi)$ are all Borel complete:

- Directed graphs;
- Symmetric graphs;
- Linear orders;
- Fields;
- Subtrees of $\omega^{<\omega}$.
Throughout the whole of this talk, $T$ will denote a complete, first order theory in a countable language.

- Interested in the Borel complexity of $(\text{Mod}(T), \cong_T)$. 
**Jumps:** Suppose $T$ is a complete $L$-theory. Let $L^+ = L \cup \{E\}$ and $T^+$ be the theory specifying:

- $E$ is an equivalence relation with infinitely many classes;
- Each $E$-class is a model of $T$.

Then $\cong_{T^+}$ is Borel equivalent to the jump $(\cong_T)^+$. 
Friedman-Stanley tower: Let

- $\cong_0$ be $id(\omega)$ [Think: Countably many non-isomorphic models.]
- $\cong_1$ be $id(2^\omega)$ [Countable sets of integers, i.e., reals]
- $\cong_2$ be $(\cong_1)^+$ [Countable sets of reals]

In general, given $\cong_\alpha$, let

- $\cong_{\alpha+1}$ be the jump $(\cong_\alpha)^+$ (i.e., ‘countable sets of $\cong_\alpha$’)

Note: $\cong_T <_B \cong_0$ iff $T$ has finitely many models.

Of special note: $\cong_2$ is ‘Countable sets of reals.’
Fundamental Dichotomy: Is $\cong_T$ (as a subset of $\text{Mod}(T) \times \text{Mod}(T)$) Borel or properly $\Sigma^1_1$?
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**Easy:** If \( \cong_T \) is Borel complete, then \( \cong_T \) is properly \( \Sigma^1_1 \).
Fundamental Dichotomy: Is $\cong_T$ (as a subset of $\text{Mod}(T) \times \text{Mod}(T)$) Borel or properly $\Sigma^1_1$?

Easy: If $\cong_T$ is Borel complete, then $\cong_T$ is properly $\Sigma^1_1$.

Note: Until recently, all known examples of $\cong_T$ properly $\Sigma^1_1$ were Borel complete, hence $\geq_B$ every $\cong_{T'}$.

This led me (and maybe others) to think of every instance of $\cong_T$ properly $\Sigma^1_1$ as being $>_{B} \cong_{T'}$ whenever $\cong_{T'}$ is Borel.

This is not always the case!
Effect of standard model-theoretic operations:
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- Borel complexity is ill-behaved under reducts.

- There are complete $T_0 \subseteq T_1 \subseteq T_2$ (in languages $L_0 \subseteq L_1 \subseteq L_2$) such that $\text{Mod}(T_0)$ is $\aleph_0$-categorical, $\text{Mod}(T_1)$ is Borel complete, and $\text{Mod}(T_2)$ has countably many models.
• **Naming (or deleting) constants** is only partially understood.

Throughout most of model theory (e.g., showing \( I(T, \aleph_0) = 2^{\aleph_0} \) or the configurations determining the spectrum \( I(T, \kappa) \) for \( \kappa > \aleph_0 \)), naming or deleting finitely many constants is **free**.
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**Open:** Can Borel completeness be gained or lost by naming a constant?

Best result so far:

**Proposition (Rast)**

Let $T$ be complete, and $T(c)$ an expansion formed by naming a constant. Then $\cong_T$ is Borel if and only if $\cong_{T(c)}$ is Borel.
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**Ulrich:** Let $M$ denote the (unique) countable random graph, and let $(M, c_n)_{n \in \omega}$ be any expansion such that $c_i \neq c_j$ for distinct $i, j$. Then $Th(M)$ is $\aleph_0$-categorical, while $Th((M, c_n)_{n \in \omega})$ is Borel complete.
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A later example will give a complete theory $T$ such that $\cong_T$ is properly $\Sigma^1_1$, but for any model $M$, the isomorphism relation $\cong_{El(M)}$ of the elementary diagram of $M$ is Borel.
Only general result to date.

**Marker:** If $T$ is not small, then $\equiv_2 \leq_B \equiv_T$, i.e., ‘countable sets of reals’ Borel reduce to $(\text{Mod}(T), \equiv_T)$. 
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**Paradigm:** ‘Independent unary predicates’ $L = \{U_n : n \in \omega\}$, $T$ says ‘Every finite boolean combination of $\pm U_n$ is consistent.’

Complete 1-types correspond to branches through $2^{<\omega}$ (i.e., reals) and for each branch, one can choose how many elements realize it.
**Theorem (Rast/Sahota)**

If $T$ is o-minimal, then $\equiv_T$ is one of the following:

- $<_B \equiv_0$ (finitely many models);
- Borel equivalent to $\equiv_1$ (reals);
- Borel equivalent to $\equiv_2$ (countable sets of reals);
- Borel complete.

**Note:** The proof of this theorem would have been massively simpler if one could name a constant!
Complete theories of linear orders with (countably many) unary predicates

Theorem (Rast)

If $T$ is a complete theory of linear orders with unary predicates, then $\cong_T$ is one of the following:

- $<_B \cong_0$ (finitely many models);
- Borel equivalent to $\cong_1$ (reals);
- Borel equivalent to $\cong_2$ (countable sets of reals);
- Borel complete.
**ω-stable theories**

**Note:** $T$ ω-stable implies $T$ small ($S_n(\emptyset)$ countable for each $n$)

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**Theorem (L-Shelah)**

*If $T$ is ω-stable and has eni-DOP or is eni-DEEP, then \( \equiv_T \) is Borel complete.*

**Note:** The proof of this would have been at least 10 pages shorter if one could name a constant!

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**Theorem (Rast, streamlining Koerwien)**

*For each ordinal $\alpha < \omega_1$, there is an ω-stable theory $T_\alpha$ such that $\equiv_{(T_\alpha)}$ is Borel equivalent to $\equiv_\alpha$ (the $\alpha$’th jump).*
\(\omega\)-stable theories (cont.)

**Theorem (Koerwien+Ulrich)**

There is an \(\omega\)-stable, depth 2 theory \(K\) for which

- \(\cong_K\) is properly \(\Sigma^1_1\) but
- \(\cong_K\) is NOT Borel complete.
Refining equivalence relations

Let \( L = \{ E_n : n \in \omega \} \) and consider \( L \)-theories \( T \) that say:

- Each \( E_n \) is an equivalence relation;
- \( E_0 \) consists of a single class;
- Each \( E_{n+1} \) refines \( E_n \), i.e., \( E_{n+1}(a, b) \) implies \( E_n(a, b) \).

In order to make \( T \) complete, need only say how many classes \( E_{n+1} \) partitions each \( E_n \)-class into.
Case 1: $REF_\omega$ says: Each $E_{n+1}$-class partitions each $E_n$-class into infinitely many classes.

- $REF_\omega$ is small, BUT
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**Case 2:** $REF_2$ says: Each $E_{n+1}$-class partitions each $E_n$-class into 2 classes.

**Theorem (L-Rast-Ulrich)**

*The isomorphism relation on $REF_2$ is properly $\Sigma^1_1$ but is not Borel complete.*
Hybrids: Given $m \leq \omega$, let $T_m$ be:

- For $n < m$, $E_{n+1}$ partitions each $E_n$-class into infinitely many classes;
- For $n \geq m$, $E_{n+1}$ partitions each $E_n$-class into 2 classes.

Then:

- $T_0$ is $\text{REF}_2$, $T_\omega$ is $\text{REF}_\omega$;
- For all $m$, $\equiv_{T_m}$ is properly $\Sigma^1_1$;
- For all $m$, $T_m$ is small;
- $\equiv_{T_0} < B \equiv_{T_1} < B \equiv_{T_2} < B \cdots < B \equiv_{T_\omega}$. 
Suppose $M \models REF_2$ is countable. Then the elementary diagram $El(M)$ is essentially the same as ‘Independent unary predicates.’ In particular:

- $\cong_{EI(M)}$ is Borel equivalent to $\cong_2$ (countable sets of reals);
- Thus, $\cong_{EI(M)}$ is Borel; BUT
- Its restriction to $L = \{E_n : n \in \omega\}$ is $REF_2$ and $\cong_{REF_2}$ is properly $\Sigma^1_1$
A final thought: It has become empirically clear that ‘Vaught’s conjecture for superstable $T$’ is much more involved than ‘Vaught’s conjecture for $\omega$-stable $T$.’
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Fact: If $T$ is superstable, but not $\omega$-stable, then $T$ is either not small, or else has a type of infinite multiplicity.
A final thought: It has become empirically clear that ‘Vaught’s conjecture for superstable $T$’ is much more involved than ‘Vaught’s conjecture for $\omega$-stable $T$.’

Fact: If $T$ is superstable, but not $\omega$-stable, then $T$ is either not small, or else has a type of infinite multiplicity.

$REF_2$ is the paradigm of a superstable theory with infinite multiplicity!