MATH 601: SOME PROBLEMS FROM PAST EXAMS

1. (a) Let $n \geq 1$. Give a projective resolution of the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$.
(b) Let N be an abelian group. Evaluate $\operatorname{Ext}^j(\mathbb{Z}/n\mathbb{Z}, N)$ for all $j \geq 0$.

2. Let M and N be free modules over an integral domain R. Let $m \in M$ and $n \in N$, and suppose $m \otimes n = 0$ in $M \otimes_R N$. Show that either $m = 0$ or $n = 0$.

3. Let R be an integral domain and let M be an injective R-module. Show that M is divisible; that is, for every $r \neq 0$ in R and every $m \in M$, there exists $n \in M$ such that $rn = m$.

4. Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \longrightarrow & 0 \\
{\scriptstyle h_1}\big\downarrow & & {\scriptstyle h_2}\big\downarrow & & & & \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \longrightarrow & 0
\end{array}$$

Prove that there is a well-defined map from A_3 to B_3 making the diagram commute.

5. Let T be the matrix (over $\mathbb{C}$)

$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}.$$

(a) Give the characteristic and minimal polynomials for T.
(b) Find a vector v such that the dimension of the span of $\{v, Tv, T^2v, T^3v, \dots\}$ is as large as possible, and prove that the dimension you get is as large as possible.

6. Let N be a 5×5 matrix with entries in $\mathbb{C}$ such that $N^n = 0$ for some positive integer n.
(a) Give all possibilities (no proofs needed) for the (i) characteristic polynomial of N, (ii) the minimal polynomial for N, and (iii) the Jordan Canonical Form of N, and indicate which possibilities for (iii) correspond to which possibilities for (i) and (ii).
(b) Give a necessary and sufficient condition for N to be diagonalizable (this should be a condition that applies specifically to the N's in this problem, not a condition that holds for arbitrary matrices).